1000 Solved Problems in Modern Physics

Ahmad A. Kamal

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Dr. Ahmad A. Kamal
425 Silversprings Lane
Murphy, TX 75094, USA
anwarakamal@yahoo.com

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Dedicated to my parents

Preface

This book is targeted mainly to the undergraduate students of USA, UK and other European countries, and the M.Sc of Asian countries, but will be found useful for the graduate students, Graduate Record Examination (GRE), Teachers and Tutors. This is a by-product of lectures given at the Osmania University, University of Ottawa and University of Tebrez over several years, and is intended to assist the students in their assignments and examinations. The book covers a wide spectrum of disciplines in Modern Physics, and is mainly based on the actual examination papers of UK and the Indian Universities. The selected problems display a large variety and conform to syllabi which are currently being used in various countries. The book is divided into ten chapters. Each chapter begins with basic concepts containing a set of formulae and explanatory notes for quick reference, followed by a number of problems and their detailed solutions.

The problems are judiciously selected and are arranged section-wise. The solutions are neither pedantic nor terse. The approach is straight forward and step-by-step solutions are elaborately provided. More importantly the relevant formulas used for solving the problems can be located in the beginning of each chapter. There are approximately 150 line diagrams for illustration.

Basic quantum mechanics, elementary calculus, vector calculus and Algebra are the pre-requisites. The areas of Nuclear and Particle physics are emphasized as revolutionary developments have taken place both on the experimental and theoretical fronts in recent years. No book on problems can claim to exhaust the variety in the limited space. An attempt is made to include the important types of problems at the undergraduate level.

Chapter 1 is devoted to the methods of Mathematical physics and covers such topics which are relevant to subsequent chapters. Detailed solutions are given to problems under Vector Calculus, Fourier series and Fourier transforms, Gamma and Beta functions, Matrix Algebra, Taylor and Maclaurean series, Integration, Ordinary differential equations, Calculus of variation Laplace transforms, Special functions such as Hermite, Legendre, Bessel and Laguerre functions, complex variables, statistical distributions such as Binomial, Poisson, Normal and interval distributions and numerical integration.

Chapters 2 and 3 focus on quantum physics. Chapter 2 is basically concerned with the old quantum theory. Problems are solved under the topics of deBroglie waves, Bohr's theory of hydrogen atom and hydrogen-like atoms, positronium and mesic atoms, X-rays production and spectra, Moseley's law and Duan–Hunt law, spectroscopy of atoms and molecules, which include various quantum numbers and selection rules, and optical Doppler effect.

Chapter 3 is concerned with the quantum mechanics of Schrodinger and Hesenberg. Problems are solved on the topics of normalization and orthogonality of wave functions, the separation of Schrodinger's equation into radial and angular parts, 1-D potential wells and barriers, 3-D potential wells, Simple harmonic oscillator, Hydrogen-atom, spatial and momentum distribution of electron, Angular momentum, Clebsch–Gordon coefficients ladder operators, approximate methods, scattering theory-phase-shift analysis and Ramsuer effect, the Born approximation.

Chapter 4 deals with problems on Thermo–dynamic relations and their applications such a specific heats of gases, Joule–Thompson effect, Clausius–Clapeyron equation and Vander waal's equation, the statistical distributions of Boltzmann and Fermi distributions, the distribution of rotational and vibrational states of gas molecules, the Black body radiation, the solar constant, the Planck's law and Wein's law.

Chapter 5 is basically related to Solid State physics and material science. Problems are covered under the headings, crystal structure, Lattice constant, Electrical properties of crystals, Madelung constant, Fermi energy in metals, drift velocity, the Hall effect, the Debye temperature, the intrinsic and extrinsic semiconductors, the junction diode, the superconductor and the BCS theory, and the Josephson effect.

Chapter 6 deals with the special theory of Relativity. Problems are solved under Lorentz transformations of length, time, velocity, momentum and energy, the invariance of four-momentum vector, transformation of angles and Doppler effect and threshold of particle production.

Chapters 7 and 8 are concerned with problems in low energy Nuclear physics. Chapter 7 covers the interactions of charged particles with matter which include kinematics of collisions, Rutherford Scattering, Ionization, Range and Straggiling, Interactions of radiation with matter which include Compton scattering, photoelectric effect, pair production and nuclear resonance fluorescence, general radioactivity which includes problems on chain decays, age of earth, Carbon dating, alpha decay, Beta decay and gamma decay.

Chapter 8 is devoted to the static properties of nuclei such as nuclear masses, nuclear spin and parity, magnetic moments and quadrupole moments, the Nuclear models, the Fermi gas model, the shell model, the liquid drop model and the optical model, problems on fission and fusion and Nuclear Reactors.

Chapters 9 and 10 are concerned with high energy physics. Chapter 9 covers the problems on natural units, production, interactions and decays of high energy unstable particles, various types of detectors such as ionization chambers, proprortional and G.M. counters, Accelerators which include Betatron, Cyclotron, Synchro-Cyclotron, proton and electron Synchrotron, Linear accelerator and Colliders.

Chapter 10 deals with the static and dynamic properties of elementary particles and resonances, their classification from the point of view of the Fermi–Dirac and Bose–Einstein statistics as well as the three types of interactions, strong, Electro-magnetic and weak, the conservation laws applicable to the three types of interactions, Gell-mann's formula, the properties of quarks and classification into supermultiplets, the types of weak decays and Cabibbo's theory, the neutrino oscillations, Electro–Weak interaction, the heavy bosons and the Standard model.

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Murphy, Texas Ahmad A. Kamal
February 2010

Contents

Chapter 1
Mathematical Physics

1.1 Basic Concepts and Formulae

Vector calculus

Angle between two vectors, $\cos\theta = \frac{\boldsymbol{A}.\boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}$
Condition for coplanarity of vectors, $\boldsymbol{A}.\boldsymbol{B} \times \boldsymbol{C} = 0$

Del

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Gradient

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

Divergence

If $V(x, y, z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, be a differentiable vector field, then $\nabla.\boldsymbol{V} = \frac{\partial}{\partial x}V_1 + \frac{\partial}{\partial y}V_2 + \frac{\partial}{\partial z}V_3$

Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{(Cartesian coordinates } x, y, z)$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\Phi^2}$$
(Spherical coordinates r, θ, Φ)

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \quad \text{(Cylindrical coordinates } r, \theta, z)$$

Line integrals

(a) $\int_C \phi \, \mathrm{d}\boldsymbol{r}$

(b) $\int_C \boldsymbol{A} \,.\, \mathrm{d}\boldsymbol{r}$
(c) $\int_C \boldsymbol{A} \times \mathrm{d}\boldsymbol{r}$

where ϕ is a scalar, $\boldsymbol{A}$ is a vector and $\boldsymbol{r} = x\hat{i} + y\hat{j} + z\hat{k}$, is the positive vector.

Stoke's theorem

$$\oint_C \boldsymbol{A} \,.\, d\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{A}) \,.\, n \, \mathrm{d}\boldsymbol{s} = \iint_S (\nabla \times \boldsymbol{A}) \,.\, \mathrm{d}\boldsymbol{s}$$

The line integral of the tangential component of a vector $\boldsymbol{A}$ taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of $\boldsymbol{A}$ taken over any surface S having C as its boundary.

Divergence theorem (Gauss theorem)

$$\iiint_V \nabla \,.\, \boldsymbol{A} \, \mathrm{d}v = \iint_S \boldsymbol{A}.\hat{n} \, \mathrm{d}s$$

The volume integral is reduced to the surface integral.

Fourier series

Any single-valued periodic function whatever can be expressed as a summation of simple harmonic terms having frequencies which are multiples of that of the given function. Let $f(x)$ be defined in the interval $(-\pi, \pi)$ and assume that $f(x)$ has the period 2π, i.e. $f(x + 2\pi) = f(x)$. The Fourier series or Fourier expansion corresponding to $f(x)$ is defined as

$$f(x) = \frac{1}{2}a_0 + \sum\nolimits_{n=1}^{\infty} (a_0 \cos nx + b_n \sin nx) \tag{1.1}$$

where the Fourier coefficient a_n and b_n are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; \mathrm{d}x \tag{1.2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; \mathrm{d}x \tag{1.3}$$

where $n = 0, 1, 2, \ldots$

If $f(x)$ is defined in the interval $(-L, L)$, with the period $2L$, the Fourier series is defined as

$$f(x) = \frac{1}{2}a_0 + \sum\nolimits_{n=1}^{\infty} (a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)) \tag{1.4}$$

where the Fourier coefficients a_n and b_n are

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n\pi x/L)\, \mathrm{d}x \tag{1.5}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(n\pi x/L)\, \mathrm{d}x \tag{1.6}$$

Complex form of Fourier series

Assuming that the Series (1.1) converges at $f(x)$,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/L} \tag{1.7}$$

with

$$C_n = \frac{1}{L} \int_{C}^{C+2L} f(x) e^{-i\pi n x/L}\, \mathrm{d}x = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ \frac{1}{2}(a_{-n} + ib_{-n}) & n < 0 \\ \frac{1}{2}a_o & n = 0 \end{cases} \tag{1.8}$$

Fourier transforms

The Fourier transform of $f(x)$ is defined as

$$\Im(f(x)) = F(\alpha) = \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\, \mathrm{d}x \tag{1.9}$$

and the inverse Fourier transform of $F(\alpha)$ is

$$\Im^{-1}(f(\alpha)) = F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} \mathrm{d}\alpha \tag{1.10}$$

$f(x)$ and $F(\alpha)$ are known as Fourier Transform pairs. Some selected pairs are given in Table 1.1.

Table 1.1

$f(x)$	$F(\alpha)$	$f(x)$	$F(\alpha)$
$\frac{1}{x^2 + a^2}$	$\frac{\pi e^{-a\alpha}}{a}$	e^{-ax}	$\frac{a}{\alpha^2 + a^2}$
$\frac{x}{x^2 + a^2}$	$-\frac{\pi i \alpha}{a} e^{-a\alpha}$	e^{-ax^2}	$\frac{1}{2}\sqrt{\frac{\pi}{a}} e^{-\alpha^2/4a}$
$\frac{1}{x}$	$\frac{\pi}{2}$	xe^{-ax^2}	$\frac{\sqrt{\pi}}{4a^{3/2}} \alpha e^{-\alpha^2/4a}$

Gamma and beta functions

The gamma function $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, \mathrm{d}x \quad (Re\ n > 0) \tag{1.11}$$

$$\Gamma(n+1) = n\Gamma(n) \tag{1.12}$$

If n is a positive integer

$$\Gamma(n+1) = n! \tag{1.13}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}; \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}; \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi} \tag{1.14}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{1.3.5\ldots(2n-1)\sqrt{\pi}}{2^n} \quad (n = 1, 2, 3, \ldots) \tag{1.15}$$

$$\Gamma\left(-n+\frac{1}{2}\right) = \frac{(-1)^n 2^n \sqrt{\pi}}{1.3.5\ldots(2n-1)} \quad (n = 1, 2, 3, \ldots) \tag{1.16}$$

$$\Gamma(n+1) = n! \cong \sqrt{2\pi n}\, n^n e^{-n} \quad \text{(Stirling's formula)} \tag{1.17}$$

$$n \to \infty$$

Beta function $B(m, n)$ is defined as

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{1.18}$$

$$B(m, n) = B(n, m) \tag{1.19}$$

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, \mathrm{d}\theta \tag{1.20}$$

$$B(m, n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, \mathrm{d}t \tag{1.21}$$

Special funtions, properties and differential equations

Hermite functions:

Differential equation:

$$y'' - 2xy' + 2ny = 0 \tag{1.22}$$

when $n = 0, 1, 2, \ldots$ then we get Hermite's polynomials $H_n(x)$ of degree n, given by

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x^2}\right) \quad \text{(Rodrigue's formula)}$$

First few Hermite's polynomials are:

$$
\begin{aligned}
&H_o(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2\\
&H_3(x) = 8x^3 - 12x,\ H_4(x) = 16x^4 - 48x^2 + 12
\end{aligned}
\tag{1.23}
$$

Generating function:

$$
e^{2tx - t^2} = \sum\nolimits_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \tag{1.24}
$$

Recurrence formulas:

$$
\begin{aligned}
&H_n'(x) = 2nH_{n-1}(x)\\
&H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)
\end{aligned}
\tag{1.25}
$$

Orthonormal properties:

$$
\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)\,\mathrm{d}x = 0 \quad m \neq n \tag{1.26}
$$

$$
\int_{-\infty}^{\infty} e^{-x^2} \{H_n(x)\}^2\,\mathrm{d}x = 2^n\, n!\sqrt{\pi} \tag{1.27}
$$

Legendre functions:

Differential equation of order n:

$$
(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1.28}
$$

when $n = 0, 1, 2, \ldots$ we get Legendre polynomials $P_n(x)$.

$$
P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2 - 1)^n \tag{1.29}
$$

First few polynomials are:

$$
\begin{aligned}
&P_o(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1)\\
&P_3(x) = \frac{1}{2}(5x^3 - 3x),\ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)
\end{aligned}
\tag{1.30}
$$

Generating function:

$$
\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum\nolimits_{n=0}^{\infty} P_n(x)t^n \tag{1.31}
$$

Recurrence formulas:

$$x P_n'(x) - P_{n-1}'(x) = n P_n(x)$$
$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1) P_n(x) \tag{1.32}$$

Orthonormal properties:

$$\int_{-1}^{1} P_m(x) P_n(x)\,\mathrm{d}x = 0 \quad m \neq n \tag{1.33}$$
$$\int_{-1}^{1} \{P_n(x)\}^2\,\mathrm{d}x = \frac{2}{2n+1} \tag{1.34}$$

Other properties:

$$P_n(1) = 1,\ P_n(-1) = (-1)^n,\ P_n(-x) = (-1)^n P_n(x) \tag{1.35}$$

Associated Legendre functions:

Differential equation:

$$(1-x^2)y'' - 2xy' + \left\{ l(l+1) - \frac{m^2}{1-x^2} \right\} y = 0 \tag{1.36}$$
$$P_l^m(x) = (1-x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_l(x) \tag{1.37}$$

where $P_l(x)$ are the Legendre polynomials stated previously, l being the positive integer.

$$P_l^o(x) = P_l(x) \tag{1.38}$$
$$\text{and } P_l^m(x) = 0 \quad \text{if } m > n \tag{1.39}$$

Orthonormal properties:

$$\int_{-1}^{1} P_n^m(x) P_l^m(x)\,\mathrm{d}x = 0 \quad n \neq l \tag{1.40}$$
$$\int_{-1}^{1} \{P_l^m(x)\}^2\,\mathrm{d}x = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \tag{1.41}$$

Laguerre polynomials:

Differential equation:

$$xy'' + (1-x)y' + ny = 0 \tag{1.42}$$

if $n = 0, 1, 2, \ldots$ we get Laguerre polynomials given by

$$L_n(x) = e^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n e^{-x}) \quad \text{(Rodrigue's formula)} \tag{1.43}$$

The first few polynomials are:

$$\begin{aligned} &L_o(x) = 1,\ L_1(x) = -x + 1,\ L_2(x) = x^2 - 4x + 2 \\ &L_3(x) = -x^3 + 9x^2 - 18x + 6,\ L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24 \end{aligned} \tag{1.44}$$

Generating function:

$$\frac{e^{-xs/(1-s)}}{1-s} = \sum_{n=0}^{\infty} \frac{L_n(x)s^n}{n!} \tag{1.45}$$

Recurrence formulas:

$$\begin{aligned} &L_{n+1}(x) - (2n + 1 - x)L_n(x) + n^2 L_{n-1}(x) = 0 \\ &xL_n'(x) = nL_n(x) - n^2 L_{n-1}(x) \end{aligned} \tag{1.46}$$

Orthonormal properties:

$$\int_0^{\infty} e^{-x} L_m(x)L_n(x)\,\mathrm{d}x = 0 \quad m \neq n \tag{1.47}$$

$$\int_0^{\infty} e^{-x} \{L_n(x)\}^2\,\mathrm{d}x = (n!)^2 \tag{1.48}$$

Bessel functions: $(J_n(x))$

Differential equation of order n

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \quad n \geq 0 \tag{1.49}$$

Expansion formula:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!\Gamma(k + 1 - n)} \tag{1.50}$$

Properties:

$$J_{-n}(x) = (-1)^n J_n(x) \quad n = 0, 1, 2, \ldots \tag{1.51}$$

$$J_o'(x) = -J_1(x) \tag{1.52}$$

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \tag{1.53}$$

Generating function:

$$e^{x(s-1/s)/2} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \tag{1.54}$$

Laplace transforms:

Definition:
A Laplace transform of the function $F(t)$ is

$$\int_0^\infty F(t)e^{-st}\,\mathrm{d}t = f(s) \tag{1.55}$$

The function $f(s)$ is the Laplace transform of $F(t)$. Symbolically, $\mathcal{L}\{F(t)\} = f(s)$ and $F(t) = \mathcal{L}^{-1}\{f(s)\}$ is the inverse Laplace transform of $f(s)$. $\mathcal{L}^{-1}$ is called the inverse Laplace operator.

Table of Laplace transforms:

$F(t)$	$f(s)$
$aF_1(t) + bF_2(t)$	$af_1(s) + bf_2(s)$
$aF(at)$	$f(s/a)$
$e^{at}F(t)$	$f(s-a)$
$F(t-a)\ t > a$; $0\ \ t < a$	$e^{-as}f(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n} n = 1, 2, 3, \ldots$
e^{at}	$\frac{1}{s-a}$
$\frac{\sin\ at}{a}$	$\frac{1}{s^2+a^2}$
$\cos\ at$	$\frac{s}{s^2+a^2}$
$\frac{\sinh\ at}{a}$	$\frac{1}{s^2-a^2}$
$\cosh\ at$	$\frac{s}{s^2-a^2}$

Calculus of variation

The calculus of variation is concerned with the problem of finding a function $y(x)$ such that a definite integral, taken over a function shall be a maximum or minimum. Let it be desired to find that function $y(x)$ which will cause the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')\,\mathrm{d}x \tag{1.56}$$

to have a stationary value (maximum or minimum). The integrand is taken to be a function of the dependent variable y as well as the independent variable x and $y' = \mathrm{d}y/\mathrm{d}x$. The limits x_1 and x_2 are fixed and at each of the limits y has definite value. The condition that I shall be stationary is given by Euler's equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0 \tag{1.57}$$

When F does not depend explicitly on x, then a different form of the above equation is more useful

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0 \tag{1.58}$$

which gives the result

$$F - y'\frac{\partial F}{\partial y'} = \text{Constant} \tag{1.59}$$

Statistical distribution

Binomial distribution

The probability of obtaining x successes in N-independent trials of an event for which p is the probability of success and q the probability of failure in a single trial is given by the binomial distribution $B(x)$.

$$B(x) = \frac{N!}{x!(N-x)!}p^x q^{N-x} = C_x^N p^x q^{N-x} \tag{1.60}$$

$B(x)$ is normalized, i.e.

$$\sum\nolimits_{x=0}^{N} B(x) = 1 \tag{1.61}$$

It is a discrete distribution.
The mean value,

$$\langle x\rangle = Np \tag{1.62}$$

The S.D.,

$$\sigma = \sqrt{Npq} \tag{1.63}$$

Poisson distribution

The probability that x events occur in unit time when the mean rate of occurrence is m, is given by the Poisson distribution $P(x)$.

$$P(x) = \frac{e^{-m} m^x}{x!} \quad (x = 0, 1, 2, \ldots) \tag{1.64}$$

The distribution $P(x)$ is normalized, that is

$$\sum_{x=0}^{\infty} p(x) = 1 \tag{1.65}$$

This is also a discrete distribution.

When NP is held fixed, the binomial distribution tends to Poisson distribution as N is increased to infinity.

The expectation value, i.e.

$$\langle x \rangle = m \tag{1.66}$$

The S.D.,

$$\sigma = \sqrt{m} \tag{1.67}$$

Properties:

$$p_{m-1} = p_m \tag{1.68}$$

$$p_{x-1} = \frac{x}{m} p_m \; and \; p_{x+1} = \frac{m}{m+1} p_x \tag{1.69}$$

Normal (Gaussian distribution)

When p is held fixed, the binomial distribution tends to a Normal distribution as N is increased to infinity. It is a continuous distribution and has the form

$$f(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2} \mathrm{d}x \tag{1.70}$$

where m is the mean and σ is the S.D.

The probability of the occurrence of a single random event in the interval $m - \sigma$ and $m + \sigma$ is 0.6826 and that between $m - 2\sigma$ and $m + 2\sigma$ is 0.973.

Interval distribution

If the data contains N time intervals then the number of time intervals n between t_1 and t_2 is

$$n = N(e^{-at_1} - e^{-at_2}) \tag{1.71}$$

where a is the average number of intervals per unit time. Short intervals are more favored than long intervals.

Two limiting cases:

$$\text{(a) } t_2 = \infty; N = N_o e^{-\lambda t} \quad \text{(Law of radioactivity)} \tag{1.72}$$

This gives the number of surviving atoms at time t.

$$\text{(b) } t_1 = 0; N = N_o(1 - e^{-\lambda t}) \tag{1.73}$$

For radioactive decays this gives the number of decays in time interval 0 and t.

Above formulas are equally valid for length intervals such as interaction lengths.

Moment generating function (MGF)

$$\begin{aligned} \text{MGF} &= Ee^{(x-\mu)t} \\ &= E\left[1 + (x-\mu)t + (x-\mu)^2 \frac{t^2}{2!} + \ldots\right] \\ &= 1 + 0 + \mu_2 \frac{t^2}{2!} + \mu_3 \frac{t^3}{3!} + \ldots \end{aligned} \tag{1.74}$$

so that μ_n, the nth moment about the mean is the coefficient of $t^n/n!$.

Propagation of errors

If the error on the measurement of $f(x, y, \ldots)$ is σ_f and that on x and y, σ_x and σ_y, respectively, and σ_x and σ_y are uncorrelated then

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + \cdots \tag{1.75}$$

Thus, if $f = x \pm y$, then $\sigma_f = \left(\sigma_x^2 + \sigma_y^2\right)^{1/2}$

And if $f = \frac{x}{y}$ then $\frac{\sigma f}{f} = \left(\frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2}\right)^{1/2}$

Least square fit

(a) Straight line: $y = mx + c$

It is desired to fit pairs of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by a straight line

Residue: $S = \sum_{i=1}^{n}(y_i - mx_i - C)^2$

Minimize the residue: $\frac{\partial s}{\partial m} = 0; \frac{\partial s}{\partial c} = 0$

The normal equations are:

$$\begin{aligned} m\sum\nolimits_{i=1}^{n} x_i^2 + C\sum\nolimits_{i=1}^{n} x_i - \sum\nolimits_{i=1}^{n} x_i y_i = 0 \\ m\sum\nolimits_{i=1}^{n} x_i + nC - \sum\nolimits_{i=1}^{n} y_i = 0 \end{aligned}$$

which are to be solved as ordinary algebraic equations to determine the best values of m and C.

(b) Parabola: $y = a + bx + cx^2$

Residue: $S = \sum_{i=1}^{n}(y_i - a - bx_i - cx_i^2)^2$
Minimize the residue: $\frac{\partial s}{\partial a} = 0; \frac{\partial s}{\partial b} = 0; \frac{\partial s}{\partial c} = 0$

The normal equations are:

$$\sum y_i - na - b\sum x_i - c\sum x_i^2 = 0$$
$$\sum x_i y_i - a\sum x_i - b\sum x_i^2 - c\sum x_i^3 = 0$$
$$\sum x_i^2 y_i - a\sum x_i^2 - b\sum x_i^3 - c\sum x_i^4 = 0$$

which are to be solved as ordinary algebraic equations to determine the best value of a, b and c.

Numerical integration

Since the value of a definite integral is a measure of the area under a curve, it follows that the accurate measurement of such an area will give the exact value of a definite integral; $I = \int_{x_1}^{x_2} y(x)\mathrm{d}x$. The greater the number of intervals (i.e. the smaller Δx is), the closer will be the sum of the areas under consideration.

Trapezoidal rule

$$\text{area} = \left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots y_{n-1} + \frac{1}{2}y_n\right)\Delta x \tag{1.76}$$

Simpson's rule

$$\text{area} = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots y_n),\ \textit{n being even.} \tag{1.77}$$

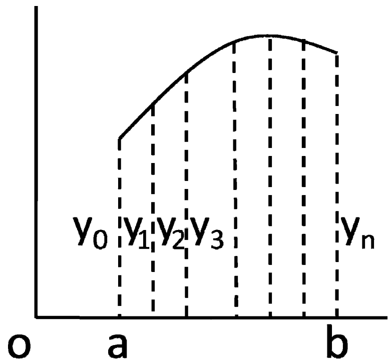

Fig. 1.1 Integration by Simpson's rule and Trapezoidal rule

Matrices

Types of matrices and definitions

Identity matrix:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.78}$$

Scalar matrix:

$$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}; \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \tag{1.79}$$

Symmetric matrix:

$$\left(a_{ji} = a_{ij}\right); \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \tag{1.80}$$

Skew symmetric:

$$\left(a_{ji} = -a_{ij}\right); \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & -a_{23} & a_{33} \end{pmatrix} \tag{1.81}$$

The *Inverse of a matrix* $B = A^{-1}$ (B equals A inverse):
if $AB = BA = I$ and further, $(AB)^{-1} = B^{-1}A^{-1}$
A commutes with B if $AB = BA$
A anti-commutes with B if $AB = -BA$

The *Transpose* (A') *of a matrix* A means interchanging rows and columns.

$$\begin{aligned} Further, (A + B)' &= A' + B' \\ (A')' &= A, (kA)' = kA' \end{aligned} \tag{1.82}$$

The *Conjugate of a matrix*. If a matrix has complex numbers as elements, and if each number is replaced by its conjugate, then the new matrix is called the conjugate and denoted by A^* or $\overline{A}$ (A conjugate)

The *Trace (Tr) or Spur* of a matix is the num of the diagonal elements.

$$Tr = \sum a_{ii} \tag{1.83}$$

Hermetian matrix

If $\overline{A'} = A$, so that $a_{ij} = \overline{a}_{ji}$ for all values of i and j. Diagonal elements of an Hermitian matrix are real numbers.

Orthogonal matrix

A square matrix is said to be orthogonal if $AA' = A'A = I$, i.e. $A' = A^{-1}$

The column vector (row vectors) of an orthogonal matrix A are mutually orthogonal unit vectors.

The inverse and the transpose of an orthogonal matrix are orthogonal.
The product of any two or more orthogonal matrices is orthogonal.
The determinant of an Orthogonal matrix is ± 1.

Unitary matrix

A square matrix is called a unitary matrix if $(\overline{A})'A = A(\overline{A})' = I$, i.e. if $(\overline{A})' = A^{-1}$.

The column vectors (row vectors) of an n-square unitary matrix are an orthonormal set.

The inverse and the transpose of a unitary matrix are unitary.
The product of two or more unitary matrices is unitary.
The determinant of a unitary matrix has absolute value 1.

Unitary transformations

The linear transformation $Y = AX$ (where A is unitary and X is a vector), is called a unitary transformation.

If the matrix is unitary, the linear transformation preserves length.

Rank of a matrix

If $|A| \neq 0$, it is called non-singular; if $|A| = 0$, it is called singular.

A non-singular matrix is said to have rank r if at least one of its r-square minors is non-zero while if every $(r + 1)$ minor, if it exists, is zero.

Elementary transformations

(i) The interchange of the ith rows and jth rows or ith column or jth column.
(ii) The multiplication of every element of the ith row or ith column by a non-zero scalar.
(iii) The addition to the elements of the ith row (column) by k (a scalar) times the corresponding elements of the jth row (column). These elementary transformations known as row elementary or row transformations do not change the order of the matrix.

Equivalence

A and B are said to be equivalent ($A \sim B$) if one can be obtained from the other by a sequence of elementary transformations.

The adjoint of a square matrix

If $A = [a_{ij}]$ is a square matrix and α_{ij} the cofactor of a_{ij} then

$$\text{adj } A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \alpha_{nn} \end{bmatrix}$$

The cofactor $\alpha_{ij} = (-1)^{i+j} M_{ij}$
where M_{ij} is the minor obtained by striking off the ith row and jth column and computing the determinant from the remaining elements.

Inverse from the adjoint

$$A^{-1} = \frac{adj\ A}{|A|}$$

Inverse for orthogonal matrices

$$A^{-1} = A'$$

Inverse of unitary matrices

$$A^{-1} = (\overline{A})'$$

Characteristic equation

$$\text{Let } AX = \lambda X \tag{1.84}$$

be the transformation of the vector X into λX, where λ is a number, then λ is called the eigen or characteristic value.
From (1.84):

$$(A - \lambda I)X = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & \cdots & a_{nn} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \tag{1.85}$$

The system of homogenous equations has non-trivial solutions if

$$|A - \lambda I| = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & \cdots & a_{nn} - \lambda \end{bmatrix} = 0 \tag{1.86}$$

The expansion of this determinant yields a polynomial $\phi(\lambda) = 0$ is called the characteristic equation of A and its roots $\lambda_1, \lambda_1, \ldots, \lambda_n$ are known as the characteristic roots of A. The vectors associated with the characteristic roots are called invariant or characteristic vectors.

Diagonalization of a square matrix

If a matrix C is found such that the matrix A is diagonalized to S by the transformation

$$S = C^{-1}AC \tag{1.87}$$

then S will have the characteristic roots as the diagonal elements.

Ordinary differential equations

The methods of solving typical ordinary differential equations are from the book "Differential and Integral Calculus" by William A. Granville published by Ginn & Co., 1911.

An ordinary differential equation involves only one independent variable, while a partial differential equation involves more than one independent variable.

The order of a differential equation is that of the highest derivative in it.

The degree of a differential equation which is algebraic in the derivatives is the power of the highest derivative in it when the equation is free from radicals and fractions.

Differential equations of the first order and of the first degree

Such an equation must be brought into the form $M\mathrm{d}x + N\mathrm{d}y = 0$, in which M and N are functions of x and y.

Type I variables separable

When the terms of a differential equation can be so arranged that it takes on the form

$$(A) \quad f(x)\,\mathrm{d}x + F(y)\,\mathrm{d}y = 0$$

where $f(x)$ is a function of x alone and $F(y)$ is a function of y alone, the process is called separation of variables and the solution is obtained by direct integration.

$$(B) \quad \int f(x)\,\mathrm{d}x + \int F(y)\,\mathrm{d}y = C$$

where C is an arbitrary constant.

Equations which are not in the simple form (A) can be brought into that form by the following rule for separating the variables.

First step: Clear off fractions, and if the equation involves derivatives, multiply through by the differential of the independent variable.

Second step: Collect all the terms containing the same differential into a single term. If then the equation takes on the form

$$XY\,\mathrm{d}x + X'Y'\,\mathrm{d}y = 0$$

where X, X' are functions of x alone, and Y, Y' are functions of y alone, it may be brought to the form (A) by dividing through by $X'Y'$.

Third step: Integrate each part separately as in (B).

Type II homogeneous equations

The differential equation

$$M\mathrm{d}x + N\mathrm{d}y = 0$$

is said to be homogeneous when M and N are homogeneous function of x and y of the same degree. In effect a function of x and y is said to be homogenous in the variable if the result of replacing x and y by λx and λy (λ being arbitrary) reduces to the original function multiplied by some power of λ. This power of λ is called the degree of the original function. Such differential equations may be solved by making the substitution

$$y = vx$$

This will give a differential equation in v and x in which the variables are separable, and hence we may follow the rule (A) of type I.

Type III linear equations

A differential equation is said to be linear if the equation is of the first degree in the dependent variables (usually y) and its derivatives. The linear differential equation of the first order is of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$$

where P, Q are functions of x alone, or constants, the solution is given by

$$ye^{\int P\mathrm{d}x} = \int Qe^{\int P\mathrm{d}x}\,\mathrm{d}x + C$$