CMS Books in Mathematics
Ouvrages de mathématiques de la SMC

1 Herman/Kučera/Šimša   Equations and Inequalities

2 Arnold   Abelian Groups and Representations of Finite Partially Ordered Sets

3 Borwein/Lewis   Convex Analysis and Nonlinear Optimization, 2nd Ed.

4 Levin/Lubinsky   Orthogonal Polynomials for Exponential Weights

5 Kane   Reflection Groups and Invariant Theory

6 Phillips   Two Millennia of Mathematics

7 Deutsch   Best Approximation in Inner Product Spaces

8 Fabian et al.   Functional Analysis and Infinite-Dimensional Geometry

9 Křížek/Luca/Somer   17 Lectures on Fermat Numbers

10 Borwein   Computational Excursions in Analysis and Number Theory

11 Reed/Sales (Editors)   Recent Advances in Algorithms and Combinatorics

12 Herman/Kučera /Šimša   Counting and Configurations

13 Nazareth   Differentiable Optimization and Equation Solving

14 Phillips   Interpolation and Approximation by Polynomials

15 Ben-Israel/Greville   Generalized Inverses, 2nd Ed.

16 Zhao   Dynamical Systems in Population Biology

17 Göpfert et al.   Variational Methods in Partially Ordered Spaces

18 Akivis/Goldberg   Differential Geometry of Varieties with Degenerate Gauss Maps

19 Mikhalev/Shpilrain /Yu   Combinatorial Methods

20 Borwein/Zhu   Techniques of Variational Analysis

21 Van Brummelen/Kinyon   Mathematics and the Historian's Craft

22 Lucchetti   Convexity and Well-Posed Problems

23 Niculescu/Persson   Convex Functions and Their Applications

24 Singer   Duality for Nonconvex Approximation and Optimization

25 Sinclair/Pimm/Higginson   Mathematics and the Aesthetic

Nathalie Sinclair, David Pimm and
William Higginson (Eds.)

Mathematics and the Aesthetic

New Approaches to an Ancient Affinity

Springer

Nathalie Sinclair
Department of Mathematics
D216 Wells Hall
Michigan State University
East Lansing, MI 48824
USA
nathsinc@math.msu.edu

David Pimm
Department of Secondary Education
341 Education South
University of Alberta
Edmonton, AB T6G 2G5
CANADA
david.pimm@ualberta.ca

William Higginson
Faculty of Education
A235 McArthur Hall
Queen's University
Kingston, ON K7L 3N6
CANADA
higginsw@educ.queensu.ca

Cover illustration: William Blake's "The Ancient of Days"

Mathematics Subject Classification (2000): 00A30

Library of Congress Control Number: 2006926457

ISBN-10: 0-387-30526-2 e-ISBN -10: 0-387-38145-7
ISBN-13: 978-0387-30526-4 e-ISBN -13: 978-0387-38145-9

Printed on acid-free paper.

9 8 7 6 5 4 3 2 1

springer.com

This book is dedicated to the memory of
Martin Schiralli (1947–2003)
philosopher, colleague, teacher and friend.

PREFACE

A majority of the chapters in this book first saw the light of day as talks at a conference organised and held at Queen's University in Kingston, Ontario, Canada in April 2001. This small, invitational meeting, tellingly entitled *Beauty and the Mathematical Beast,* brought together a range of academics interested in and committed to exploring connections between mathematics and aesthetics. The enthusiastic response of participants at this gathering encouraged the presenters to expand upon their initial contributions and persuaded the organisers to recruit further chapters in order to bring a greater balance to the whole.

The timing of this event was not arbitrary. The preceding decade had seen a resurgence in serious writing dealing with deeper relations between mathematics (and science) and 'the beautiful'. In many ways, we the editors of this volume found these contributions to the literature were revisiting and drawing on themes that had been prominent over two thousand five hundred years ago, in certain writings of the Pythagoreans. While not intending to offer a historical reappraisal of these ancient thinkers here, we have none the less chosen to invoke this profound interweaving of the mathematical and the aesthetic to which this reputedly secretive philosophical sect was extensively attuned.

This book is divided into three sections comprising three chapters each, each with its own short introduction discussing the particular chapters within. These nine chapters in all are flanked by an introductory and a concluding chapter, both of which written by ourselves, which we describe now.

The opening Chapter α describes the ancient affinity between the mathematical and the aesthetic referred to in the book's title, an affinity we aim to illuminate as well as cultivate and advocate by means of this collection. Chapter α also provides a brief history of the mathematical aesthetic, beginning with the Pythagoreans but flowering in the twentieth century, while highlighting some of the themes and issues that subsequent chapters raise. These include attention to the following familiar questions: *can criteria for mathematical beauty be discovered?*, *is mathematics created or discovered?* and *is mathematics an art or a science?*

The final chapter of this book, Chapter ω, revisits some of these questions posed in Chapter α in light of the nine chapters in between. It provides some insights into those initial questions while raising further ones of its own. In particular, it offers three strong themes which stretch the mathematical aesthetic beyond the boundaries set by previous inquiries, all of which are related to potential sources of pleasure and desire for the mathematician: desire for distance and detachment; longing for certainty and perfection; pleasure in melancholy.

The ten authors of the various chapters in this book come from Canada, the US and Europe. Two who were born in Britain now live and work in Canada, while one from Latvia and one from Canada are now in the US. Each anglophone country has its own slight variants of spoken and written English, as well as punctuation conventions. Is the em-dash a thing of beauty or an abhorrence three times wider than any other character in the set? Is that extra ‘u’ in *colour* redundant, that repeated ‘l’ in ‘travelled’ an unnecessary extravagance (as a number of spell-checkers suggest)? Should the issue of the mathematical scope of variables enter into discussions of where to place commas and full-stops in relation to quotation marks? Is an ‘s’ or a ‘z’ to be preferred in generalisations? [1] What seem to be matters of convention (and are therefore, at root, arbitrary) none the less raised a number of exercising aesthetic issues. As editors, we have decided on a position of plurality and respect for individual heritage, rather than impose a completely specified geographic orthography.

One of the considerable satisfactions we the editors have received in creating this book has arisen from drawing on the diverse expertise of the contributors to this volume, both mathematical and otherwise. Another has been the extended opportunity for the three of us to work alongside one another, exploring matters large and small.

We specifically want to mention here the breadth of scholarship that Martin Schiralli (the author of Chapter 5) brought to this project. Tragically, Martin died before this book was completed, aged only 56. His depth of philosophical knowledge, combined with his fresh perspective on mathematics, added considerably to many elements of this collection.

Nathalie Sinclair
David Pimm
William Higginson

January, 2006

[1] An entertaining discussion of some related issues can be found in *Eats, Shoots & Leaves* (Truss, 2004).

ACKNOWLEDGEMENTS

We, the editors, wish to acknowledge the considerable assistance provided by the combined resources of the following library systems:

Michigan State University
Queen's University
Simon Fraser University
University of Alberta
University of British Columbia
Vancouver Public Library

We also wish to thank Rachelle Painchaud-Nash (of Fine Line Design) for her meticulous and thoughtful design and setting of this book.

Chapter 3

Figure 1: © David W. Henderson.
Figure 2: © Daina Taimina.
Figure 10: © 2004 The M.C. Escher Company – Baarn – Holland. All rights reserved.
Figure 18: © 2004 The M.C. Escher Company – Baarn – Holland. All rights reserved.

Chapter 9

Figure 2: Photographer W. Ritter, courtesy of Dumbarton Oaks, Byzantine Photograph and Fieldwork Archives,Washington, DC.
Figure 4: © Photograph: The National Gallery, London.
Figure 5: © Picasso Estate (Paris)/SODRAC (Montreal) 2005.
Figure 6: © Museo Poldi Pezzoli, Milan.
Figure 7: © René Magritte Estate/ADAGP (Paris)/SODRAC (Montreal) 2005 © Tate, London 2004.
Figure 8: © Scala/Art Resource, NY.
p. 209: Photograph by Ken Saunders.
p. 213: Photograph by Graham Challifour.
Figure 9: © The Trustees of the National Museums of Scotland.
p. 215: Photograph © Lacock Abbey Publications, reproduced by permission of the William Henry Fox Talbot Trustees.
Figure 10: © Réunion des Musées Nationaux/Art Resource, NY. Photograph by H. Lewandowski.

Chapter ω

Figure 1: © The Trustees of The British Museum.

We are very grateful to the following for permission to reproduce poetic material in this book.

p. 45: Jet Wimp (now Jet Foncannon), co-editor of the anthology *Against Infinity,* which contains the poem 'Poet as mathematician' by Lillian Morrison.

p. 182: Sharon Nelson, for the lines quoted from her collection *This Flesh These Words*.

p. 226: Patrick Lane, for the lines quoted from his collection *Old Mother.*

p. 248: Faber and Faber Ltd (London) and HaperCollins (New York), for the lines quoted from Sylvia Plath's collection *Ariel.*

Contents

Notes about Authors

Jonathan M. Borwein was the founding Director of the Centre for Experimental and Constructive Mathematics at Simon Fraser University (SFU). In 2004, he (re-)joined the Faculty of Computer Science at Dalhousie University as a Canada Research Chair in Distributed and Collaborative Research, cross-appointed in Mathematics, while preserving an adjunct appointment at Simon Fraser. Jonathan was born in St Andrews, Scotland, in 1951 and received his D.Phil. from Oxford in 1974 as a Rhodes Scholar. His interests span pure (analysis), applied (optimization) and computational (numerical and computational analysis) mathematics and high-performance computing. He has authored ten books (most recently two on experimental mathematics – www.expmath.info – and a monograph on variational analysis) and over 250 journal articles. He is currently Governor at large of the Mathematical Association of America (2004–07) and a past President of the Canadian Mathematical Society (2000–02).

David W. Henderson was born in Walla Walla, Washington State, and graduated from Ames (Iowa) High School, Swarthmore College (mathematics, physics, philosophy) and the University of Wisconsin (with a Ph.D. in geometric topology under R. H. Bing). After a two-year stint at the Institute for Advanced Study in Princeton, he joined the mathematics faculty at Cornell University in 1966 and has been there ever since. David's great love in mathematics is geometry of all sorts. His interests have widened into issues of what he calls educational mathematics. He has been directing Ph.D. theses in both mathematics and mathematics education. These interests led him to be invited to the ICMI Study Conference on the Teaching of Geometry in Sicily in 1995, where he met Daina Taimina. David has written four textbooks on geometry (three with Daina) and they have collaborated on many other activities. David has had visiting academic positions in India, Moscow, Warsaw, West Bank (Palestine), South Africa, USA and Latvia. Currently, he is Professor of Mathematics at Cornell University.

William C. Higginson is a member, and former Coordinator, of the Mathematics, Science and Technology Education Group in the Faculty of Education at Queen's University, Kingston, Ontario. A graduate of Queen's, Cambridge and Exeter Universities, as well as the University of Alberta, he has taught at Queen's since 1973. In 1983-84, he was a visiting professor in the Department of Architecture at the Massachusetts Institute of Technology and was a founding member of the Media Laboratory there. He returned to MIT as visiting professor of media technology in 1988. His research interests centre on the interaction between the subject of mathematics and various other disciplines, particularly literature as well as the visual and plastic arts. He is one of the authors of *Creative Mathematics* (Upitis, Phillips and Higginson, 1997), in which he documented a constructive–aesthetic approach to the teaching of mathematics.

R. Nicholas Jackiw is the original designer and developer of *The Geometer's Sketchpad.* An educational software environment for the creation, visualization, exploration, and analysis of mathematical models, Sketchpad provides a rare example of effective educational software that has made a successful transition from academic research lab to widespread commercial impact. He began work on the software as an undergraduate at Swarthmore College, under the direction of Eugene Klotz and Doris Schattschneider, and presently serves as the Chief Technology Officer of KCP Technologies, the software affiliate of Key Curriculum Press. In addition to designing software and directing the project's programming staff, he is active in the pre-service and in-service professional development of teachers, conducting workshops and institutes across the country; and has overseen the program's translation into more than a dozen foreign languages. He leads NSF software research projects in Principal Investigator or Senior Researcher capacities and is a frequently invited speaker to schools, software and geometry conferences, as well as meetings of the National Council of Teachers of Mathematics.

David Pimm is currently a professor of mathematics education at the University of Alberta. A Fulbright scholar, he has authored three books, edited four more and written many research articles which explore the inter-relationships between language and mathematics. His work has focused both on analyses of mathematics classroom language and on producing theoretical accounts of linguistic aspects of mathematics itself. He is particularly interested in the roles of metaphor and metonymy in creative mathematical endeavour. His secondary research interest is in the potential influence of studies of the history and philosophy of mathematics on the teaching of mathematics. He was editor of the international journal *For the Learning of Mathematics* from 1997 until 2003.

Doris Schattschneider holds a Ph.D. in mathematics from Yale University and is Professor Emerita of Mathematics at Moravian College, where she taught since 1968. She was Geometer and Senior Associate on the NSF-funded Visual Geometry Project that produced the software *The Geometer's Sketchpad,* along with videos and activity books on polyhedra and symmetry. She has lectured widely on the topics of tiling, polyhedra, symmetry, dynamic geometry, geometry and art (especially the art of M. C. Escher) and visualization in teaching. She is the author of more than 40 articles, and author, co-author or editor of several books, including *M. C. Escher: Visions of Symmetry* (Freeman, 1990; Abrams, 2004), *Geometry Turned On: Dynamic*

Software in Learning, Teaching, and Research (MAA, 1997), *A Companion to Calculus* (Brooks/Cole, 1995) and *M. C. Escher's Legacy* (Springer, 2003). Doris has been active in the Mathematical Association of America at all levels and was editor of *Mathematics Magazine* 1981–1985. In 1993, she received the national MAA Award for Distinguished Teaching of College or University Mathematics.

Martin Schiralli, until his untimely death in 2003, was associate professor of the philosophy of education at Queen's University where he specialised in epistemology and aesthetics. His book, C*onstructive Postmodernism* (1999), analysed the epistemological challenges and aesthetic opportunities presented in newer ways of thinking. In keeping with the renewed postmodern interest in relaxing the epistemological demarcations between established disciplines and fields of inquiry, his most recent research involved the concept of 'pattern' as a means of representing those underlying affinities between the mathematical and the aesthetic that are of particular relevance to contemporary art and technology.

Nathalie Sinclair is currently an Assistant Professor at Michigan State University, cross-appointed between the Department of Mathematics and the Department of Teacher Education. Her work on the mathematical aesthetic has extended to the K-12 domain, and, more recently, to the post-secondary level where she is investigating the aesthetic development of young mathematicians. She is also interested in dynamic mathematics environments and the roles of visualisation, intuition and experimentation in the development of mathematical understanding.

Dick Tahta has taught both in schools and universities. Now retired, he still has a love–hate relationship with mathematics. He often feels iconoclastic about mathematics – especially the teaching of it to the unwilling. But sometimes a theorem can excite him as much as a painting or a poem.

Daina Taimina was born and received all her formal education in Riga, Latvia. In 1977, she started to teach at the University of Latvia and continued for more than twenty years. Her Ph.D. thesis was in theoretical computer science (under Rusins Freivalds), but later she became more involved with geometry, history of mathematics and mathematics education. These interests led to her being invited to the ICMI Study Conference on the Teaching of Geometry in Sicily in 1995 where she met David Henderson. She has written a book on the history of mathematics (in Latvian) and (with David) three recent geometry textbooks. Daina was a Visiting Associate Professor at Cornell 1997-2003. Currently, Daina is a Senior Research Associate at Cornell University.

CHAPTER α
A Historical Gaze at the Mathematical Aesthetic

Nathalie Sinclair and David Pimm

> No matter how correct a mathematical theorem may appear to be, one ought never to be satisfied that there was not something imperfect about it until it gives the impression of also being beautiful. (George Boole, in MacHale, 1993, p. 107)

The ancient Greeks, primarily by way of the Pythagoreans, established and celebrated a fundamental affinity between the mathematical and the aesthetic. This affinity was nothing about surface charm or happy coincidences. It had deep roots, integral as it was to the world-view of the Pythagoreans, to their beliefs about knowledge and learning. It closely connected the raw world of sense and experience to the divine world of perfection and beauty. Number was the principle that governed all things, rather than being simply useful for counting or measuring – as modern minds might think, if indeed they stop to consider this omnipresent convenience at all. Through number, one could come to know the world, and through the harmonies found in numerical patterns and in geometrical forms, one could gain access to the clearest and most indubitable essence – the real.

This ancient affinity started losing sway early on, even with Plato and Aristotle. Nevertheless, traces of this Pythagorean perception have remained, resurfacing at various times, such as at the beginning of the twentieth century. For instance, in the second volume of his book *On Growth and Form,* D'Arcy Thompson (1917/1968) wrote:

> For the harmony of the world is manifest in Form and Number, and the heart and soul and all the poetry of Natural Philosophy are embodied in the concept of mathematical beauty. (pp. 1096-1097)

Thompson went on to add that this is what the Pythagoreans taught us, Philolaus in particular (a Pythagorean whose influence is also discussed in Chapter 5 of this book), before remarking:

> Moreover, the perfection of mathematical beauty is such (as Colin Maclaurin learned of the bee), that whatsoever is most beautiful and regular is also found to be most useful and excellent. (p. 1097)

While not all would agree with his attributing to the beautiful the most utilitarian properties (or at least the most 'useful', however that may be seen at different times), Thompson is, in this passage, apparently identifying mathematics as possessing the highest form of perfection – a theme we shall find recurring repeatedly. Finally, Thompson seconded the view of a certain Monsieur Henri Fabré, who wrote that one sees in Number "le comment et le pourquoi des choses" [the how and the why of things] and finds in it "la clef de voûte de l'Univers" [the keystone of the universe] (p. 1097).

If Thompson signalled rapprochement, we can also identify periods of disjunction or even denial of this ancient affinity. For instance, T. S. Eliot (1921/1932) wrote of his sense of a 'dissociation of sensibility' (the loss of the direct fusion of thought and feeling) in much of the poetry of eighteenth- and nineteenth-century England. (This, as well as other instances, including the nineteenth-century English Romantics' scorn of mathematics and science, is touched on in Chapter 9.)

And one of the more recent accounts of this process of the scientific/artistic affinity dissolving (at least symbolically), from the mid-twentieth century this time, was given by scientist and novelist C. P. Snow (1959) in his essay naming and exploring aspects of 'the two cultures'. Since then, however, these two cultures – the arts and the sciences – have once again started to find an intermittent, yet growing rapport, as evidenced by the number of books, conferences and courses seeking common behaviour and beliefs.

This recent work has included many productive marriages between the sciences (including mathematics) and the arts, such as, for example, contemporary sculptures of numerical patterns (Dickson, 1993) and mathematical analyses of Jackson Pollock's paintings (Taylor *et al.,* 1999): this is further discussed in Chapter 6. Ethnographically-oriented scholars have taken interest in revealing the mathematical dimension of past artistic artifacts, such as the geometry of Pueblo pottery (Campbell, 1989) and the symmetry of Islamic design (Chorbachi, 1989). And, of course, the plethora of books on the Dutch artist Maurits Escher, particularly the recently published *M.C. Escher's Legacy: a Centennial Celebration* (Schattschneider and Emmer, 2003), has shown how his prints were born out of the artist's mathematical *and* artistic interests and how his work continues to inspire both mathematical *and* artistic analyses.

Scholars working in this interdisciplinary, 'cross-cultural' arena provide concrete examples of the ways in which mathematics and the arts can both inspire each other, not only in contemporary settings, but also in historical ones. Increasingly, however, scholars have also been working to reveal the close relationship between scientific and artistic creativity and have succeeded in defying popular beliefs that feed the antagonistic 'two culture' worldview, including that which holds that scientists operate exclusively rationally and artists solely intuitively or emotionally.

Some aspects of this ancient affinity even seem to be seeping into other, non-academic cultural milieux. Images of fractals, for instance, which have become increasingly widespread (who has never found themselves staring at a fractal screensaver?), have provided many non-mathematicians with opportunities to encounter compelling, visually beautiful mathematical artifacts.

And though mathematics is far from being seen as playing the central epistemological role it did for the Pythagoreans, it has nonetheless made some inroads into more mainstream culture.

The proliferation of mathematical films and plays, such as *A Beautiful Mind, Pi, Arcadia, Breaking the Code* and *Proof,* harken back to ancient times when playwrights such as Aristophanes could refer to then-current mathematical problems (such as the squaring of the circle) as easily as political ones. Publishers have apparently realised that the once-sullen, esoteric line of pure mathematics books might be gaining in appeal, as titles such as *Fermat's Last Theorem, The Code Book* and *The Honor Roll: Hilbert's Problems and their Solvers* populate bookstore shelves. Instead of offering accounts of mathematics using the formal, abstract language to be found in research journals – and often imposed upon reluctant schoolchildren – these books tell exciting, sometimes heart-wrenching and very human stories of mathematicians and their discoveries, seeking to convey the sense of beauty and elegance to which mathematicians are drawn. Once again, we are being provided with glimpses of the way in which mathematics connects experience and abstraction, connects the senses with structures, connects the human with the divine.

The scale of the recent rapprochement among mathematics, science and the arts, as well as the apparently growing appeal of mathematics in more mainstream culture, are both manifestations of a re-emergent affinity between the mathematical and the aesthetic, one that might be coming closest to the golden era of the Pythagoreans. In keeping with the philosophy of the Pythagoreans, the chapters in this book focus on this affinity at a deeper level, beyond surface applications (as might be suggested by geometricised paintings or musical fractals), to more fundamental, epistemological connections. They attempt to articulate a common sub-stratum between the mathematical and the aesthetic, one that is integrally related to human sense-making and to learning.

The goal of this opening chapter is to provide a brief historical sense of the development of ideas around the mathematical aesthetic. Readers with backgrounds in the aesthetics branch of classical philosophy will find the equivalent branch of mathematics rather young and comparatively uncritical. Nevertheless, a certain amount of grappling with difficult challenges can be found, though not with the systematic or cumulative attention that has built and continues to build the mathematical edifices so cherished by mathematicians themselves.

We begin by looking at some fragments of these challenges, as found in the long period stretching from the ancient Greeks up to the beginning of the twentieth century. We then turn to the twentieth century itself and find there an explosion of interest in the mathematical aesthetic, particularly around questions such as: *is mathematics an art or a science?* and *can criteria for mathematical beauty be identified?*

Some Pre-Twentieth-Century Fragments Concerning the Mathematical Aesthetic

The extant writings attributed to Pythagoras and his followers reveal that the Pythagorean school, if not Pythagoras himself, found in the beauty of mathematics the very highest order of aesthetic interest. In fact, the Pythagoreans were overwhelmed by the aesthetic appeal of the theorems they discovered and were perennially preoccupied with the interconnectedness of the mathematical and the aesthetic. This interconnectedness permeated their worldview, which saw reality as ultimately revealed in mathematically harmonious concepts.

Mathematical studies were thus seen as furnishing ladders and bridges to the divine, because they shared a perfection and beauty that was considered true of the divine, but felt lacking in the physical world. Unlike Plato, who separated number, an abstract entity, from the things numbered, the Pythagoreans saw number as being tied up with the actual procedure of counting and thus closely connected with things. Number reached out or down into the world of sense and experience. As such, the Pythagoreans saw the roots of Plato's *exclusively* abstract entities in the 'real', the human, the sensory world.

Both Plato and Aristotle, though philosophically divergent in many ways, were much influenced by the ideas of Pythagoras, particularly with respect to the connection between mathematics and the beautiful. Plato saw mathematics as providing the most fundamental of all ideas and believed in mathematical objects as perfect forms. As he wrote in *Philebus*:

> By 'beauty of figures' I mean in this context not what most would consider beautiful – not, that is, the figures of creatures in real life or in pictures. I mean a straight line, a curve and the plane and solid figures that lathes, rulers and squares can make from them. I hope you understand. I mean that, unlike other things, they are not *relatively* beautiful: their nature is to be beautiful in any situation, just as they are, and to have their own special pleasantness, which is utterly dissimilar to the pleasantness of scratching. (51d; 1982, p. 121; *italics in Waterfield*)

And Aristotle, in his *Metaphysics,* wrote that the mathematical sciences have much to say about the beautiful and the good, and that:

> the chief forms of beauty are order and symmetry and definiteness, which the mathematical sciences demonstrate in a special degree. (Book M, 1078b; 1966, p. 218)

(Martin Schiralli, in Chapter 5, however, discusses important differences among the views of Plato, Aristotle and the Pythagoreans.)

Once into the Christian era, by no means all were comfortable with linking the mathematical and the divine, of humans equating investing the mathematical and investing the divine with the qualities of perfection, thereby perhaps equating the two. For instance, St Augustine, in his twelve-volume work *De Genesi ad Litteram,* warned:

> Hence, the good Christian should beware of mathematicians and all those who make empty prophecies, especially when they tell the truth, for fear of leading his soul into error by consorting with demons. (Book II, *23,* 35-36)

When reading this observation, however, it is important to realise that the most common connotative meaning of the word 'mathematician' in St Augustine's day was not what it would be today, including as it did those engaged in astrology, alchemy, *gematria* and magic. And, as Chapter 9 speculatively explores, in Byzantium and in mediaeval Europe at least, the drive to mathematise may have been 'side-tracked' into theology, until Renaissance artists found an alternative outlet in their work.

With regard to Islam, Endress (2003) informs us that the only mediaeval mathematics-related dissertation on the aesthetically beautiful can be found in Ibn al-Haytham's *Optics,* a discipline that was seen as the converse of geometry by mediaeval mathematicians. It may be true that such mathematicians were less inclined to talk directly about the beauty of mathematics; nonetheless, they certainly wrote about some of its other aesthetic qualities. For example, the tenth-century mathematician Abu Salh al-Kuhi – according to Berggren perhaps "the last mathematician to look on mathematics with the eyes of the great Hellenistic geometry" (cited in Endress, p. 193) – extolled the certainty of mathematics. He wrote of the rules of geometry as being "consistent and unchanging" and eschewed the kind of 'bad' mathematics that was based on numerical, imperfect approximations.

The eleventh-century Islamic theologian Al-Ghazzali warned of mathematics – and particularly its predilection for aesthetic qualities such as precision and clarity – leading to harmful things other than magic. One additional drawback of mathematics, he wrote, was that:

> every student of mathematics admires its precision and the clarity of its demonstrations. This leads him to believe in the philosophers and to think that all the sciences resemble this one in clarity and demonstrative power. (in Hoodbhoy, 1991, p. 105)

Such caveats against misplaced or even idolatrous authority, whether to be located in a particular author or within mathematics itself, have been echoed time and time again down the centuries. For instance, we note in passing that one of the more notable complaints concerning Isaac Newton's extensive biblical chronology (which occupied much of the latter part of his life) was its being credited with more credence than its due, because of the reputation of its creator in quite another area of human endeavour.

So, over a millenium after St Augustine's expression of concern, we still find Archbishop François de Fénélon (1697/1845) in Paris expressing a not-dissimilar unease:

> Surtout ne vous laissez point ensorceler par les attraits diaboliques de la géométrie. [Above all, do not allow yourself to be bewitched by the diabolical attractions of geometry.] (p. 493)

In the Christian West, right up to the time of Fibonacci (and beyond, into the sixteenth and even seventeenth centuries), the more likely meaning for 'mathematician' was astrologer (and, even worse, 'conjuror'). It is worth recalling that such an Augustinian pejorative description of 'mathematician' (or its common equivalent of 'geometer') was almost as fitting of Isaac Newton (see, for instance, Gleick, 2003, on the 'alternative' Newton) as the Elizabethan neoplatonist mathematician and magus John Dee (1527–1608) of an England a century earlier, whose magnificent personal academic library was perhaps the best in England at that time (see Yates, 1969).

Dee lived in very complex political, religious and intellectual times. Similar concerns linking mathematics with devil-worship surfaced in England, very soon after the English Reformation started, with Henry VIII asserting the King as head of the new Church of England (via the 1534 Act of Supremacy denying the authority of the Pope). In 1550, three years after the death of Henry VIII, government commissioners ('Visitors') went destructively through Oxford University college libraries, casting more than a suspicious glance at books containing mathematical diagrams, consigning many volumes to destruction. [1]

Twenty years after this book-burning event, Dee published his extensive and very influential 'fruitfull præface' (which ran to ninety-five printed pages) to the first English-language version of Euclid's *Elements*. Following a highly Pythagorean discussion of the nature of mathematics in terms of number, Dee asserted:

> For, [*Things Mathematicall*], being (in a manner) middle, between things supernaturall and naturall: are not so absolute and excellent as things supernaturall; Nor yet so base and grosse, as things naturall: But are things immateriall, and neverthelesse, by material things able somewhat to be signified. And though their perticular Images, by Art, are aggregable and divisible: yet the generall *Forms* notwithstanding, are constant, unchangeable, untransformable and incorruptible. Neither of the sense, can they, at any time, be perceived or judged. Nor yet, for all that in the royall mind of man, first conceived. But surmounting the imperfection of conjecture, weening and opinion: and comming short of high intellectuall conception, are the Mercuriall fruit of *Dianœticall* discourse, in perfect imagination subsisting. A marvellous newtrality have these things *Mathematicall,* and also a strange participation between things supernaturall, immortall, intellectuall, simple and indivisible: and things naturall, mortall, sensible, compounded and divisible. Probability and sensible proof, may well serve things naturall, and is commendable: In Mathematicall reasonings, a probable Argument is nothing regarded: nor yet the testimony of sens[e], any whit credited: But onely a perfect demonstration, of truths certain,

> necessary, and invincible: universally and necessarily concluded: is allowed as sufficient for an Argument exactly and purely Mathematicall. (1570; in Rudd, 1651, pp. 4-5)

There are a number of resonances between the above quotation of Dee's and themes addressed in this book. First, in placing mathematics neither of this world nor the next, but somehow hovering between the two with connections and links to both, Dee calls attention to the Janus-faced nature of mathematics, as well as presciently identifying mathematics as a 'mediating third' between the two.

To evoke in the context of this quotation the tension between 'pure' and 'applied' mathematics, to cast it in this modern frame (that is, to worry about Eugene Wigner's (1960) claim about 'the unreasonable effectiveness of mathematics'), is to assert the gap between mathematical and natural. But some of the protestors cited above are at equally great pains to maintain the separation between mathematical and what Dee terms 'the supernaturall', identified by some (but not all) with 'the divine'.

We would also like to draw on this quotation in order to make some links with themes explored in this book. To a considerable extent, quite a number of chapters in this book – in particular, Chapters 3, 4, 5, 8 and 9 – explore different ways of disagreeing with Dee's remark "Neither of the sense, can they, at any time, be perceived or judged". Additionally, in Chapter 1, Jonathan Borwein takes (indirect) exception to Dee's assertion that "a probable Argument is nothing regarded". David Pimm, in Chapter 8, discusses aspects of what Dee termed the "Art" of "perticular Images", as well as exploring the connection between 'Popish' catholicism and concern about mathematical images in the twentieth century (prefaced, as we saw above, in the sixteenth). Finally, in Chapter 9, Dick Tahta centrally examines the nature of "sensible" objects in relation to mathematics.

The Mathematical Aesthetic in the Twentieth Century

Though the eighteenth and nineteenth centuries were extremely fruitful in terms of mathematical discoveries and advances, it seems that mathematicians infrequently, at least in print, reflected on issues related to the mathematical aesthetic. This is not to say, however, that they did not think about or mention aesthetic values. Gauss's mathematical diary (see Gray, 1984), for example, contains many references to the beauty or elegance of his own mathematical ideas and discoveries. For instance, as a nineteen-year-old in 1796, Gauss wrote about a new proof obtained "all at once, from scratch, different, and not a little elegant" (p. 108). In another entry, this time made in 1800, he described his work on the arithmetic–geometric means as being "most beautifully bound together and increased infinitely" (p. 122) to the theory of transcendental quantities. In addition to beauty and elegance, Gauss made reference to aesthetic qualities such as a "charming theorem" (p. 125) and to a "most simple and expeditious method" (p. 124).

However, for some reason, the turn of the past century brought about a comparative flurry of interest in the nature of mathematics. In particular, there were concerted efforts to ascertain whether mathematics belonged more to the arts or to the sciences, from which it had not long ago been divorced (during the latter part of the nineteenth century, not least due to developments in connection with non-Euclidean geometry). It also marked the beginning of sustained inquiries into the development of mathematical knowledge and the extent to which it is fuelled by some aesthetic as well as utilitarian or logical considerations (which, *pace* D'Arcy Thompson, were usually seen as relatively distinct).

Finally, and early on in this flurry of activity, mathematicians became interested once more in the psychology of mathematical discovery. [2] Some twentieth-century mathematical writers on the aesthetic turned to the central question of the extent to which affective responses and aesthetic sensibilities were involved in the process of mathematical creation. Their attention to the aesthetic was not as intense and all-encompassing as that of the earlier Pythagoreans, but they each began, in their own way, to rekindle the embers of this ancient affinity. Here, we examine each of these themes in turn, tracing out, when possible, aspects of their historical developments.

The aesthetics of mathematical creation

In 1908, Henri Poincaré began to bring renewed attention to the aesthetic dimension of mathematical creation, but his focus was more pragmatic and markedly different from that of the ancient Greeks. He was most interested in probing the aesthetic influences that affect the process of mathematical discovery. This focus proved unlike that of many of the mathematicians who would follow him, who attended more to the aesthetic values or principles that exist in mathematical ideas or products (the discoveries themselves). By analysing the process of mathematical creation, Poincaré tried to show that mathematical invention depends upon the often sub-conscious choice and selection of 'beautiful' combinations of ideas, those best able to "charm this special sensibility that all mathematicians know" (1908/1956, p. 2048).

In his book *The Psychology of Invention in the Mathematical Field,* Jacques Hadamard (1945) proposed the first expansion of Poincaré's aesthetic heuristic theory, additionally claiming that aesthetic sensibilities often guide a mathematician's general choices about which line of investigation to pursue. He wrote specifically about the "sense of beauty" (p. 130) which can inform the mathematician that "such a direction of investigation is worth following; we feel that the question *in itself* deserves interest" (p. 127; *italics in original*). Hadamard also added to Poincaré's ideas on the role of the mathematical pre-conscious in mathematical thinking, locating the period in which it is most operative – the *incubation* period – within a larger theory of mathematical inquiry. Through both historical and empirical studies, he supported his account from mathematicians such as Pierre de Fermat, Evariste Galois, Bernhard Riemann, George Birkhoff, George Pólya and Norbert Wiener.

Morris Kline (1953) subsequently pointed out that aesthetic concerns not only guide the direction of an investigation, but motivate the search for new proofs of theorems already correctly established but lacking in aesthetic appeal – by means of their ability to "woo and charm the intellect" (p. 470) of the mathematician. Kline took this aesthetic motivation as a definitive sign of the artistic nature of mathematics. Wolfgang Krull (1930/1987) illustrated how aesthetic preferences – such as a mathematician's desire for simple, symmetric structures – can seriously influence the further development of mathematics, as well as the derivation of new properties and the creation of new theories.

In his earlier attempt to define mathematics as the "classification and study of all possible patterns" (p. 12), Warwick Sawyer (1955) implied that the heuristic value of mathematical beauty stems from mathematicians' sensitivity to pattern and originates in their belief that *"where there is pattern there is significance"* (p. 36; *italics in original*). Sawyer went on to explain the heuristic value of this trust in pattern:

> If in a mathematical work of any kind we find that a certain striking pattern recurs, it is always suggested that we should investigate *why* it occurs. It is bound to have some meaning, which we can grasp as an idea rather than as a collection of symbols. (p. 36; *italics in original*)

Sawyer might well have explained Poincaré's special aesthetic sensibility as a sensibility toward pattern, viewed broadly as any regularity that can be recognised by the mind. For him, the mathematician is not only able to recognise regularities and symmetries, but is also attuned to look for and respond to them with further investigation.

Poincaré's writing on the mathematical aesthetic, which was definitely excluding of most everyone (more so than Sawyer's account) suggested that only the very creative mathematicians had access to this aesthetic guide. This claim may have provoked the "literary superstition" that Alfred North Whitehead (1926) mentioned, which views the aesthetic appreciation of mathematics as being a "monomania confined to a few eccentrics in each generation" (in Hardy, 1940, p. 85). Hardy quoted Lancelot Hogben (1940) "the aesthetic appeal of mathematics may be very real for a chosen few" (p. 86) and accused him of echoing this "superstition".

Indeed, Bertrand Russell's (1917) famous quotation, "Mathematics, rightly viewed, posseses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture" (p. 57), does seem to suggest that mathematics exercises a coldly impersonal attraction, one not meant for normal individuals. As we shall see, Russell's frigid tastes are not the only ones that mathematics can satisfy. But this theme of the exclusiveness of mathematical aesthetic judgements (concerning who is able to make them), to be found in the writings of Poincaré, Russell and Hardy, persists in the mathematics literature.

Armand Borel (1983) was faced with overcoming a different kind of exclusivity in his attempt to convey the nature of mathematics and the mathematical aesthetic to a wider audience, of both mathematicians and non-mathematicians. He began by arguing that the development of mathematics was "derived from, guided by, and judged according to aesthetic criteria" (p. 11), thereby astutely acknowledging both Poincaré's heuristic aesthetic and Hadamard's aesthetic of choice. However, he then attempted to show how what may seem like the "pure and esoteric" aesthetics of mathematicians are actually bound up with "more earthly yardsticks" (p. 15), such as applicability and usefulness, values that Borel hoped non-mathematicians would find more recognisably mathematical than beauty or elegance.

Almost eighty years after Poincaré, the philosopher Harold Osborne (1984) wrote:

> the reliance on the heuristic value of mathematical beauty in scientific theory has become something of a commonplace. (p. 291)

This indicates the extent to which scientists – and especially physicists – had placed their trust in Poincaré's notion of the mathematical aesthetic sensibility as a kind of muse who, if listened to carefully, would both guide and inspire creativity. [3] Indeed, scientists have been much more prolific than mathematicians in cataloguing and inspecting the effect of this trust on the development of scientific theories (see, for example, Chandrasekhar, 1987; Curtin, 1982; Farmelo, 2002; McAllister, 1996; Wechsler, 1978).

Yet few scholars have explicitly discussed the *differences,* in terms of their aesthetic dimensions, between mathematics and the (other) sciences. There is certainly a common belief among physicists that what they find beautiful in their theories is ultimately mathematical. In fact, it would seem that mathematics plays a key bridging role between the sciences and the arts, at times transforming scientific ideas into forms and patterns that afford aesthetic attention. But science and mathematics have different aims, as well as different measures of success. This leads McAllister (1996), for instance, to warn that the nature and role of the scientific aesthetic cannot be blindly transferred to the domain of mathematics.

Mathematics: an art or a science?

The mathematics literature has long been replete with questions about the nature of mathematics and its place in the plural world of the arts and sciences. While Gauss's claim that mathematics is the queen of the sciences has often been repeated, so has the claim that mathematics belongs more properly to the arts. The British scholar J. W. N. Sullivan made the latter argument in 1925, claiming that mathematics is the product of a free creative imagination, unconditioned by the external world. It is, he argued, just as 'subjective' as the other arts, even though it can be used to illuminate natural phenomena.

Moreover, Sullivan (1925/1956, p. 2020) claimed that mathematicians are impelled by the same incentives as artists, citing as evidence the fact that the “literature of mathematics is full of aesthetic terms” and that many mathematicians are “less interested in results than in the beauty of the methods” (p. 2020) by which those results are found. His interest in the mathematical aesthetic experience, which he saw giving rise to the same satisfactions as the artistic experience, was distinct from Poincaré’s focus on the mathematical aesthetic sensibility, which acts as a guide. Yet Sullivan saw neither mathematics nor art as existing to satisfy “aesthetic emotions”: rather, he saw both art and mathematics as means by which humans can “rise to a complete self-consciousness” (p. 2021).

The philosopher Rom Harré (1958) was more interested in the aesthetic *differences* between mathematics and the arts. He pointed out the uniqueness of mathematical aesthetic judgements by comparing them with *bona fide* aesthetic judgements. He described mathematical appraisals of beauty and elegance as *quasi-aesthetic,* since they use “words from our regular aesthetic vocabulary, which fall outside the normal range of aesthetic judgements” (p. 133). In fact, for Harré, “quasi-aesthetic appraisals are not a queer sort of aesthetic appraisal but simply not aesthetic appraisals at all” (p. 136). Quasi-aesthetic appraisals are “essentially second-order” because of two factors (p. 137). First, appraisals such as ‘beautiful’ and ‘elegant’ do not betoken success in mathematics the way they do in artistic fields: “If an object doesn’t move us it has failed altogether aesthetically, but if a proof doesn’t move us it does not for that reason fail altogether mathematically” (p. 137). Second, quasi-aesthetic appraisals require *comparing* an object with very specific other objects of the same kind. Harré contended that, in mathematics, the elegance of a proof “can only be judged by a comparison, explicit or implicit, between alternative proofs of the *same* result” (p. 137): in contrast, “Ordinary aesthetic appraisals are essentially non-comparative” (p. 137).

Harré’s formalist stance forced him to trivialise almost completely the importance of aesthetic appraisals in mathematics. Because the aesthetic is only secondary to achievement, it is thereby robbed of any epistemic interest. Furthermore, contemporary philosophers and art theorists would challenge Harré’s claim about the aesthetic bar of success inherent in the arts and the non-comparability of aesthetic judgements. The philosopher of mathematics Thomas Tymoczko (1993) may well have pointed out the most operative difference between aesthetic judgements in mathematics and those at work in the arts. This is that the mathematics community does not have many (any?) ‘mathematics critics’ to parallel the strong role played by art critics in appreciating, interpreting and arguing about the aesthetic merit of artistic products.

In 1933, the American mathematician George Birkhoff approached the connection between mathematics and beauty from the reverse direction, proposing a theory by which mathematics could be used to describe beauty. According to Whittaker (1945), Birkhoff wanted to create:

> a general mathematical theory of the fine arts, which would do for aesthetics what had been achieved in another philosophical subject, logic, by the symbolisms of Boole, Peano, and Russell. (p. 127)

Birkhoff admitted that the aesthetic feeling was "intuitive" and *"sui generis"*, but held nevertheless that the attributes upon which aesthetic values depend are accessible to measurement. He proposed three main variables constituting typical aesthetic experiences: the complexity of the object (C), the feeling of value or aesthetic measure (M) and the property of harmony, symmetry or order (O). With the following equation, $M = O/C$, he presented to us his hypothesis that the aesthetic measure is determined "by the density of order relations in the aesthetic object" (1933/1956, p. 2186). He also provided equations that could define both the variables O and C more formally.

Birkhoff's formula never gained much currency in the world of art criticism, nor in the world of mathematics. After all, the terms O and C are not straightforward to measure: can the square grid, which is highly ordered with little complexity, be considered of great aesthetic value? What about a fractal image? The difficulty in measuring O and C makes the formula almost impossible to use. And perhaps artists and mathematicians alike were unimpressed by Birkhoff's formula for its tacit presumption that aesthetic value can be measured in some absolute way (regardless of personal, social or cultural styles), based on a set of accurate rules. Regardless of his formulaic approach, Birkhoff did identify qualities such as order, harmony and complexity as being relevant to aesthetic value, thus echoing the ancient Greeks while at the same time anticipating the work of several of the mathematicians we have yet to discuss.

Criteria for the mathematically beautiful

In 1940, G. H. Hardy published what became arguably the most widely-read inquiry into the mathematical aesthetic. Unlike either Poincaré or Hadamard, Hardy was primarily interested in defining mathematical beauty as it exists in mathematical products, particularly in proofs. He proposed a somewhat complex scheme that distinguished 'trivial' beauty – which can be found in chess – from 'important' beauty, which can only be found in serious mathematics. But, for Hardy, serious involved significant, which in turn required generality – scope or reach – and depth. *Generality* and *depth* are both difficult to define, but can, according to Hardy, be immediately recognised by those with a "high degree of mathematical sophistication" (p. 103). Such mathematicians will find a mathematical idea significant when it can be "connected, in a natural and illuminating way, with a large complex of other mathematical ideas" (p. 89). Hardy illustrated his notion of mathematical beauty with two examples: Euclid's proof of the infinity of primes and the Pythagorean proof of the irrationality of $\sqrt{2}$. These two proofs appear frequently in the literature as particularly fine examples of beautiful proofs (for example, see Dreyfus and Eisenberg, 1986, or King, 1992).

Having defined mathematical beauty in terms of significance and seriousness, Hardy went on to say that the triviality of ideas (such as those found in chess problems, but not in beautiful mathematics) "disturbs any more purely aesthetic judgement" (p. 113). Hardy proposed that purely aesthetic qualities are unexpectedness, inevitability and economy. Considerably later, Roger Penrose (1974) would add to Hardy's list the criterion of "unexpected simplicity" (p. 267). Hardy advanced a formalist perspective of mathematical beauty by only acknowledging responses to formal properties. For Hardy, and many others, formalism represents the dominant 'public aesthetic' of mathematics; if mathematics presents *any* aesthetically relevant qualities, these qualities *must* be formal in nature.

Shortly after Hardy's publication, François Le Lionnais (1948/1971) proposed a completely different, non-formalist way of approaching the problem of mathematical beauty – without making reference to either Hardy or Poincaré. Le Lionnais was not interested in the process-oriented aesthetic sensibilities that Poincaré was, but his scope was wider than Hardy's, including as it did various kinds of 'facts' and 'methods' as potential objects of mathematical beauty. Le Lionnais effectively enlarged the sphere of mathematical entities that can have aesthetic appeal, including not only entities such as definitions, shapes, proofs, solutions and theorems, which are appreciated after the fact, but also the various methods and processes used to work with mathematical entities, which can be appreciated *while* doing mathematics.

In addition, Le Lionnais emphatically drew attention to the subjectivity of aesthetic responses, by classifying mathematicians' orientations as either 'classical' or 'romantic', thus allowing for degrees of appreciation – banned by Hardy – according to personal preference. These categories represent two styles of human endeavour: on the one hand, a desire for equilibrium, harmony and order; and, on the other, a yearning for lack of balance, form obliteration and pathology. A very similar distinction was made by Freeman Dyson (1982), who distinguished between 'unifiers' and 'diversifiers', the former finding and cherishing symmetries, the latter enjoying the breaking of them.

In addition, Harold Osborne (1984), in tracing aspects of the aesthetic in the sciences, also recognised the human dimension of mathematical aesthetic response, arguing that aesthetic satisfaction derives from the common human desire to impose order on chaos. Citing Davis and Hersh's (1980) observation, "to some extent, the whole object of mathematics is to create order where previously chaos seemed to reign, to extract structure and invariance from the midst of disarray and turmoil" (p. 172), Osborne implied that mathematics provides an optimal context in which to gain aesthetic satisfaction.

Le Lionnais's stance on the subjectivity of aesthetic responses did not do much to quell the belief, common among mathematicians especially, that most mathematicians will agree on their aesthetic judgements. This common belief was fuelled in part by the exclusivity of Poincaré and Hardy, which seemed to imply that if your aesthetic judgement did not agree with that of a great mathematician, then you were simply not a great mathematician. It was also fuelled by the enormous discrepancies of taste and judgement found in the arts which, by any mathematician's definition of subjectivity, dwarfed the differences identified in the mathematical world.

Jerry King (1992), like Hardy, presumed the supposed homogeneity of mathematicians' aesthetic response and further concluded that mathematicians work from some set of commonly-accepted aesthetic principles. Moreover, he assumed that mathematicians' judgements are not subjective, but instead depend solely upon the mathematics itself, making it possible to formulate decisive criteria. In his book *The Art of Mathematics,* King drew on aesthetic theories of philosophy and art criticism in order to articulate "a complete aesthetic theory of mathematics" (p. 157).

Rather than expanding Hardy's or Osborne's list of factors that contribute to aesthetic appeal, King's primary goal was to identify general-level aesthetic criteria that would help distinguish 'good' mathematics from 'bad' (thereby conflating Hardy's distinction between the beautiful and the aesthetic). He thus proposed two definitive criteria: the principle of minimal completeness and the principle of maximal applicability. King illustrated both principles using the Pythagorean proof of the irrationality of $\sqrt{2}$. The principle of minimal completeness, in effect, functions as a super-class to Hardy's aesthetic qualities. However, King's principle of maximal applicability resonates more with Hardy's notions of significance, depth and generality.

Finally, David Wells's (1990) survey of contemporary mathematicians has most convincingly illuminated the subjectivity question. He asked the readers of *The Mathematical Intelligencer* to rate, on a scale of one to ten, twenty-four theorems according to their mathematical beauty. From the seventy-six responses, many from top mathematicians mostly from North America, he drew a number of inferences. First, mathematicians do not always agree on their aesthetic judgements – at least not in terms of evaluating the beauty of theorems.

Wells identified many factors that contribute to the differences in judgement: field of interest; preferences for certain mathematical entities such as problems, proofs or theorems; past experiences or associations with particular theorems; even mood. He also pointed out that aesthetic judgements change over time: this was particularly evident in the rating of Euler's formula, which was historically considered "the most beautiful formula of mathematics" (p. 38), but is now, according to Wells's respondents at least, considered too obvious even to elicit an aesthetic response.

The inferences made by Wells correspond to a contextualist view of aesthetic appreciation and are summed up by this respondent: "beauty, even in mathematics, depends upon historical and cultural contexts, and therefore tends to elude numerical interpretation" (p. 39). Indeed, John von Neumann had already spoken of the phenomenon of mathematical 'styles' back in 1947, arguing that, it is "hardly possible to believe in the existence of an absolute, immutable concept of mathematical rigor, dissociated from all human experience" (p. 190). He used as evidence the changes in styles of mathematical proofs and fashionable areas of interest over the past two millennia.

One might wonder why these changes in style appear so much less dramatic than the ones found in the arts. Are the styles necessarily more confined in mathematics, owing to the handful of aesthetic commitments that ultimately define the discipline? Or does the study of mathematics attract a small enough number of like-minded people that aesthetic revolutionaries such as Picasso, Pollock or Cage do not have mathematical equivalents?

Some Final Comments

It is certainly tempting to wonder why the twentieth century witnessed such an explosion in thinking about the mathematical aesthetic, if only to help predict what might be in store for the twenty-first. Will this book be unawarely documenting the close of an active period of investigation or will it serve as a springboard for further, fruitful inquiry?

It would be hard to overlook the fact that at turn of the twentieth century, not long after the discovery of non-Euclidean geometries and just as Cantor's work on trigonometric series and the continuum was emerging, foundational concerns were mounting and questions about axiomatisation were beginning to press. These concerns would incite mathematicians to begin seriously inspecting the nature of mathematics and for some, such as David Hilbert, Herman Weyl and L. E. J. Brouwer, to turn their attention to 'meta-mathematical' questions, albeit with markedly different responses.

Few mathematicians since the ancient Greeks have stepped back from the exhilarating momentum of creating mathematics to consider larger epistemological questions. Hilbert's famous list of unsolved problems, which essentially came to define much of what would be considered 'interesting' to work on, also came at the turn of the twentieth century. His lengthy list either prompted or nourished a broader consideration of the whole field of mathematics – its goals, methods, and successes – yielding yet more 'meta'-mathematical thinking, which could hardly ignore the important aesthetic dimension of mathematics.

It is striking to us that mathematicians often mention 'beauty', yet there seems to be a relative dearth of further amplification. One might have expected those past mathematicians who thought in these terms to have been capable of developing ideas of, say harmony, proportion, fit, rhythm, etc, more precisely. To some extent, this is what Hardy (1940) tried to do, though only by connecting 'beauty' to other barely less opaque terms such as 'elegance', 'depth', 'seriousness' and 'significance'. We do see instances here and there, for instance with Alfred North Whitehead (on rhythm), with Warwick Sawyer (on pattern) and in an overly-mathematised attempt by George Birkhoff.

But it may well be that many mathematicians simply do not consider this to be a serious enterprise, one worthy of their time and attention. Even

Hardy (1940) expressed a sense that such reflection 'about' mathematics (offering a different sense of 'meta'-mathematical activity) is not really the preferred activity or even the very business of mathematicians.

> It is a melancholy experience for a professional mathematician to find himself writing about mathematics. The function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done. (p. 61)

There seem to be some 'inevitable' combinations of aesthetic words that are mathematically invoked as if conjoined: for instance, beauty *and* elegance, perfection *and* beauty. 'Elegance', in particular, seems to have been co-opted by mathematicians in their rather restricted aesthetic language, as conveying a sense of both succinctness and sophistication. In ordinary parlance, 'elegance' might be seen as a classical, class-ridden term – not so much socio-economic 'class' perhaps as intellectual 'class' (though Bertrand Russell, for example, certainly partook of both). Of course, there must be a sociological proviso here – it was only very few (privileged) Greeks, and then for a long time a very few (privileged) other individuals, who could sit and think as opposed to practice or teach.

There might also be more subtle reasons for this explosion of aesthetic consideration and writing. It was also around the turn of the last century that the field of mathematics made its final separation from the sciences, its increasing abstractions having less and less to do with the kind of questions that drove the development of calculus, for example. Mathematics was carving itself out as a distinct, self-sufficient field with famously little to do with the 'real' world. But how, then, could it justify its existence? This was a question that those both inside and outside the tall, opaque walls guarding the mathematical terrain asked.

Perhaps this question prompted some mathematicians to search for some varied connections – ones that many ancient Greek mathematicians would have assumed – between mathematics and the arts, another field which offers few practical applications, though is admired on the whole for its display of creativity and its production of aesthetically pleasing artifacts. Like the modernist art movement, which was burgeoning with a sense of art for art's sake *(l'art pour l'art)* during this time in the early twentieth century, mathematics was now being done for mathematics' sake.

One instance of this modernist mathematical ethos is apparent in van der Waerden's (1930/1991) *Modern Algebra,* in which any questions of algebra's utility had completely vanished (see Chapter 8). John Dee's (1570) survey of various branches of 'the tree of mathematics' in his *Mathematicall præface* referred positively to his identification of which parts of mathematics were, to use his term, 'commodious': van der Waerden felt no such compunction.

This twentieth-century expansion of interest in the aesthetic did not only occur in mathematics, of course. As Denis Donoghue (2003) observes:

> Interest in beauty and aesthetics was greatly stimulated in the first years of the twentieth century — as C. K. Ogden, I. A. Richards, and James Wood noted in *The Foundations of Aesthetics* (1922) — by a wider knowledge of non-European art, especially of Eastern and primitive art, and by the rapid development of psychology as an accredited practice. (p. 36)

Additionally, in the latter part of the twentieth century, developments mostly outside mathematics itself may have further contributed to this most recent explosion within mathematics. For example, scholars in a number of different fields have become increasingly attracted to sociobiologist E. O. Wilson's (1998) notion of 'consilience'. In his view, unification or connection is sought among the increasingly-fractured disciplines and ways of knowing. He is also concerned, more generally, with the breaking down of long-standing dichotomies between *mind* and *body,* between *rational* and *emotional,* between *logical* and *intuitive.* Although few scholars have contributed to the search for affinities in the still often-isolated and inhospitable world of mathematics, these intellectual changes are at least supportive of such endeavours, as we hope this book will show.

Notes

[1] Writing about eighty years after this event, Oxford University historian Anthony à Wood observed:

> Many MSS, guilty of no other superstition than red letters in their fronts or title, were either condemned to the fire or jakes. [...] sure I am that such books wherein appeared Angles, or Mathematical Diagrams, were thought sufficient to be destroyed, because accounted Popish, or diabolical, or both.
>
> What was done to the public Library I shall elsewhere shew: as for those belonging to Colleges, they suffered the same fate almost as the public, though not in so gross a manner. From Merton Coll. Library a cart load of MSS and above were taken away, such that contained the Lucubrations (chiefly of controversial Divinity, Astronomy and Mathematicks) of divers of the learned Fellows thereof, in which Studies they in the two last centuries obtained great renown. (in Gutch, 1796, pp. 106-107)

[2] This twentieth-century return to the question of the psychology of the mathematician connects for us with thirteenth-century Henry of Ghent's potent phrase 'the melancholy disposition of the mathematical mind'. It was coined in the light of much Aristotelean writing and the then-dominant Galenic theory of 'humours', a means of linking human psychology with the cosmos. It also relates closely to Albrecht Dürer's famous engraving *Melencolia I* – see Yates (1979). This is a topic we explore further in the closing chapter of this book.

[3] The Pythagoreans, who celebrated the Muses as "the keepers of the knowledge of harmony and the principles of the universe which allowed access to the everlasting gods" (see Comte, 1994, p. 135), would have been delighted by the trust that scientists, and mathematicians, have come to place on this aesthetic muse.