Thomas Sonar

3000 Years of Analysis

Mathematics in History and Culture

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Dedicated to

Eberhard Knobloch

in gratefulness.

'I conceive that these things, king Gelon, will appear incredible to the great majority of people who have not studied mathematics ...'

ARCHIMEDES [Archimedes 2002, p. 360]

About the Author

Thomas Sonar was born 1958 in Sehnde next to Hannover. After studying Mechanical Engineering at the University of Applied Sciences ('Fachhochschule') Hannover he became a laboratory engineer in the Laboratory for Control Theory of the same University for a short time, and founded an engineering office. He then studied mathematics at the University of Hannover (now Leibniz University), after which he worked from 1987 until 1989 at the German Aerospace Establishment DLR (then DFVLR) in Brunswick for the orbital glider project HERMES as a scientific assistant. Next he went to the University of Stuttgart to work as a PhD student under Prof. Dr. Wolfgang Wendland while spending some time studying under Prof. Keith William Morton, at the Oxford Computing Laboratory. His PhD thesis was defended in 1991 and Thomas Sonar went to Göttingen to work as a mathematician ('Hausmathematiker') at the Institute for Theoretical Fluid Mechanics of the DLR; there he developed and coded the first version of the TAU-code for the numerical computation of compressible fluid fields, which is now widely used. In 1995 the postdoctoral lecture qualification for mathematics was obtained from the TU (then TH) Darmstadt on the basis of a habilitation treatise. From 1996 until 1999 Thomas Sonar was full professor of Applied Mathematics at the University of Hamburg and is professor for Technical and Industrial Mathematics at the Technical University of Brunswick since 1999 where he is currently the head of a work group on partial differential equations. In 2003 he declined an offer of a professorship at the Technical University of Kaiserslautern connected with a leading position in the Fraunhofer Institute for Industrial Mathematics ITWM. In the same year Sonar founded the centre of continuing education for mathematics teachers ('Mathelok') at the TU Brunswick which stays active with regular events for pupils also.

Early in his career Thomas Sonar developed an interest in the history of mathematics, publishing in particular on the history of navigation and of logarithms in early modern England, and conducted the widely noticed exhibitions in the 'Gauss year' 2005 and in the 'Euler year' 2007 in Brunswick. Further publications concern Euler's analysis, his mechanics and fluid mechanics, the history of mathematical tables, William Gilbert's magnetic theory, the history of ballistics, the mathematician Richard Dedekind, and the death of Gottfried Wilhelm Leibniz. In 2001 Sonar published a book on Henry Briggs' early mathematical works after intense research in Merton College, Oxford. In 2011 his book *3000 Jahre Analysis* (3000 years of analysis) was published in this series and in December 2014 he edited the correspondence of Richard Dedekind and Heinrich Weber. Altogether Thomas Sonar has published approximately 150 articles and 15 books – partly together with colleagues. He has established a regular lecture on the history of mathematics at the TU Brunswick and has for many years held a lectureship on this topic at the University of Hamburg. Many of his publications also concern the presentation of mathematics and the history of mathematics to a wider public and the improvement of the teaching of mathematics at secondary schools.

Thomas Sonar is a member of the Gesellschaft für Bildung und Wissen e.V. (Society for Education and Knowledge), the Braunschweiger Wissenschaftliche Gesellschaft (Brunswick Scientific Society), a corresponding member of the Academy of Sciences in Hamburg, and an honorary member of the Mathematische Gesellschaft in Hamburg (Mathematical Society in Hamburg).

Preface of the Author

As an author I was very glad that the German edition of this book which was first published in 2011 was very well received. Indeed, a second edition with corrections and additions was published in 2016. After the English translation *The History of the Priority Dispute between Newton and Leibniz* of my book *Die Geschichte des Prioritätsstreits zwischen Leibniz und Newton* was published by Birkhauser in 2018 and my revered 'language editor' Pat Morton did not want to retire but was keen to start on another translation I began translating the second edition of *3000 Jahre Analysis*. Here it is.

The German edition of this book has some precursors in a series of books on the history of mathematics: '6000 Jahre Mathematik', '5000 Jahre Geometrie', '4000 Jahre Algebra', of which only the volume on geometry has been translated into English up to now. It seemed logical to add a volume on the history of analysis to this series and thereby making the history of analysis available to interested non-specialists and a broader audience.

The current volume stands out in the series for the following reason. All books in the series were designed to present scientifically reliable facts in a readable form to convey the delight of mathematics and its historical development. But while a cultural history of mathematics can be presented without much mathematical details, while geometry can be described in the history of its constructions in beautiful drawings, and while the history of algebra, at least until the 19th century, can be developed from quite elementary mathematical reflections, this concept naturally has to fail in the case of analysis. In essence analysis is the science of the infinite; namely the infinitely large as well as the infinitely small. Its roots lie already in the fragments of the Pre-Socratic philosophers and their considerations of the 'continuum', as well as in the burning question of whether space and time are made 'continuously' or made of 'atoms'. Thin threads of the roots of analysis reach even back to the realms of the Pharaohs and the Babylonians from which the Greek received some of their knowledge. But not later than with Archimedes (about 287–212 BC) analysis reached a maturity which asks for the active involvement of my readership. Not by any stretch of imagination can one grasp the meaning of the Archimedean analysis withouth studying some examples thoroughly and to comprehend the mathematics behind them with pencil and paper. Although after Archimedes this knowledge was buried in the dark again it came back to life at latest with the Renaissance where analysis progressed in giant steps; and again this science calls for the attention of the reader! To put it somehow poetically: Analysis turns out to be a demanding beloved and one has to succumb to her in order to gain some understanding.

But have no fear! My remarks are not meant to discourage you; on the contrary: they are meant to increase the excitement concerning the contents of this book. You are required to think from time to time, but then deep and satisfying insights into one of the most important disciplines of mathematics wait as a reward. Without analysis the Technical Revolution and the developments of our highly engineered world relating thereto would have been unthinkable.

There are several books on the history of analysis on the market and the reader deserves a few remarks concerning the position of this book in relation to others. I do not claim to publish the latest and hitherto unknown research results. However, the present book differs significantly from others. First of all historical developments in the settings are given much attention as is usual for books in our German book series. Furthermore I have put weight on the Pre-Socratic philosophers and the Christian middle ages in which the discussion of the nature of the continuum had been decisive. Finally the common clamp encompassing all areas described in this book is the infinite. This clamp allows me not to surrender to the unbelievable breadth of developments in the 20th century; functional analysis, measure theory, theory of integration, and so on, but rather to end in the nonstandard analysis in which we again find infinitely small and large quantities and in which the continuum of the Pre-Socratic philosophers is honoured again. In this sense we come to full circle which connects Zeno of Elea (about 490–about 430 BC), Thomas Bradwardine (about 1290–1349), Isaac Newton (1643–1727), Gottfried Wilhelm Leibniz (1646–1716), Leonhard Euler (1707–1783), Karl Weierstraß(1815–1897), Augustin Louis Cauchy (1789–1857), and finally Abraham Robinson (1918–1974) and Detlef Laugwitz (1932–2000).

It is also due to this encompassing clamp that I have included the development of set theory in the history of analysis which is unusual. In the light of the history of the handling of infinity set theory certainly belongs here.

This book has been made possible by the project group 'History of Mathematics' of the University of Hildesheim, Germany, which I want to thank with gratitude. In particular I have to thank the late Heinz-Wilhelm Alten, my friend Klaus-Jürgen Förster, and Karl-Heinz Schlote for their confidence in me. Heiko Wesemüller-Kock has taken care of the design of this book in his usual, professional manner. One can only sense the enormity of his task if one has drawn pictures and sketches, modified or corrected existing diagrams, and designed hundreds of legends of pictures by oneself. The results of his extensive work can be seen in this book and in all other books in our series. Without the publisher, who encouraged this translation, the book would not have come to life. I have to thank Mrs. Sarah Annette Goob and Mrs. Sabrina Hoecklin of Birkhauser Publishers in particular.

However, I am not a trained historian. My continuing and long-standing interest in history has helped a lot, of course, but reliable books like the 'Der große Ploetz' [Ploetz 2008] or the wonderful little volumes of Reclam Publishers starting their titles with 'Kleine Geschichte ...' or 'Kleine ... Geschichte' [Maurer 2002], [Altgeld 2001], [Dirlmeier et al. 2007], [Haupt et al. 2008] were indispensable. In case of doubts however, only an informed historian is of real help and I am very lucky that my friend and colleague

Gerd Biegel of the 'Institut für Braunschweigische Regionalgeschichte' was at my side although permantly suffering from an overload of work. While we smoked many a cigar and drank innumerable cups of espressos he provided insights into many historical contexts.

Although in the meantime he finished his studies and is currently working on his PhD thesis my LaTeX-wizard Jakob Schönborn, who already cared for the second German edition of this book, the first German edition of *Die Geschichte des Prioritätsstreits zwischen Leibniz und Newton*, and its English version *The History of the Priority Dispute between Newton and Leibniz* stood at my side to also supervise the LaTeXnical side of this book. I can only thank him wholeheartedly for his commitment to this book project!

I am particularly grateful to Prof. Dr. Eberhard Knobloch, not only for his precious time he sacrified while proofreading the German edition but also for numerous constructive criticism and hints concerning correct translations from ancient Greek and Latin. Since he is a true role model not only for me but for a whole generation of scientists this book is dedicated to him.

I am most grateful to the wonderful Patricia (Pat) Morton who offered again to turn my 'Germanic English' into her lovely Oxford English. Without her encouragement to continue our work on book projects I would have dared to even start working on this book.

A book like this costs time; much time! My decision to write this book therefore had serious consequences in particular for my wife Anke. I had to spend a lot of time in my study and in libraries while life went on without me. A lot of money was spend to buy new and second-hand books to enrich my private library on the history of analysis. All this as well as the now meter deep piles of books and manuscripts in our living room, on couches and chairs and on the floor my beloved wife Anke has put up with and she has reacted with humour and only with a few biting remarks. After the two volumes '6000 Jahre Mathematik' by Hans Wußing had been published and were presented to the public during a small ceremony at the town hall of Hildesheim, my wife got to the heart of it: 'My husband has a mistress who is 6000 years old, and he loves her dearly!' For this and since she bears with me and my old mistress I thank her with all my heart.

Thomas Sonar

Preface of the Editors

With great pride we can present here the English translation of the second edition of the German book '3000 Jahre Analysis' for which our author deserves gratitude and appreciation.

Concerning the contents of this books it has to be noted that it is not just a translation of the German book. Again some sections have been reworked and some new material has been carefully added. We have to thank again Heiko Wesemüller-Kock and Anne Gottwald for their ernormous work concerning the pictures and the gathering of publication rights at different license suppliers. Their work ensures that the illustrations appearing in this book share the same high quality as in all other books in our series.

We are also greatful to Birkhauser Publishers and in particular to our partner Sarah Annette Goob who supported us actively. My gratitude also extends to further persons which Thomas Sonar has already mentioned in his preface.

After '5000 Years of Geometry' and 'The History of the Priority Dispute between Newton and Leibniz' the '3000 Years of Analysis' is the third book of our German book series appearing in the English language. We wish this book to also become a real success enjoying a wide distribution. May this book find many readers and may it convey an impression of the beauty and meaning of mathematics in our culture. It may perhaps even arouse their interest in mathematics.

Hildesheim, July 2020, for the editors

Karl-Heinz Schlote *Klaus-Jürgen Förster*

Project group 'History of Mathematics' at the University of Hildesheim

Advice to the reader

Parentheses contain additional insertions, biographical details, or references to figures.

Squared brackets contain

- omissions and insertions in quotations
- references to the literature within the text
- references to sources in legends of figures

In the figure legends squared brackets mark the author/creator of the particular work. Further specifications appear in common paranthesis.

Figures are numbered following chapters and sections, e.g. Fig. 10.1.4 means the fourth figure in section 10.1 of chapter 10.

The original titles of books and journals appear in italic type, likewise quotations. Further reading or explanations of only shortly described circumstances are marked by references like '(cp. more detailed in. . .)'.

Literally or textually quoted literature as well as further reading can be found in the bibliography.

Contents

1 Prologue: 3000 Years of Analysis

T. Sonar, *3000 Years of Analysis*, https://doi.org/10.1007/978-3-030-58223-4_1

since 3000 BC	Nomads from the north immigrate to southern Mesopotamia. Sumerian city states emerge and the cuneiform writing on clay tablets. The realms at the Nile unite. Emergence of hieroglyphs
about 2707–2170	Old Kingdom in Egypt. Emergence of pyramids; the step pyramid at Saqqara, the bent pyramid at Dahshur, the great pyramids of Khufu, Chefren and Mykerinos
2170–2020	First interim period in Egypt
about 2235–2094	Realm of Akkad in Mesopotamia founded by Sargon of Akkad
about 2137–1781	Middle Kingdom in Egypt. Mathematical papyri
1850	Presumable time of origin of the Moscow Papyrus
1793–1550	Second interim period in Egypt
1650	Ahmes writes the Rhind Mathematical Papyrus
2000–1595	Ancient Babylonic period in Mesopotamia. Emergence of the first legislative texts of mankind under King Hammurabi (about 1700)
1675	In Mesopotamia a clay tablet is inscribed with the length of the diagonal in a square
about 1550–1070	New Kingdom in Egypt. Temple of Hatshepsut and royal tombs in Thebes. Temple of Amun in Karnak. Sun worship of Akhenaten in Amarna
1279–1213	Ramsses II, Temple of Abu Simbel
1070–525	Third interim period and late period in Egypt
about 1700–609	Assyrian realm. Mathematical cuneiform texts; zikkurates
about 750–620	Neo-Assyrian realm, the first great empire in the history of the world; residences in Nimrud and Nineveh
625–539	Neo-Babylonian realm, heyday of astrology and astronomy
539	Cyrus the Great conquers Babylon
525	Persians conquer Egypt
332	Alexander the Great conquers Egypt

Remark: There are differing chronologies in the literature

Fig. 1.0.2. Egypt and Mesopotamia in the pre-Christian era

1.1 What is 'Analysis'?

Three thousand years of analysis? Did analysis not emerge in the 17th century by Newton and Leibniz?

To answer this question satisfactorily we should look at a definition of 'analysis' first. On the internet the following definition[1] can be found:

> *'Mathematical analysis formally developed in the 17th century ...'*

There you go! According to this definition analysis would be approximately 400 years old, but beware: The definition goes on:

[1] https://en.wikipedia.org/wiki/Mathematical_analysis

> *'… but many of its ideas can be traced back to earlier mathematicians.'*

But how far do we have to 'trace back'?

Good old reliable Encyclopaedia Britannica defines 'Analysis (mathematics)' as

> *'a branch of mathematics that deals with continuous change and with certain general types of processes that have emerged from the study of continuous change, such as limits, differentiation, and integration. Since the discovery of the differential and integral calculus by Isaac Newton and Gottfried Wilhelm Leibniz at the end of the 17th century, analysis has grown into an enormous and central field of mathematical research, with applications throughout the sciences and in areas such as finance, economics, and sociology.'*

I do not have any problems whatsoever to follow this definition! Analysis is concerned with the mathematics of continuous changes from which problems of tangents, quadrature problems (i.e. the computation of areas below crooked curves), and eventually the actual differential and integral calculus of Newton and Leibniz developed.

In a narrower sense analysis is but the mathematical branch of infinite processes and of 'infinitely small quantities' and this sense should be the ribbon accompanying us on our journey through history as a kind of Ariadne's thread. However, this is not possible consistently. The notion of 'function' is certainly central to analysis but for a start has nothing to do with infinitely small quantities. Nevertheless a discussion of the concept of functions certainly belongs to the history of analysis.

How come the 3000 years? Well, special numbers like π or $\sqrt{2}$ play a certain role and such numbers (or the approximations thereof) can in fact be found in ancient Egypt and in the cultural region of Mesopotamia.

1.2 Precursors of π

Already in the famous Papyrus Rhind[2] an approximate computation of the area of a circle can be found. Papyrus Rhind was written by a scribe named Ahmes about the year 1650 BC who wrote that he only copied mathematical problems which were at least 200 years older.

In Problem 48 of his papyrus Ahmes depicted a circle which is inscribed in a square. We can infer from the calculations following that the square of edge length of 9 units results in an area of 81 square units, and that the circle with diameter 9 units has an area of 64 square units. In Problem 50 a precise instruction to compute a circle area can be found [Gericke 2003, p. 55]:

[2] Named after the Scotsman Alexander Henry Rhind who bought the papyrus in 1858 in Luxor.

Fig. 1.2.1. The start of the Papyrus Rhind. The Papyrus is 5,5 m long and has a height of 32 cm. It contains problems concerning mathematical themes which nowadays would be called algebra, fractional arithmetic, geometry and trigonometry. It is itself a copy of an original from the 12th dynasty (19th century BC). Scribe Ahmes copied this original about 1650 BC in hieratic writing. (Department of Ancient Egypt and Sudan, British Museum EA 10057, London [Photo: Paul James Cowie])

> *'Example of the computation of a circular field of (diameter) 9. What is the amount of its area? Take 1/9 away from it (the diameter). The remainder is 8. Multiply 8 by 8. It becomes 64.'*
>
> (Beispiel der Berechnung eines runden Feldes vom (Durchmesser) 9. Was ist der Betrag seiner Fläche? Nimm 1/9 von ihm (dem Durchmesser) weg. Der Rest ist 8. Multipliziere 8 mal 8. Es wird 64.)

This calculation rule allows us to conclude that the Egyptians used $\pi_{\text{Egypt}}/4 = (8/9)^2$ as the value for $\pi/4$. Since they did neither know the nature nor the role of π we may ask ourselves how this value was achieved. One possibility would be the use of a grid. Circumscribe a square with edge length d around a circle with diameter d units and divide the square into 9 evenly spaced subsquares as shown in figure 1.2.2 (left). The area of the square would grossly overestimate the area of the circle, hence we divide the four subsquares in the corners of the square into two triangles each and count only one each as contributing to the area as in figure 1.2.2 (right). Therefore 5 subsquares and 4 triangles remain and the area of the circle is approximated by

$$A_{\text{circle}} \approx 5 \cdot \left(\frac{d}{3}\right)^2 + 4 \cdot \frac{1}{2}\left(\frac{d}{3}\right)^2 = \frac{7}{9}d^2.$$

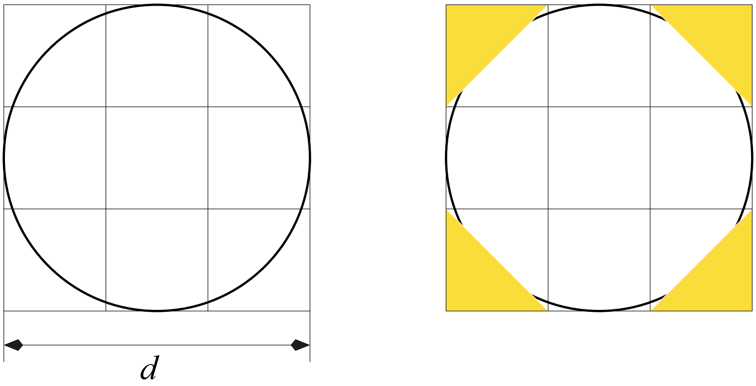

Fig. 1.2.2. Approximation of the area of a circle from the outside

However, Ahmes gives the approximation

$$A_{\text{circle}} \approx \frac{64}{81} d^2 = \left(\frac{8}{9} d\right)^2 .$$

He apparently enlarged the (correct) approximation $\frac{7}{9}d^2 = \frac{63}{81}d^2$ by an area of $\frac{1}{81}d^2$ to finally arrive at square numbers in numerator and denominator! But did he? Somewhat frustrated Otto Neugebauer (1899 – 1990) commented [Neugebauer 1969a, p. 124]:

> *'And it is not understandable how one comes from this term* [$\frac{7}{9}d^2$ for the area of the circle] *to the Egyptian formula. Without new sources it therefore makes little sense to express presumptions concerning this formula since the obvious way obviously does not lead directly to the desired result'*
>
> (Und es ist nicht einzusehen, wie man von diesem Ausdruck zu der ägyptischen Formel hinüberkommen kann. Ohne neues Textmaterial hat es also wenig Sinn, über die Entstehungsgeschichte dieser Formel Vermutungen zu äußern, da der naheliegende Weg offenbar nicht direkt zum Ziel führt.)

Since the true area of a circle is given by $A_{\text{circle}} = \pi r^2 = (\pi/4)d^2$ the ancient Egyptians worked with the approximate value

$$\pi_{\text{Egypt}} = 3.16049$$

which is by no means a bad approximation! At least the relative error is only

$$\frac{\pi_{\text{Egypt}} - \pi}{\pi} \approx 0.00601643,$$

hence about 0.6%!

In the TV production 'The Story of Maths' [Du Sautoy 2008] which is well worth watching, mathematician Marcus du Sautoy (b. 1965) gave another explanation of how the Egyptians might have come up with their formula for the area of a circle. Following his explanation the approximation $\pi_{\text{Egypt}}/4 = (8/9)^2$ stems from an ancient Egyptian board game in which spheres filling hemispherical depressions in a wooden board have to be moved around. Using these spheres a circle can be formed having a diameter corresponding to 9 spheres. Redistributing the spheres so that they form a square then this square happens to have an edge length of 8. If du Sautoy's interpretation is correct then we have here an early attempt to 'square the circle'. This problem, also called 'quadrature of the circle', will occupy us later on.

Fig. 1.2.3. Queen Neferarti (19th dynasty, wife of Ramesses II) playing the game Senet. The rules of the game Senet could be roughly reconstructed. The rules of other games like 'Hounds and Jackals' or the Snake Game are mostly unknown (Wall painting in the burial chamber of Nefertari, West Thebes)

1.3 The π of the Bible

The ancient Egyptian value for π was already much more accurate than the 'biblical' value. In the first Book of Kings, Chapter 7:23 we read

> *'And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and its height was five cubits: and a line of thirty cubits did compass it round about.'*

And in the Second Book Chronicles, Chapter 4:2 we find

> *'Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about.'*

Hence the form of the sea is indeed a circle with diameter $d = 10$ cubits and circumference of $U = 30$ cubits. Since the relation between circumference and diameter of every circle is $U = \pi d$ we arrive at

$$\pi_{\text{Bible}} = \frac{U}{d} = \frac{30}{10} = 3.$$

This was the value which was also used by the Babylonians and Edwards (b. 1937) in [Edwards 1979, p. 4, Ex.5] gave an attempt to explain it which I find appealing. Instead of approximating the area of a circle in the Egyptian manner one could have come up with the idea of not only circumscribing a square to the circle but also to *inscribe* another square as in figure 1.3.1. Then the area of the circle should be approximated by the arithmetic mean of the areas of the squares. The area of the circumscribed square apparently is $A_1 := d^2 = 4r^2$. According to Pythagoras' theorem which was well known in Mesopotamia it follows that the edge length of the inner square is $\sqrt{r^2 + r^2} = \sqrt{2}r$, hence the area of the inscribed square turns out to be $A_2 := 2r^2$. Since the area of the circumscribed square overestimates the area of the circle while the area of the inscribed square underestimates it one can hope that the arithmetic mean might yield a useful approximation to the area of the circle:

$$A_{\text{Circle}} \approx \frac{A_1 + A_2}{2} = 3r^2.$$

And in fact here the biblical value of π appears!

But that seems not to be the end of the story as far as the Babylonians are concerned. According to Beckmann (1924 – 1993) [Beckmann 1971, p. 21 f.] and Neugebauer [Neugebauer 1969b, p. 46 f.] clay tablets were excavated in 1936 some 200 miles east of Babylon at Susa including computations concerning some geometrical figures. One of the tablets was concerned with a regular hexagon inscribed in a circle and stated that the ratio of the perimeter of the hexagon to the circumference of the circumscribed circle would be

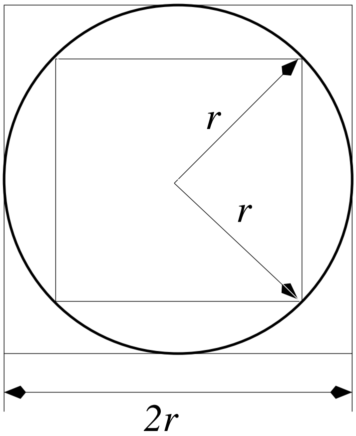

Fig. 1.3.1. Approximating the area of the circle from within

$$\frac{57}{60} + \frac{36}{60^2}.$$

The Babylonians knew that the perimeter of the hexagon is $6r$ if the radius is denoted by r, see figure 1.3.2.

The ratio sought therefore is $6r/C$ if C denotes the circumference of the circle. Since

$$\pi = C/2r$$

we conclude that

$$\frac{3}{\pi} = \frac{57}{60} + \frac{36}{60^2}$$

and hence $\pi = 31/8 = 3.125$.

This shows that also the Babylonians knew better approximations to π than just 3.

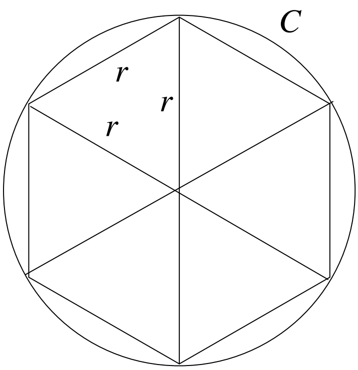

Fig. 1.3.2. Approximating the area of the circle by means of a regular hexagon

1.4 Volume of a Frustum of a Pyramid

In the so called Moscow Mathematical Papyrus located at the Pushkin Museum in Moscow one finds Problem 14 which almost points to one of the basic tasks of analysis. In this problem the volume of a frustum of a pyramid is computed.

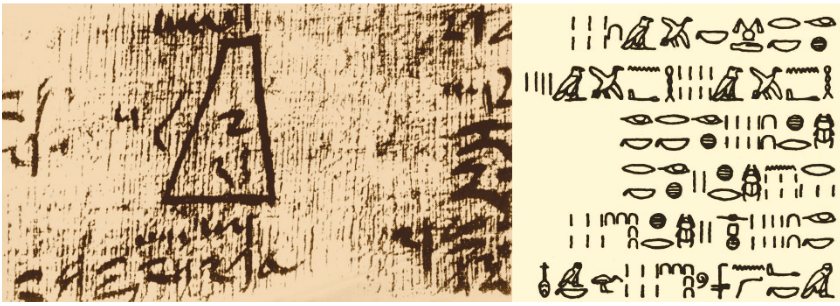

Fig. 1.4.1. Computation of the volume of a frustum of a pyramid (Moscow Mathematical Papyrus) in hieratic writing and in hieroglyphs

For the master builders of the pyramids this calculation must have been of particular importance since the pyramids were built in layers. A pyramid is therefore nothing more than the sum of frusta of pyramids with a pyramidal part on top. We do not want to speculate here how the Egyptians arrived at their (correct) formula of the volume of a frustum of a pyramid but refer our readers to the corresponding sections in [Gillings 1982] (see also [Scriba/Schreiber 2000, p. 14 ff.], [Wußing 2008, p. 99 f.]).

Although Problem 14 is only concerned with the computation using concrete numbers the Egyptians must have been aware of the correct formula

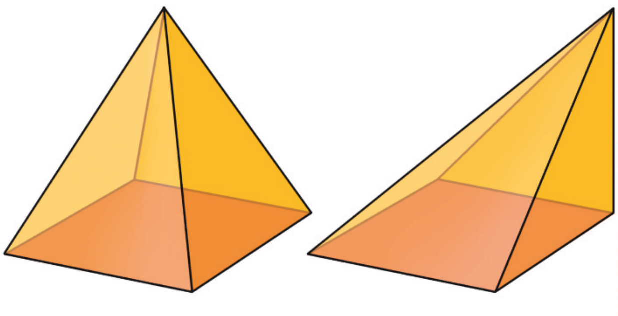

Fig. 1.4.2. A symmetric and a right-angled pyramid with identical base areas and identical heights share the same volume

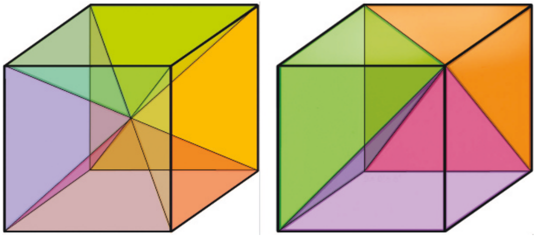

Fig. 1.4.3. Decomposition of a cube into six symmetric pyramids of half the height with the tip points in the centre (left), and in three right-angled pyramids (right)

$$V = \frac{h}{3}\left(a^2 + ab + b^2\right) \tag{1.1}$$

for the volume where a and b are the edge lengths of the two deck areas and h denotes the height of the frustum. Neugebauer [Neugebauer 1969a, p. 126] calls this a 'gem' (Glanzstück) of Egyptian mathematics.

Pointing towards mathematical analysis is the method with which the volume of a pyramid was probably actually computed (in [Gillings 1982] yet another method can be found). For this purpose the Egyptian scribes considered a pyramid in which the top point is located exactly above one of the edge points.

Three of those right-angled (or 'oblique') pyramids with identical height and base edge together form a cube with identical height and base edge; i.e. the volume of each of the pyramids is a third of the volume of the cube. One can alternatively build a cube from 6 congruent symmetric pyramids with half the height of their base edges. Place one of these pyramids top point to top point above another and fill the free space with the remaining four pyramids as

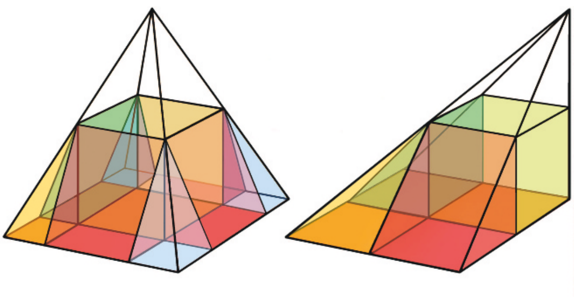

Fig. 1.4.4. The calculation of a frustum of a pyramid can graphically be understood by division into its geometric basic forms: 1 cuboid in the middle, 4 prisms at the sides, and 4 right-angled pyramids at the edges; in case of the right-angled pyramid the same cuboid but only 2 prisms of twice the volume at the sides and 1 right-angled pyramid of fourfold volume at the edge, so that both frusta have the identical volume of $V = \frac{h}{3}(a^2 + ab + b^2)$

Fig. 1.4.5. Step Pyramid of Pharao Djoser in Saqqara (about 2600 BC) [Photo: H.-W. Alten]

shown in figure 1.4.3 (left). Imagining now the right-angled pyramid in figure 1.4.2 cut into very many thin slices parallel to the base area and shifting these slices then a symmetric pyramid of the same volume results where its top point is now above the centre of the square base area. The same is valid in case of the frusta of pyramids in figure 1.4.4, of course. Indeed pyramids have emerged in ancient Egypt from many layers. Already the Mastabas of the kings of the first two dynasties (about 3000 – 2700 BC) show these layers.

King Djoser, second king of the 3rd dynasty, ordered his original three-stage Mastaba to be increased by three further stages where each of the stages consists of many thin layers of stone cuboids. Hence emerged the famous Step Pyramid of Djoser about 2680 BC in Saqqara. Under the rule of King Sneferu of the 4th dynasty the transition from layers of frusta towards the abstract geometrical form of the pyramid took place. That seems to have been a kind of a great gamble since in the first phase of the building up to a height of approximately 49 meters the construction by means of inwardly inclined layers proved unstable. This was the result of a too steep slope angle of approximately 58 degrees as well as the inclination of the layers. In the second phase the base area was enlarged, the slope angle was decreased to 54 degrees, but the techniques of the inwardly inclined layers was kept. As this also turned out to lead to instabilities the slope angle was further decreased to 43 degrees in a third phase and horizontal layers were put on the present frustum of a pyramid. Hence emerged the Bent Pyramid of Sneferu about