Progress in Probability
75

Sergio I. López
Víctor M. Rivero
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Arno Siri-Jégousse
Editors

# XIII Symposium on Probability and Stochastic Processes

UNAM, Mexico, December 4-8, 2017

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# Progress in Probability

Volume 75

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# XIII Symposium on Probability and Stochastic Processes

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# Preface

This volume contains research contributions and lecture notes from the XIII Symposium on Probability and Stochastic Processes, held at the Universidad Nacional Autónoma de México (UNAM), México in December 4–8, 2017.

Since the first edition of this symposium, held in December 1988 at CIMAT, the initial and main goal of the event is to create a forum where ideas are exchanged and recent progress in the field are discussed, by gathering national and international researchers as well as graduate students, making it one of the main events in the field in Mexico. Held in Mexico City, preserving the tradition of taking place almost biannually at different Mexican institutions all over the country for the last three decades, it gathered academics from Argentina, Chile, France, and the USA, many of whom are academically close to the probability community in Mexico. A wide range of topics were covered, from consolidated to emergent areas in probability, in both theoretical and applied probability. It is also worth mentioning that this event also hosted a number of conferences for the celebration of the 75th anniversary of the Instituto de Matemáticas, UNAM. This undoubtedly gave a great impulse to the symposium's program, in particular by ensuring the participation of further prestigious mathematicians and enlarging the scope of the event.

This scientific program included two courses: Reflected (degenerate) Diffusions and Stationary Measures organized by Mauricio Duarte and Soliton Decomposition of Box–Ball System and Hydrodynamics of N-Branching Brownian Motions by Pablo Ferrari. We thank both, Duarte and Ferrari for putting together these interesting courses and the lecture notes. The event also benefited from five plenary talks that were given by Octavio Arizmendi, Florent Benaych-Georges, Rolando Cavazos, Joaquín Fontbona, and Brian Rider, as well as four plenary talks shared with the Instituto de Matemáticas, given by Luis Caffarelli, Pierre-Louis Lions, Sylvie Méléard, and Nizar Touzi. Another four thematic sessions on Random Matrices, Random Trees, Risk Theory, and Stochastic Control and four contributed talks completed the outline of the symposium.

This volume is split into two main parts: the first one presents lecture notes of the course provided by Mauricio Duarte. It is followed by its second part which contains research contributions of some of the participants. Pablo Ferrari, along with Davide Gabrielli, wrote a research contribution paper instead of lecture notes.

The lecture notes of Mauricio Duarte give an insight about diffusions with instantaneous reflection when hitting a boundary. After defining properly these processes, two main tools are developed to analyze them: stochastic differential equations and the submartingale problem. After studying in detail the existence and uniqueness of the stationary measure of such processes, the author illustrates his lecture with two examples inspired from his own research: the Brownian motion reflected when hitting a falling particle (gravity versus Brownian motion) and the spinning Brownian motion.

The research contributions start with an article written by Osvaldo Angtuncio-Hernández where an alternative construction of multidimensional random walks conditioned to stay ordered is provided. This is the multidimensional version of the standard random walk conditioned to stay positive. This new construction is inspired by the one-dimensional case, where random walks are conditioned to stay positive until a geometric time, and has the advantage of requiring only a minimal restriction on the h-function, relaxing the hypotheses of previous works. The h-function is also studied in detail and a characterization, when the limit is Markovian or sub-Markovian, is provided.

A Berry-Essen-type theorem is presented by Octavio Arizmendi and Daniel Perales for finite free convolution of polynomials. They investigate the rate of convergence in the central limit theorem for finite free convolution of polynomials indicating, as for the classical and free case, also a rate of order $n^{-1/2}$. Cumulants for finite free convolution, introduced by the authors in another paper, are used to approach free cumulants in order to obtain their result. This approach provides a nice example of the many properties in free probability which already appear in finite free probability.

Erik Bardoux and José Pedraza pose the problem of finding an optimal time which minimizes the $L^1$ distance to the last time when a spectrally negative Lévy process $X$ (with positive drift) is below the origin barrier. Since last-passage times are not stopping times, a direct approach to solve the problem is not suitable. Based on existing results, the authors rewrite the problem as a classical stopping time problem. Using that reformulation, they succeed in linking the optimal stopping level with the median of the convolution with itself of the distribution function of the negative of the running infimum of the process. Three qualitative different examples are explicitly computed: Brownian motion with drift, Cramér–Lundberg risk process, and a process with infinite variation without a Gaussian component. These computations give a strong flavor of how these results can be used in applications.

With the initial intention of writing a survey article, Pablo Ferrari and Davide Gabrielli present a novel decomposition of configurations of solitons. After reviewing recent results about equivalent decompositions of solitons as different interacting particles systems, they construct a decomposition equivalent to the branch decomposition of the tree associated with the excursion. Using this new decomposition, they are able to obtain combinatorial results and to find explicitly the joint distribution of branches in the Bernoulli independent case. Besides the new results, the paper is a nice introduction to the topic of Box–Ball systems and its soliton conservation property.

Finally, Orimar Sauri studies the invertibility of continuous-time moving average processes driven by Lévy process. The author provides sufficient conditions for the recovery of the driving Lévy process, motivated by the discrete-time moving average framework. The paper reviews some invertibility results on discrete-time moving average processes and gives a detailed overview of stochastic integrals of a deterministic kernel regarding Lévy process. Then, continuous-time moving average processes are invertible whenever the Fourier transform of the kernel never vanishes, and a regularity condition on the characteristic triplet of the background driving Lévy process is imposed. Several examples are discussed including Ornstein–Uhlenbeck processes.

All of the papers, including the lecture notes from the guest course, were subjected to a strict peer-review process with high international standards. We are very grateful to the referees, experts in their fields, for their professional and useful reports. All of their comments were considered by the authors and substantially improved the material presented herein.

We would also like to express our gratitude to all of the authors whose contributions are published in this book, as well as to all of the speakers and session organizers of the symposium for their stimulating talks and support. Their valuable contributions show the interest and activity in the area of Probability and Stochastic Processes in Mexico.

We hold in high regards the editors of the book series Progress in Probability, Steffen Dereich, Davar Khoshnevisan, Andreas E. Kyprianou, and Sidney I. Resnick, for giving us the opportunity to publish this special volume in the aforementioned prestigious series.

Special thanks to the symposium venue Facultad de Ciencias at UNAM and its staff for providing great hospitality and excellent conference facilities.

We are also indebted to the Local Committee of the conference, formed by Clara Fittipaldi, Yuri Salazar-Flores, and Geronimo Uribe-Bravo, whose organizational work allowed us to focus on the academic aspects of the conference. The symposium as well as this volume would not have been possible without the great support of our sponsors:

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We hope that the reader will enjoy learning about the various topics addressed in this volume.

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# Contents

# Part I
# Lecture Notes

# Reflected (Degenerate) Diffusions and Stationary Measures

**Mauricio Duarte**

**Abstract** These notes were written with the occasion of the XIII Symposium on Probability and Stochastic Processes at UNAM. We will introduce general reflected diffusions with instantaneous reflection when hitting the boundary. Two main tools for studying these processes are presented: the submartingale problem, and stochastic differential equations. We will see how these two complement each other. In the last sections, we will see in detail two processes to which this theory applies nicely, and uniqueness of a stationary distribution holds for them, despite the fact they are degenerate.

## 1 Reflected(-ing) Brownian Motion

In this notes,[1] we will introduce some aspects of the theory of Reflected diffusions. We start by considering a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a one dimensional Brownian motion $B_t$ in this space. For details on the construction of Brownian motion and some other introductory material on stochastic processes related to these notes, the reader can consult the books [20, 22, 35].

[1] These notes were written for a four-lecture mini-course at the XIII Symposium on Probability and Stochastic Processes, held at the Faculty of Sciences, Universidad Nacional Autónoma de México, from December 4–8, 2017.

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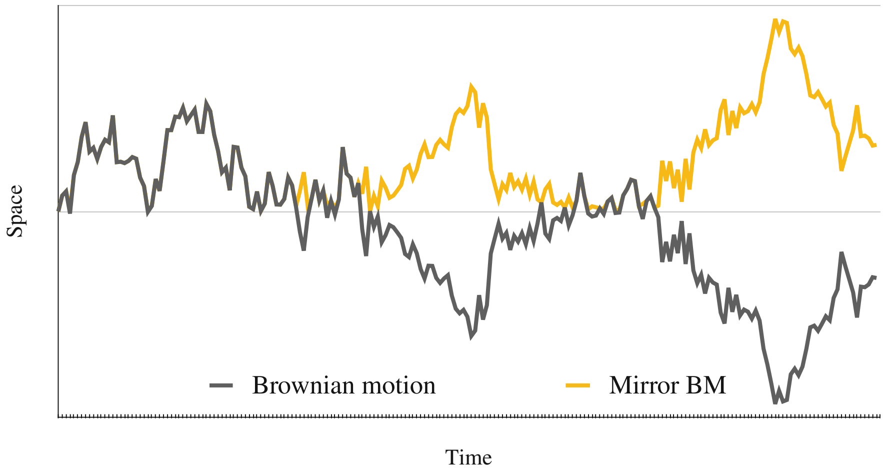


**Fig. 1** A Brownian motion and its mirrored reflection

A celebrated result of Lévy (1948), established that the following two processes have the same law:

$$X^1 = \{|B_t|\,,\ 0 \le t < \infty\}$$

$$X^2 = \left\{ \sup_{0 \le u \le t} (B_u - B_t),\ 0 \le t < \infty \right\}.$$

Is intuitive to picture the graph of $X_t^1$ as a Brownian motion that is mirrored on the horizontal axis. We will call the process $X_t^1$ a mirror Brownian motion (Fig. 1).

Observe that the mirror Brownian motion carries less information than its original counterpart.

**Proposition 1.1** *The filtration $\mathcal{F}_t^B$ generated by $B_t$ is strictly larger than the filtration $\mathcal{F}_t^{|B|}$ generated by $|B_t|$.*

***Proof*** Consider the process $S_t = \operatorname{sgn}(B_t)$, that is $S_t = 0$ if $B_t = 0$, and $S_t = B_t/|B_t|$ for $B_t \neq 0$. It is clear form the definition that $S_t$ is adapted to $\mathcal{F}_t^B$.

To prove our claim, it suffices to show that $S_t$ is not adapted to $\mathcal{F}_t^{|B|}$. Assume that $S_t = \varphi(|B_u|, u \le t)$, for some measurable function $\varphi$. Define $\widetilde{B}_t = -B_t$, and $\widetilde{S}_t = \operatorname{sgn}(\widetilde{B}_t)$. It follows that

$$\widetilde{S}_t = \varphi(|\widetilde{B}_u|, u \le t) = \varphi(|B_u|, u \le t) = S_t,$$

a contradiction. Therefore, $S_t$ is not adapted to $\mathcal{F}_t^{|B|}$, and the claim is proved. $\square$

**Definition 1.2** Any stochastic process with the same law as $X^1$ will be called a (one dimensional) Reflected Brownian motion. Typically, we will abbreviate it by RBM.

A necessary remark is that one way to obtain a RBM is as a mirrored BM. We will explore other ways to realize this process in what follows.

Our aim is to describe RBM improving the measurability restrictions discussed in Proposition 1.1. In order to do so, we need an extension of the well known Itô formula. The proof of the following proposition can be found in Theorem 1.1, Chapter VI [35].

**Proposition 1.3** *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and $X$. If $X$ is a continuous semimartingale, there exists a continuous increasing process $A^f$ such that*

$$f(X_t) = f(X_0) + \int_0^t D^- f(X_u) dX_u + A_t^f, \tag{1.1}$$

*where $D^- f$ corresponds to the left derivative of $f$.*

If we take $f(x) = |x|$, then the functional given by the previous proposition is known as the Local time of Brownian motion at zero, and is typically denoted by $L_t^0$. The resulting equation is known as Tanaka's formula:

$$|B_t| = |B_0| + \int_0^t \mathrm{sgn}(B_u) dB_u + L_t^0. \tag{1.2}$$

This equation shows that $|B_t|$ is also a semimartingale, and its martingale part is given by $\beta_t = \int_0^t \mathrm{sgn}(B_u) dB_u$. We can easily check from Itô's isometry that the quadratic variation of $\beta$ is $\langle \beta \rangle_t = t$, which implies that $\beta_t$ is a Brownian motion by Levy's characterization of Brownian motion (see [22, Chapter 3, Theorem 3.16].)

Lévy also proved that the law of the process given by $M_t^B = \sup_{0 \le u \le t} B_u$ is the same as the law of Local time at zero. The following proposition justifies the terminology "local time" for the process $L_t^0$.

**Proposition 1.4** *Given a Brownian motion $B_t$, its local time at zero satisfies the following formula a.s.*

$$L_t^0 = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{[-\varepsilon,\varepsilon]}(B_u) du, \tag{1.3}$$

***Proof*** The proof uses Itô's formula to approximate equation (1.2). We will consider a sequence of functions $f_n \in C^2(\mathbb{R})$ such that $f_n(x) \to |x|$ for all $x \in \mathbb{R}$, and for $x \neq 0$ it holds that $f_n'(x) \to \mathrm{sgn}(x)$, and $|f_n'(x)| \le 2$.

Take $\varepsilon > 0$. Let $g_n(x)$ be the even extension of the function

$$g_n(x) = \mathbb{1}_{[0,\varepsilon]}(x) + (1 - n(x-\varepsilon))\mathbb{1}_{\left(0,\frac{1}{n}\right]}(x-\varepsilon), \qquad x \ge 0,$$

and let $f_n(x)$ be the $C^2(\mathbb{R})$ function such that $f_n(0) = 0$, $f_n'(0) = 0$, and $f_n''(x) = g_n(x)$. For $x \geq 0$ we have

$$f_n'(x) = x\mathbb{1}_{[0,\varepsilon]}(x) + (\varepsilon + (x-\varepsilon)(2 - n(x-\varepsilon))/2)\mathbb{1}_{\left(0,\frac{1}{n}\right]}(x-\varepsilon) + \left(\varepsilon + \frac{1}{2n}\right)\mathbb{1}_{\left(\frac{1}{n},\infty\right]}(x-\varepsilon),$$

and for $x < 0$ we have $f_n'(x) = -f_n'(-x)$. We readily check that $|f_n'(x)| \leq \varepsilon + 1$, and that $f_n'(x) \to (x-\varepsilon)\mathbb{1}_{[0,\varepsilon]}(x) + \varepsilon$. Using these facts, we prove that there is a function $f_\infty(x) = \lim_{n\to\infty} f_n(x)$, satisfying

$$f_\infty(x) = \int_0^{|x|\wedge\varepsilon} (t-\varepsilon)\,dt + \varepsilon|x| = \frac{1}{2}(|x|-\varepsilon)^2\mathbb{1}_{[0,\varepsilon]}(x) + \varepsilon|x| = o(\varepsilon) + \varepsilon|x|$$

Applying Itô's formula to $f_n$, and taking the limit as $n \to \infty$, we obtain

$$o(\varepsilon) + \varepsilon|B_t| = \varepsilon|B_0| + \int_0^t (|B_u| - \varepsilon)\mathbb{1}_{[-\varepsilon,\varepsilon]}(B_u) + \operatorname{sgn}(B_u)\varepsilon\,dB_u + \frac{1}{2}\int_0^t \mathbb{1}_{[-\varepsilon,\varepsilon]}(B_u)du$$

For fixed $t \geq 0$, it is easy to see that the martingale term in the last equation has quadratic variation of order $o(\varepsilon)$, and so, by dividing that equation by $\varepsilon$, it converges as $\varepsilon$ goes to zero. We obtain,

$$|B_t| = |B_0| + \int_0^t \operatorname{sgn}(B_u)dB_u + \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{[-\varepsilon,\varepsilon]}(B_u)du.$$

Comparing this equation to (1.2), we obtain the desired result. □

Moving forward, if we denote $X_t = |B_t|$, then, the last display in the proof of Proposition 1.4 motivates the definition of the Local time at zero of $X_t$, namely,

$$L_t = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{[0,\varepsilon]}(X_u)du. \tag{1.4}$$

This yields the decomposition

$$X_t = X_0 + \beta_t + L_t, \tag{1.5}$$

where $\beta_t = \int_0^t \operatorname{sgn}(B_u)dB_u$ is a Brownian motion. The local time $L_t$ is then defined in terms of $X_t$ only. Let's summarize the most relevant properties we already know about $L_t$:

(1) it is increasing,
(2) it is a continuous additive functional,
(3) it only increases on the set $\{t : X_t = 0\}$.

Properties (1) and (2) are direct form Proposition 1.3, while (3) can be deduce directly form Proposition 1.4. Our next step will be to turn around the decomposition $X_t = X_0 + \beta_t + L_t$, that is, we will prove an important lemma that says that given any Brownian motion $\beta_t$, we can find a Reflected Brownian motion $X_t$, and a process $L_t$ satisfying properties (1)–(3) such that (1.5) holds.

**Lemma 1.5 (Skorohod)** *Let $y(\cdot)$ be a real valued, continuous function on $[0, \infty)$ such that $y(0) \geq 0$. There exists a unique pair of functions $(z, \varphi)$ on $[0, \infty)$ such that:*

*(a)* $z(t) = y(t) + \varphi(t)$ *for all* $t \geq 0$,
*(b)* $z(t) \geq 0$,
*(c)* $\varphi(\cdot)$ *is increasing, continuous,* $\varphi(0) = 0$, *and the corresponding measure* $d\varphi(t)$ *is carried by the set* $\{t : z(t) = 0\}$.

*Moreover,*

$$\varphi(t) = \sup_{u \leq t} (-y(t) \vee 0). \tag{1.6}$$

***Proof*** Define $\varphi(t)$ by (1.6) above, and $z(t)$ by condition (a) in the statement. Then, it is clear that (a) and (b) are satisfied. To check condition (c), note that Eq. (1.6) shows that $\varphi(0) = 0$ and that $\varphi$ is continuous and increasing. Since $z(t)$ is continuous, the set $\mathbb{Z} = \{t : z(t) = 0\}$ is closed. Hence, its complement $\mathbb{Z}^c$ is a union of open intervals. Let $(t_1, t_2) \subset \mathbb{Z}^c$. We claim that $\varphi(t_2) = \varphi(t_1)$. If not, we have $\varphi(t_2) > \varphi(t_1)$, which means by (1.6) that there is $t_3 \in (t_1, t_2)$ such that $\varphi(t_3) = -y(t_3) > \varphi(t_1)$. But then we obtain $t_3 \in \mathbb{Z}$, a contradiction. It follows that $d\varphi(t) = 0$ for $t \in \mathbb{Z}^c$.

To check uniqueness, let $(z_1, \varphi_1)$ be another pair of functions satisfying conditions (a), (b), and (c). Note that $z_1(t) - z(t) = y_1(t) - y(t)$ is a difference between two increasing processes and so it is an absolutely continuous function. We have

$$\begin{aligned}\frac{1}{2}|z(t) - z_1(t)|^2 &= \int_0^t (z(u) - z_1(u))d(\varphi(u) - \varphi_1(u)) \\ &= -\int_0^t z(u)d\varphi_1(u) - \int_0^t z_1(u)d\varphi(u) \leq 0,\end{aligned}$$

where we have used (c) in the second equality, and (b) and (c) in the inequality. It follows that $z_1(t) = z(t)$, and $y_1(t) = y(t)$. □

The Skorohod Lemma enables us to provide a strong construction of RBM, that is, given a Brownian motion $B_t$, we can define a RBM $X_t$ such that the filtrations generated by these two processes coincide.

**Theorem 1.6** *Let $B_t$ be a Brownian motion starting from 0. There are unique stochastic process $(X_t, L_t)$ such that $X_t = X_0 + B_t + L_t$, and $L_t = \sup_{u \leq t}(-(X_0 +$

$B_t) \wedge 0)$. *The law of* $(X_t, L_t)$ *is the same as* $(|B_t|, L_t^0)$*, that is,* $X_t$ *is a Reflected Brownian motion. Equation* (1.4) *holds, and also* $\mathcal{F}^X = \mathcal{F}^B$.

***Proof*** Since trajectories of $B_t$ are a.s. continuous, we can define $(X_t, L_t)$ as the unique solution to the Skorohod problem associated to $y(t) = X_0 + B_t$, from where the first assertion of the theorem follows.

From uniqueness in Skorohod's lemma, we have that $(|B_t|, L_t^0)$ is the unique solution for $y(t) = X_0 + \beta_t$ (with $\beta_t$ as before). The law of the solution $(X_t, L_t)$ is completely determined by the explicit solution to the Skorohod lemma, and so it coincides with the law of $(|B_t|, L_t^0)$, because $B_t$ and $\beta_t$ have the same law. This proves that $X_t$ is a RBM.

To show (1.4), recall the sequences of functions $f_n$ and $g_n$ defined in Proposition (1.4). Since $X_t$ is a semimartingale with martingale part $B_t$, it follows from Itô formula that

$$\begin{aligned} f_n(X_t) &= f_n(X_0) + \int_0^t f_n'(X_u)dB_u + \int_0^t f_n'(X_u)dL_u + \frac{1}{2}\int_0^t g_n(X_u)du \\ &= f_n(X_0) + \int_0^t f_n'(X_u)dB_u + \frac{1}{2}\int_0^t g_n(X_u)du, \end{aligned}$$

because $dL_t$ is supported on $\{X_t = 0\}$, and $f_n'(0) = 0$. Recall that $X_t \geq 0$. Taking the limit as $n$ goes to infinity, and justifying the limit procedures an is Proposition (1.4), we arrive at

$$o(\varepsilon) + \varepsilon X_t = \varepsilon X_0 + \int_0^t \big((X_u - \varepsilon)\mathbb{1}_{[0,\varepsilon]}(X_u) + \varepsilon\big)\ dB_u + \frac{1}{2}\int_0^t \mathbb{1}_{[0,\varepsilon]}(X_u)du. \tag{1.7}$$

We claim that the martingale $\varepsilon^{-1}\int_0^t (X_u - \varepsilon)\mathbb{1}_{[0,\varepsilon]}(X_u)dB_u$ converges to zero as $\varepsilon \to 0$. Indeed, it has quadratic variation

$$\varepsilon^{-2}\int_0^t (X_u - \varepsilon)^2\mathbb{1}_{[0,\varepsilon]}(X_u)du \leq \int_0^t \mathbb{1}_{[0,\varepsilon]}(X_u)du.$$

It follows that

$$\mathbb{E}\left(\varepsilon^{-2}\int_0^t (X_u - \varepsilon)^2\mathbb{1}_{[0,\varepsilon]}(X_u)du\right) \leq \mathbb{E}\left(\int_0^t \mathbb{1}_{[0,\varepsilon]}(|B_u|)du\right) = \int_0^t \mathbb{P}(|B_u| \leq \varepsilon)du$$

The last quantity converges to zero as $\varepsilon$ goes to zero. This can be seen, for instance, by dominated convergence. Dividing (1.7) by $\varepsilon$ and taking $\varepsilon$ to zero, yields

$$X_t = X_0 + B_t + \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{[0,\varepsilon]}(X_u)du.$$

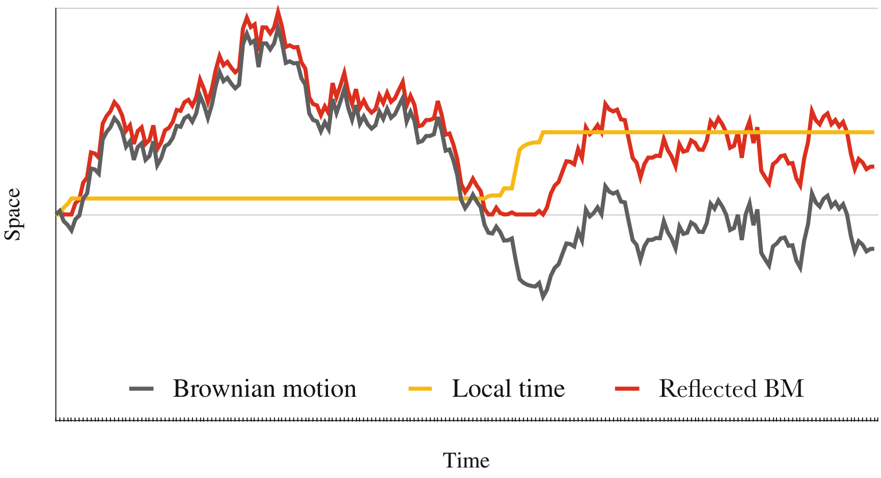


**Fig. 2** Typical paths of functions in Skorohod reflection of Brownian motion

Comparing this equation with $X_t = X_0 + B_t + L_t$, we obtain (1.4).

It is clear from the explicit solution to the Skorohod Lemma that $X_t$ is $\mathcal{F}_t^B$ measurable, and so $\mathcal{F}_t^X \subset \mathcal{F}_t^B$. To show the converse, note that Eq. (1.4) shows that $L_t$ is $\mathcal{F}_t^X$ measurable, and so it is $X_t - L_t = B_t$ (Fig. 2). $\square$

The Skorokhod Lemma redefines our notion of Reflected Brownian motion. We started with specular reflection, and ended with a constrained process, which, away from the boundary of its domain, behaves as a Brownian motion.

Can we extend this idea to other driving processes? Namely, instead of reflecting a BM, can we constrain in a similar way other diffusions?

**Exercise** Let $X_t$ be a RBM. Prove that

$$\int_0^\infty \mathbb{1}_{\{0\}}(X_t)dt = 0.$$

## 2 Stochastic Differential Equations with Reflection

Let $B_t$ be a Brownian motion. We are interested in finding a process $X_t \geq 0$ satisfying the following equation:

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt + dL_t, \tag{2.1}$$

where the coefficients $\sigma(\cdot)$ and $b(\cdot)$ are Lipschitz continuous in $\mathbb{R}_+$, and $\sigma(x) \geq 0$. The process $L_t$ must be increasing, continuous, and it must satisfy $dL_t = \mathbb{1}_{\{0\}}(X_t)dL_t$.

**Definition 2.1** A process $X_t$ satisfying (2.1) will be called a Reflected diffusion.

*Reflected Brownian Motion with Drift* Let's consider the case $\sigma \equiv \sigma_0 > 0$, and $b \equiv b_0 \in \mathbb{R}$, In this case, we can use Skorohod lemma again, to find the processes $(X_t, L_t)$. This time, we will write it a little different, but equivalently as in Eq. (1.6):

$$L_t = \sup_{u \le t} \left(-X_0 - \sigma_0 B_t - b_0 t \vee 0\right).$$

**Theorem 2.2** *A unique strong solution exists for Eq.* (2.1).

Our approach to finding a solution to (2.1) will be analytical. But let's make some remarks first:

- There is no reason to expect that $X_t = |X_0 + B_t|$ solves the equation.
- Because of the dependence of the coefficients on $X_t$, Skorohod lemma only provides an implicit solution. More precisely, provided we have a solution $X_t$, then Skorohod lemma ensures that

$$L_t = \sup_{u \le t} \left(-X_0 - \int_0^t \sigma(X_u) dB_u - \int_0^t b(X_u) du \vee 0\right)$$

- The process is an Itô diffusion up to the first time it hits level zero. The essential problem is how to define the local time after this time.

***Proof*** Fix $T > 0$, and consider the Banach space $\mathcal{H}_T$ of continuous adapted processes such that for all $t > 0$, $\mathbb{E}\left(\sup_{u \le T} |X_u|\right) < \infty$, equipped with the norm

$$\|X\|_T = \mathbb{E}\left(\sup_{u \le T} |X_u|^2\right)^{1/2}.$$

For $X \in \mathcal{H}_T$, let $\mathcal{S}(X)$ be the first component of the solution to the Skorohod problem with input $y(t) = X_0 + \int_0^t \sigma(X_u) dB_u + \int_0^t b(X_u) du$.

We will show that $\mathcal{S}(\cdot)$ is a contracting map if $T$ is small enough. Applying Itô formula, if $X, Y \in \mathcal{H}_T$, we deduce that [2]

$$\begin{aligned}
|\mathcal{S}(X)_t - \mathcal{S}(Y)_t|^2 = 2 \int_0^t & (\mathcal{S}(X)_u - \mathcal{S}(Y)_u)(\sigma(X_u) - \sigma(Y_u)) dB_u + \\
&+ 2 \int_0^t (\mathcal{S}(X)_u - \mathcal{S}(Y)_u)(b(X_u) - b(Y_u)) du + \\
&+ 2 \int_0^t (\mathcal{S}(X)_u - \mathcal{S}(Y)_u) d(L_u^X - L_u^Y) + \int_0^t |\sigma(X_u) - \sigma(Y_u)|^2 du.
\end{aligned}$$

[2]This is a good exercise in stochastic calculus. I recommend to follow it closely, and fill the minor gaps.