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Asymptotic, Algebraic and Geometric Aspects of Integrable Systems

In Honor of Nalini Joshi On Her
60th Birthday, TSIMF, Sanya, China,
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*Dedicated to Nalini Joshi on the occasion of
her 60th birthday*

Contents

Quadrangular Sets in Projective Line and in Moebius Space, and Geometric Interpretation of the Non-commutative Discrete Schwarzian Kadomtsev–Petviashvili Equation	1
Adam Doliwa and Jarosław Kosiorek	
Complexity and Integrability in 4D Bi-rational Maps with Two Invariants	17
Giorgio Gubbiotti, Nalini Joshi, Dinh Thi Tran, and Claude-Michel Viallet	
A Non-linear Relation for Certain Hypergeometric Functions	37
Gerd Schmalz and Vladimir Ezhov	
An Algebraically Stable Variety for a Four-Dimensional Dynamical System Reduced from the Lattice Super-KdV Equation	43
Adrian Stefan Carstea and Tomoyuki Takenawa	
Opers for Higher States of the Quantum Boussinesq Model	55
Davide Masoero and Andrea Raimondo	
Nonsingular Rational Solutions to Integrable Models	79
Gegenhasi, Xing-Biao Hu, Shi-Hao Li, and Bao Wang	
Stokes Phenomenon Arising in the Confluence of the Gauss Hypergeometric Equation	101
Calum Horrobin and Marta Mazzocco	
Periodic Trajectories of Ellipsoidal Billiards in the 3-Dimensional Minkowski Space	159
Vladimir Dragović and Milena Radnović	
Analogues of Kahan’s Method for Higher Order Equations of Higher Degree	175
A. N. W. Hone and G. R. W. Quispel	

On Some Explicit Representations of the Elliptic Painlevé Equation 191
Masatoshi Noumi

On the Lattice Potential KP Equation 213
Cewen Cao, Xiaoxue Xu, and Da-jun Zhang

Quadrangular Sets in Projective Line and in Moebius Space, and Geometric Interpretation of the Non-commutative Discrete Schwarzian Kadomtsev–Petviashvili Equation



Adam Doliwa and Jarosław Kosiorek

Abstract We present geometric interpretation of the discrete Schwarzian Kadomtsev–Petviashvili equation in terms of quadrangular set of points of a projective line. We give also the corresponding interpretation for the projective line considered as a Moebius chain space. In this way we incorporate the conformal geometry interpretation of the equation into the projective geometry approach via Desargues maps.

Keywords Discrete Schwarzian KP equation · Desargues maps · Projective line · Chain geometry · Moebius–Veblen configuration

2010 Mathematics Subject Classification 51B10, 51A20

1 Introduction

In the present paper we address two questions concerning geometric interpretation of the following discrete integrable system

$$(\phi_{(jk)} - \phi_{(k)})(\phi_{(jk)} - \phi_{(j)})^{-1}(\phi_{(ij)} - \phi_{(j)})(\phi_{(ij)} - \phi_{(i)})^{-1}(\phi_{(ik)} - \phi_{(i)})(\phi_{(ik)} - \phi_{(k)})^{-1} = 1, \quad (1)$$

where $\phi: \mathbb{Z}^N \rightarrow \mathbb{F}$ is a map from N -dimensional integer lattice to a division ring \mathbb{F} , and indices in brackets denote shifts in the corresponding variables, i.e. $\phi_{(i)}(n_1, \dots, n_N) = \phi(n_1, \dots, n_i + 1, \dots, n_N)$. The above equation appeared first as the generalized lattice spin equation in Nijhoff and Capel (1990), and was called the

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non-commutative discrete Schwarzian Kadomtsev–Petviashvili (SKP) equation in Bogdanov and Konopelchenko (1998); Konopelchenko and Schief (2005). As being one of various forms of the discrete Kadomtsev–Petviashvili (KP) system (Hirota 1981; Kuniba et al. 2011), Eq. (1) plays pivotal role in the theory of integrable systems and its applications.

Relation between geometry of submanifolds and integrable systems is an ongoing research subject which can be dated back to second half of XIX-th century (Darboux 1887–1896). In fact, geometric approach to discrete integrable systems initiated in Bobenko and Pinkall (1996), Doliwa and Santini (1997), Konopelchenko and Schief (1998), see also Bobenko and Suris (2009) for a review, demonstrates that the basic principles of the theory are encoded in incidence geometry statements, some of them known in antiquity.

For example, complex $\mathbb{F} = \mathbb{C}$ version of Eq. (1) was identified in Konopelchenko and Schief (2002) as a multi-ratio condition which describes generalization to conformal geometry of circles of the Menelaus theorem in the metric geometry (Coxeter and Greitzer 1967). Quaternionic version of the equation was studied in Konopelchenko and Schief (2005, 2009), see also King and Schief (2003) for other geometric interpretations of the multi-ratio condition in relation to integrable discrete systems.

The more recently introduced notion of Desargues maps (Doliwa 2010), as underlying property of discrete KP equation considers collinearity of three points. This approach works in projective geometries over division rings and leads directly to the linear problem for the equation in its non-Abelian Hirota–Miwa form (Nimmo 2006). We remark that the Desargues maps give new understanding (Doliwa 2010, 2013) of the previously studied discrete conjugate nets (Doliwa and Santini 1997). These are characterized by planarity of elementary quadrilaterals (see also Sauer 1937; Doliwa 1997). The compatibility condition for Desargues maps gives projective Menelaus theorem, but leaves open the following **Question 1: Can the conformal geometry interpretation of the discrete Schwarzian Kadomtsev–Petviashvili equation be incorporated into the Desargues map approach?** Notice that the recent generalization of the Desargues theorem to context of conformal geometry (King and Schief 2012) may suggest something opposite.

When studying reductions of the Desargues maps, as for example in Doliwa (2013), one is forced to restrict dimension of the ambient projective space up to “Desargues maps into projective line”. Even if the linear problem is well defined there the geometric condition, which defines the maps, is empty. This leads to **Question 2: What should replace the Desargues map condition for the ambient space being projective line?** We remark that the analogous problem for discrete conjugate nets in the ambient space being a plane was successfully solved in Adler (2006).

Our answer for both questions is based on the notion of the quadrangular sets, which was introduced by von Staudt in his seminal work (von Staudt 1847) as a tool to provide axiomatization of the projective geometry. We remark that quadrangular sets of points appeared in integrable discrete geometry in theory of the B -quadrilateral lattice (Doliwa 2007a), but in the context of the Pappus theorem and the Moebius pair of tetrahedra, which is outside of the interest of the present paper.

In Sect. 2 we first recall basic ingredients of the geometry of the projective line, in particular the notion of cross-ratio in the general non-commutative case (Baer 1952). We also formulate the corresponding concept of the multi-ratio of six points on the projective line over a division ring, which generalizes the definition known for commutative case in terms of two cross-ratios or determinants. We show that quadrangular sets of points are fully characterized by the “multiratio equals one” condition also in the non-commutative case (as the commutative case is well known Richter-Gebert 2011). This gives our answer to Question 2, which we present in Sect. 3.

Our answer to Question 1, which we present in Sect. 4, is also implied by geometry of the projective line, but this time the line is equipped with additional structure. When the division ring \mathbb{F} contains a subfield \mathbb{K} in its center then \mathbb{F} -projective images of the canonically embedded \mathbb{K} -line form the so called chains. This leads to the concept of Moebius chain geometry (Benz 1973; Herzer 1995). We show that in such spaces certain quadrangular sets have particular interpretation in terms of the so called Moebius–Veblen chain configuration. In the simplest case of the classical Moebius geometry, where $\mathbb{K} = \mathbb{R} \subset \mathbb{C} = \mathbb{F}$ the chains are circles (homographic images of the real line in the complex conformal plane), and our approach gives that of Konopelchenko and Schief (2002).

2 Projective Geometry of a Line

2.1 Cross-Ratio and Multi-ratio in Projective Geometry over Division Rings

A *right linear space* (\mathbb{F}, \mathbb{V}) consists of a division ring \mathbb{F} and an additive abelian group \mathbb{V} such that \mathbb{F} acts on \mathbb{V} from the right satisfying usual axioms. The corresponding projective geometry studies linear subspaces of the \mathbb{F} -space \mathbb{V} . The points of the corresponding projective space $\mathbb{P}(\mathbb{F}, \mathbb{V})$ are one dimensional subspaces of (\mathbb{F}, \mathbb{V}) .

Remark For simplicity we assume that \mathbb{F} is of characteristic zero, but we expect that also finite characteristic may be relevant and give interesting results (Białecki and Doliwa 2005).

A *collineation of the linear space* (\mathbb{F}, \mathbb{V}) upon the linear space (\mathbb{F}, \mathbb{W}) is a bijective and order preserving mapping σ of the partially ordered (by inclusion) set of subspaces of \mathbb{V} upon the set of subspaces of \mathbb{W} . When dimension of \mathbb{V} is at least three, any such collineation is given by a semi-linear map, i.e. linear map $\mathbb{V} \rightarrow \mathbb{W}$ and supplemented by an automorphism of the division ring.

The case of two dimensional linear spaces (i.e. projective lines) needs a special treatment. Then any bijection of projective line can be called collineation. There arises the problem of characterizing those maps which are induced by semi-linear

maps of (\mathbb{F}, \mathbb{V}) . The first step in that direction (the full answer can be found in Baer 1952) makes use of a generalization of the classical notion of cross ratio.

Definition 1 (Baer 1952) Suppose that P, Q, R, S are four distinct points on the line L . Then the number $c \in \mathbb{F}$ belongs to the cross ratio $\begin{bmatrix} P & Q \\ S & R \end{bmatrix} \subset \mathbb{F}$ if there exist elements $\mathbf{p}, \mathbf{q} \in \mathbb{V}$ such that

$$P = \langle \mathbf{p} \rangle, \quad Q = \langle \mathbf{q} \rangle, \quad R = \langle \mathbf{p} + \mathbf{q} \rangle, \quad S = \langle \mathbf{p} + \mathbf{q}c \rangle.$$

Below we present the known expression of the cross-ratio in terms of non-homogeneous coordinates.

Theorem 1 If \mathbf{k}, \mathbf{l} are two independent elements of \mathbb{V} , and p, q, r, s are four distinct elements of \mathbb{F} then

$$\begin{bmatrix} \langle \mathbf{k} + \mathbf{l}p \rangle & \langle \mathbf{k} + \mathbf{l}q \rangle \\ \langle \mathbf{k} + \mathbf{l}s \rangle & \langle \mathbf{k} + \mathbf{l}r \rangle \end{bmatrix} = [(s - q)^{-1}(p - s)(p - r)^{-1}(r - q)],$$

where by for $c \in \mathbb{F}$ by $[c] = \{aca^{-1} \mid a \in \mathbb{F} \setminus \{0\}\}$ we denote the equivalence class of conjugate elements.

Remark Given three distinct points $P, Q, R \in L$ and given $c \in \mathbb{F} \setminus \{0, 1\}$ there exists the fourth point $S \in L$, distinct from the previous ones, such that $c \in \begin{bmatrix} P & Q \\ S & R \end{bmatrix}$; $c = 0$ corresponds to $S = P$, $c = 1$ corresponds to $S = R$, while to admit $S = Q$ we need to give the value $c = \infty$.

Remark When points P, Q, R have been taken as projective basis of the line, i.e. $p = 0, q = \infty, r = 1$, then $\begin{bmatrix} P & Q \\ S & R \end{bmatrix} = [s]$.

The following result for $\dim \mathbb{V} \geq 3$ justifies the use of cross-ratio in describing geometry of the projective line.

Theorem 2 Suppose that (P, Q, R, S) and (P', Q', R', S') are quadruples of distinct collinear points. There exists collineation π of the linear space (\mathbb{F}, \mathbb{V}) , $\dim \mathbb{V} \geq 3$, such that $\pi(P) = P', \pi(Q) = Q', \pi(R) = R', \pi(S) = S'$ if and only if there exists an automorphism α of the division ring \mathbb{F} such that

$$\begin{bmatrix} P & Q \\ S & R \end{bmatrix}^\alpha = \begin{bmatrix} P' & Q' \\ S' & R' \end{bmatrix}.$$

We present below the analogous geometric notion of multi-ratio in the non-commutative case, which we adapted from known definition in the commutative case in terms of two cross-ratios (Morley and Musselman 1937; King and Schief 2003; Richter-Gebert 2011). Like for the non-commutative cross-ratio our geometric definition leads to a class of conjugate elements of the division ring.

Definition 2 Suppose that P, Q, R, S, T, U are six distinct points on the line L . Then the number $m \in \mathbb{F}$ belongs to the multi-ratio $\left[\begin{smallmatrix} P & Q & S \\ U & R & T \end{smallmatrix} \right] \subset \mathbb{F}$ if there exist elements $p, q, s \in \mathbb{V}$ such that

$$\begin{aligned} P &= \langle p \rangle, & Q &= \langle q \rangle, & S &= \langle s \rangle, & R &= \langle p + q \rangle, & T &= \langle p + s \rangle, \\ U &= \langle p + qa \rangle, & U &= \langle p + sb \rangle & m &= ab^{-1}. \end{aligned}$$

Proposition 3 If k, l are two independent elements of \mathbb{V} , and p, q, r, s, t, u are six distinct elements of \mathbb{F} then

$$\left[\begin{smallmatrix} \langle k + lp \rangle & \langle k + lq \rangle & \langle k + ls \rangle \\ \langle k + lu \rangle & \langle k + lr \rangle & \langle k + lt \rangle \end{smallmatrix} \right] = \left[(r - p)^{-1}(q - r)(q - u)^{-1}(s - u)(s - t)^{-1}(t - p) \right]. \quad (2)$$

Proof Let the vectors $p, q \in \mathbb{V}$ be such as in Definition 2, define the factors $\lambda, \mu, \sigma, \rho \in \mathbb{F} \setminus \{0\}$ for points P, Q, R, U by

$$(k + lp)\lambda = p, \quad (k + lq)\mu = q, \quad (k + lr)\sigma = p + q, \quad (k + lu)\rho = p + qa.$$

Elimination of the factors μ, σ, ρ leads to the relation

$$a = \lambda^{-1}(r - p)^{-1}(q - r)(q - u)^{-1}(u - p)\lambda.$$

Similar reasoning, but for points P, S, T, U leads to similar relation

$$b = \lambda^{-1}(t - p)^{-1}(s - t)(s - u)^{-1}(u - p)\lambda,$$

which, combined with the previous one, concludes the proof. \square

Remark When the division ring \mathbb{F} is commutative, our definition of the multi-ratio $\left[\begin{smallmatrix} P & Q & S \\ U & R & T \end{smallmatrix} \right] \in \mathbb{F}$ reduces to the product $\left[\begin{smallmatrix} P & Q \\ U & R \end{smallmatrix} \right] \left[\begin{smallmatrix} P & S \\ U & T \end{smallmatrix} \right]^{-1}$ of two cross-ratios.

Proposition 4 The multi-ratio is an invariant of the collineations induced by linear transformations of the space (\mathbb{F}, \mathbb{V}) .

Proof Fix coordinate system, like in Proposition 3, and use the fact, that such collineations are generated by affine transformations $\phi \mapsto a\phi + b$ and inversions $\phi \mapsto \phi^{-1}$

$$(a\phi + b)(c\phi + d)^{-1} = ac^{-1} + (b - ac^{-1}d)(c\phi + d)^{-1}.$$

By direct calculation, both transformations preserve the multi-ratio (2) understood as the class of conjugate elements. \square

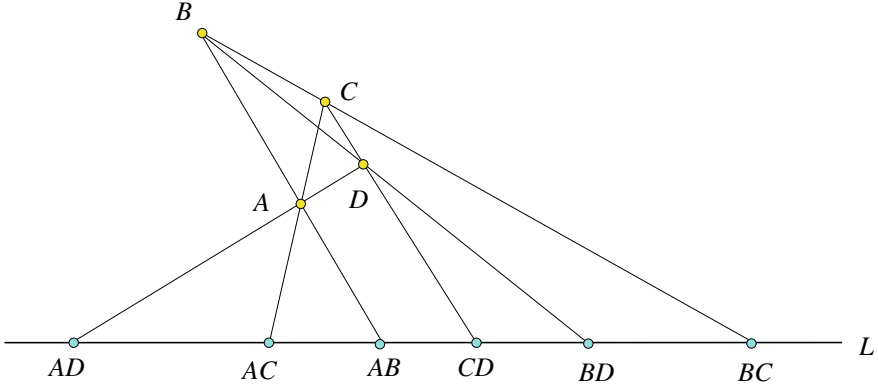


Fig. 1 Quadrangular set of points

2.2 Quadrangular Set of Points on Projective Line

A *complete quadrangle* is a projective figure formed by four points (vertices) A, B, C, D in the plane, no three of which are collinear, and the six distinct lines (sides) that are produced by joining them pairwise. The intersection points AB, \dots, CD of the six lines with a line not incident with vertices of the quadrangle form the *quadrangular set* (von Staudt 1847), see Fig. 1.

It is known that, by Desargues theorem, any five points of the quadrangular set (labeling fixed) determine uniquely the sixth point of the set. Moreover, collineations map quadrangular sets into quadrangular sets.

Remark The ordering of the points is important, up to permutation of the letters A, B, C, D . By combinatorial arguments one can show that given five generic points of the projective line there are 30 positions of the sixth point such that for appropriate ordering the six points form a quadrangular set.

Proposition 5 *The six distinct points AB, \dots, CD form a quadrangular set if and only if their non-homogeneous coordinates $\Phi_{AB}, \dots, \Phi_{CD}$ satisfy the multi-ratio condition*

$$(\Phi_{AD} - \Phi_{AC})(\Phi_{AD} - \Phi_{AB})^{-1}(\Phi_{BD} - \Phi_{AB})(\Phi_{BD} - \Phi_{BC})^{-1}(\Phi_{CD} - \Phi_{BC})(\Phi_{CD} - \Phi_{AC})^{-1} = 1. \quad (3)$$

Proof Given five points of the set, we reconstruct the planar quadrilateral which allows to obtain the sixth point. It is known (Veblen and Young 1910) that the construction is independent on the freedom in choice of the quadrilateral. Because we were not able to find the multi-ratio characterization of the quadrangular sets in the non-commutative case we present its detailed derivation.

In the general case fix coordinate system on the line $L = \{(a, 0) | a \in \mathbb{F}\}$. Choose an arbitrary point $A \neq L$, which can be given then non-homogeneous coordinates

$(0, 1)$. The last freedom in the construction is the choice of a point B on the line $\langle A, AB \rangle$, which fixes its coordinates

$$(\Phi_B^1, \Phi_B^2) = (\Phi_{AB}(1 - \sigma), \sigma), \quad \sigma \in \mathbb{F} \setminus \{0, 1\}.$$

Then the coordinates of the point $C = \langle A, AC \rangle \cap \langle B, BC \rangle$ are

$$(\Phi_C^1, \Phi_C^2) = (\Phi_{AC}(1 - \mu), \mu), \quad \mu = \left[\Phi_{AC} - \Phi_{AB} + (\Phi_{AB} - \Phi_{BC})\sigma^{-1} \right]^{-1} (\Phi_{AC} - \Phi_{BC});$$

notice identity

$$\mu^{-1} - 1 = (\Phi_{AC} - \Phi_{BC})^{-1} (\Phi_{AB} - \Phi_{BC}) (\sigma^{-1} - 1). \quad (4)$$

Similarly, the coordinates of the point $D = \langle A, AD \rangle \cap \langle C, CD \rangle$ read

$$(\Phi_D^1, \Phi_D^2) = (\Phi_{AD}(1 - \lambda), \lambda), \quad \lambda = \left[\Phi_{AD} - \Phi_{AC} + (\Phi_{AC} - \Phi_{CD})\mu^{-1} \right]^{-1} (\Phi_{AD} - \Phi_{CD}),$$

and then

$$\lambda^{-1} - 1 = (\Phi_{AD} - \Phi_{CD})^{-1} (\Phi_{AC} - \Phi_{CD}) (\mu^{-1} - 1), \quad (5)$$

moreover

$$(\Phi_{CD} - \Phi_{AC})^{-1} (\Phi_{AD} - \Phi_{AC}) = \mu^{-1} (\mu - \lambda) (1 - \lambda)^{-1}. \quad (6)$$

Finally, the non-homogeneous coordinates $(\Phi_{BD}, 0)$ of the point $BD = L \cap \langle B, D \rangle$ are given by

$$\Phi_{BD} = \Phi_{AB}(1 - \sigma) + [\Phi_{AD}(1 - \lambda) - \Phi_{AB}(1 - \sigma)] (\sigma - \lambda)^{-1} \sigma, \quad (7)$$

what implies

$$(\Phi_{BD} - \Phi_{AB})^{-1} (\Phi_{AD} - \Phi_{AB}) = \sigma^{-1} (\sigma - \lambda) (1 - \lambda)^{-1}. \quad (8)$$

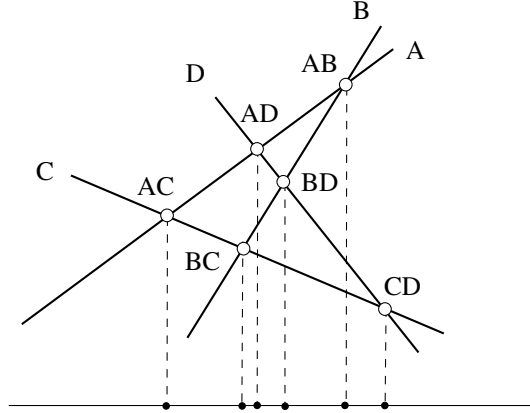
Equations (4)–(8) give then

$$\begin{aligned} & (\Phi_{BD} - \Phi_{BC}) - (\Phi_{CD} - \Phi_{BC})(\Phi_{CD} - \Phi_{AC})^{-1} (\Phi_{AD} - \Phi_{AC})(\Phi_{AD} - \Phi_{AB})^{-1} (\Phi_{BD} - \Phi_{AB}) = \\ & [(\Phi_{AD} - \Phi_{CD})(\lambda^{-1} - 1) - (\Phi_{AB} - \Phi_{BC})(\sigma^{-1} - 1) + (\Phi_{CD} - \Phi_{BC})(\mu^{-1} - 1)] \lambda (\sigma - \lambda)^{-1} \sigma = 0, \end{aligned}$$

which concludes the first part of the proof.

Because Eq. (3) is uniquely solvable for any of its six points, once other five are given, and by the analogous property of the quadrangular set, the condition described by the equation completely characterizes quadrangular sets of the projective line. \square

Fig. 2 Veblen configuration



3 Desargues Maps into Projective Line

3.1 The Veblen Configuration and the Multi-ratio

Consider Veblen or Menelaus (Konopelchenko and Schief 2002) configuration $(6_2, 4_3)$ in projective space, i.e. six points and four lines with two lines incident with each point, and three points incident with each line, see Fig. 2. We label points of the configuration by two element subsets of the four element set $\{A, B, C, D\}$, and lines by single elements of the same set. A point is incident with the line if its label contains the label of the line.

Let us present an algebraic description of the Veblen configuration, which can be considered as a non-commutative version of the theorem of Menelaus (Coxeter and Greitzer 1967).

Proposition 6 *Given points $AB \in \langle BC, BD \rangle$, $AD \in \langle BD, CD \rangle$, $AC \in \langle BC, CD \rangle$ on three sides of the triangle BC, BD, CD , and distinct from its vertices. These three points are collinear if and only if the corresponding proportionality coefficients $a, b, c \in \mathbb{F} \setminus \{0, 1\}$ between their non-homogeneous coordinates, as defined by*

$$\begin{aligned} (\Phi_{BC} - \Phi_{AB}) &= (\Phi_{BD} - \Phi_{AB})a, \\ (\Phi_{CD} - \Phi_{AD}) &= (\Phi_{BD} - \Phi_{AD})b, \quad (\Phi_{BC} - \Phi_{AC}) = (\Phi_{CD} - \Phi_{AC})c, \end{aligned} \quad (9)$$

satisfy condition

$$a = bc. \quad (10)$$

Proof To show that the collinearity implies condition (10) assume that the vectors $\Phi_{AB}, \Phi_{AC}, \Phi_{AD}$, as calculated from the above linear relations, satisfy the constraint of the form

$$\Phi_{AB} - \Phi_{AC} = (\Phi_{AD} - \Phi_{AC})\lambda.$$

The linear independence of vectors $\Phi_{BD} - \Phi_{CD}$ and $\Phi_{BC} - \Phi_{CD}$ implies then

$$\lambda = 1 - (c - 1)(a - 1)^{-1} = (b - 1)c(a - 1)^{-1},$$

which gives Eq. (10).

From the other side, insert the condition (10) into first of the above linear equations, which in conjunction with other two gives

$$(\Phi_{AB} - \Phi_{AD})(1 - bc) = (\Phi_{AC} - \Phi_{AD})(1 - c),$$

thus showing the collinearity. \square

Corollary 7 *Assume that for fixed coordinate number i all components $\Phi_{AB}^i, \dots, \Phi_{CD}^i$ of the points of the Veblen configuration are distinct (see Fig. 2), then the components satisfy the following multi-ratio condition*

$$(\Phi_{CD}^i - \Phi_{AC}^i)(\Phi_{CD}^i - \Phi_{AD}^i)^{-1}(\Phi_{BD}^i - \Phi_{AD}^i)(\Phi_{BD}^i - \Phi_{AB}^i)^{-1}(\Phi_{BC}^i - \Phi_{AB}^i)(\Phi_{BC}^i - \Phi_{AC}^i)^{-1} = 1.$$

Proof Insert expressions

$$\begin{aligned} a &= (\Phi_{BD}^i - \Phi_{AB}^i)^{-1}(\Phi_{BC}^i - \Phi_{AB}^i), \\ b &= (\Phi_{BD}^i - \Phi_{AD}^i)^{-1}(\Phi_{CD}^i - \Phi_{AD}^i), \\ c &= (\Phi_{CD}^i - \Phi_{AC}^i)^{-1}(\Phi_{BC}^i - \Phi_{AC}^i), \end{aligned}$$

into the condition (10). \square

We conclude this Section with a result, which justifies the statement that *quadrangular sets should be considered as Veblen configurations in the geometry of projective line.*

Proposition 8 *In the plane of the Veblen configuration consider point O not on lines of the configuration. The intersection points of lines joining O to vertices of the configuration with an arbitrary line not incident with O form a quadrangular set.*

Proof Take the point O as the first vertex of the quadrangle, fix a line of the Veblen configuration, and use three remaining points of the configuration as three remaining vertices of the quadrangle. On the line we have built then a quadrangular set. The lines joining the points of the Veblen configuration with point O are the lines joining O to points of the quadrangular set. Any transversal section of the lines by another line gives six points perspective with the quadrangular set. Because such transformations map quadrangular sets into quadrangular sets (Veblen and Young 1910) we obtain the statement. \square

Remark The case when O is a point at infinity and the line is a coordinate line is actually visualized in Fig. 2.

3.2 Desargues Maps

From point of view of difference equations usually one considers maps of \mathbb{Z}^N lattice. Recently integrable systems on other regular lattices are also of some interest, see for example (Doliwa et al. 2007). In particular, the Desargues maps, although initially defined on multidimensional integer lattice, allow for an interpretation (Doliwa 2011) as maps from multidimensional root lattice of type A . Such an approach from the very beginning takes into account the corresponding affine Weyl group symmetry of the discrete KP system.

Recall that the $N \geq 2$ -dimensional root lattice $Q(A_N)$ is generated by vectors along the edges of regular N -simplex in Euclidean space (Conway and Sloane 1988; Moody and Patera 1992). Vertices of the lattice tessellate the space into N types of convex polytopes $P(k, N), k = 1, \dots, N$, called *ambo-simplices*. It is known that the corresponding affine Weyl group $W(A_N)$ acts on the Delaunay tiling by permuting tiles within each class $P(k, N)$. The 1-skeleton of $P(k, N)$ is the so called Johnson graph $J(N + 1, k)$: its vertices are labeled by k -point subsets of $\{1, 2, \dots, N + 1\}$, and edges are the pairs of such sets with $(k - 1)$ -point intersection.

The tiles $P(1, N)$ are congruent to the initial N -simplex which generates the vertices of the root lattice. We color its faces $P(1, 2)$ in black. The faces $P(2, 2)$ of the simplex $P(N, N)$ we color white. Then $P(2, 3)$ which is regular octahedron has four black and four white triangular faces, see Fig. 3.

Definition 3 By Desargues map $\phi: Q(A_N) \rightarrow \mathbb{P}(\mathbb{F}, \mathbb{V})$ we mean a map for which images of vertices of simplices $P(1, 2)$ with black triangular faces are collinear.

To avoid degenerations it is implicitly assumed that images of vertices of simplices $P(2, 2)$ with white triangular faces are in generic position. Then the octahedra $P(2, 3)$

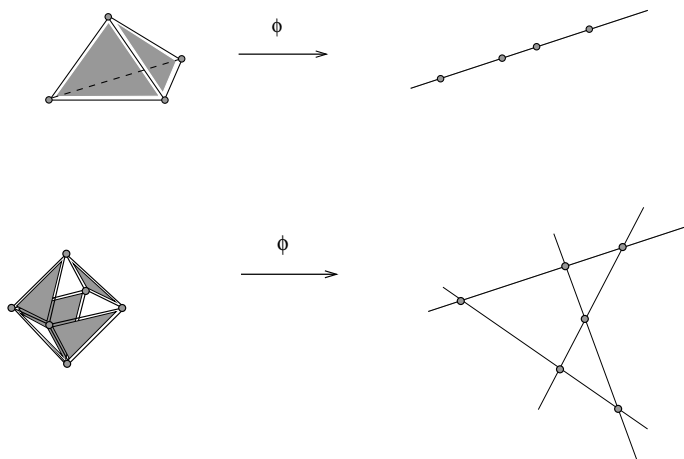


Fig. 3 Desargues map

is mapped into Veblen configurations, see Fig. 3. Moreover, after introduction of appropriate \mathbb{Z}^N coordinates in the lattice the non-homogeneous coordinates of the map satisfy Eq. (1), compare with Corollary 7. Guided by Proposition 8 we give our answer to Question 2.

Definition 4 By Desargues map of root lattice $Q(A_N)$ into a projective line we mean a map for which images of vertices of ambo-simplices $P(2, 3)$ are quadrangular sets with labeling induced by that of Johnson graph $J(4, 2)$.

4 Desargues Maps in Moebius Chain Geometry

Below we describe the concept of Moebius chain geometry of a projective line, which allows for a construction of quadrangular sets which satisfy certain additional property, adjusted to the structure. This will give our answer to Question 1.

4.1 The Concept of Moebius Chain Geometry

Assume that the division ring \mathbb{F} contains a proper subfield \mathbb{K} in its center. The division ring \mathbb{F} can be considered then as a division \mathbb{K} -algebra. Correspondingly, the projective line over \mathbb{F} inherits additional structure, best described within the concept of chain geometry (Benz 1973; Herzer 1995; Blunck and Herzer 2005). Define the chains as images of the canonically embedded \mathbb{K} -line (called the standard chain) under action of the group of collineations induced by linear maps of the \mathbb{F} -line. Points are called cocatenal if they belong to a common chain.

Remark The simplest example for $\mathbb{K} = \mathbb{R} \subset \mathbb{C} = \mathbb{F}$ is the classical conformal Moebius geometry of circles (as chains) in the Riemann sphere (complex line or conformal plane). The notion of chain geometry applies actually to any \mathbb{K} -algebra, however we will be dealing exclusively with division algebras (i.e. with the so called Herzer 1995 Moebius chain geometries).

The \mathbb{K} -vector space \mathbb{F} can be given natural affine space structure. The straight lines are the chains which pass through the infinity point of the projective line $\mathbb{F} \cup \{\infty\}$. Notice that the notion of “straight line” depends actually on the particular choice of the infinity point. Two chains are called tangent in point P when after sending P to infinity the chains are parallel in corresponding affine space. Notice following results of the Moebius chain geometry:

- (1) Any three distinct points of the \mathbb{F} -line are contained in exactly one chain.
- (2) Four distinct points are cocatenal if and only if their cross-ratio is well defined element of $\mathbb{K} \setminus \{0, 1\}$.
- (3) (Miquel condition) Given four chains $\mathcal{C}_i, i = 1, 2, 3, 4$, no three of which have a common point, but $\mathcal{C}_i \cap \mathcal{C}_{i+1} = \{P_i, Q_i\}$ for every i (subscripts are taken

modulo 4). Then the four points P_i are cocatenal if and only if the four points Q_i are cocatenal.

Remark One can consider (Blunck and Havlicek 2000) generalizations of the Moebius chain geometry for which \mathbb{K} is a subdivision ring of \mathbb{F} not necessarily contained in its center (like for example in the case $\mathbb{C} \subset \mathbb{H}$, where chains are two dimensional spheres S^2 in $S^4 \equiv \mathbb{H} \cup \{\infty\}$). However such a generalization may violate some of the properties above.

4.2 Moebius–Veblen Configuration

Let us present an analogue of the Veblen axiom/theorem, see also Blunck and Herzer (2005) for its general version (in a slightly different formulation) for chain geometries.

Proposition 9 *Given five distinct points AB, AC, AD, BC, BD of the Moebius chain space such that the chains $\mathcal{C}(AB, AC, AD)$ and $\mathcal{C}(AB, BC, BD)$ have in common additional point $I \neq AB$. Then also the chains $\mathcal{C}(I, AC, BC)$ and $\mathcal{C}(I, AD, BD)$ have in common additional point $CD \neq I$, or alternatively the chains are tangent in I .*

Proof By sending the intersection point I to infinity, see Fig. 4, we obtain five points of the classical Veblen configuration in the affine space (\mathbb{K}, \mathbb{F}) what allows to construct the sixth point of the “straightened configuration”, and then eventually to go back to the original one. □

Remark Notice that in the classical Moebius geometry of the complex projective line the assumption about existence of the point I is not needed. However this assumption is essential, if we would like to perform the construction in the case of quaternionic projective line with $\mathbb{R} \subset \mathbb{H}$.

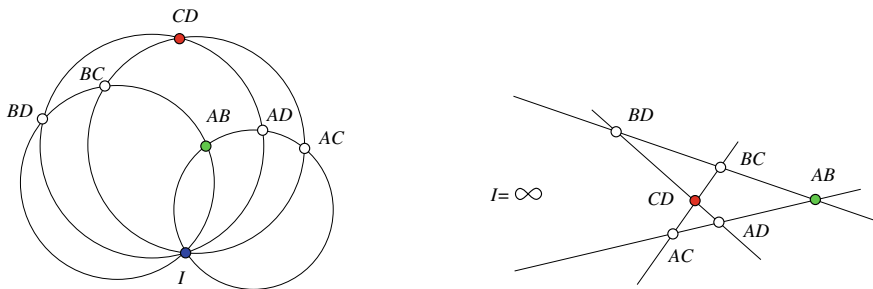


Fig. 4 Moebius–Veblen configuration in Moebius chain geometry

The points AB, AC, AD, CD, BC, BD will be called ordinary points of the Moebius–Veblen configuration, while I is called the infinity point of the configuration.

Proposition 10 *In the Moebius–Veblen configuration the constructed point CD is the sixth quadrangular point of the initial five ordinary points in the standard labeling.*

Proof In the “straightened configuration” apply Proposition 5 to get coefficients a, b, c which belong to $\mathbb{K} \setminus \{0, 1\}$ and satisfy condition (10). But now one can eliminate the coefficients directly on the level of Eq. (9) in order to get the multi-ratio condition (3) in \mathbb{F} . Proposition 4, which gives invariance of the condition with respect to projective collineations of the \mathbb{F} -line, and Proposition 5 imply the statement. \square

Remark Even if we do not have the point I to our disposal the sixth point CD of the quadrangular set exists, by the general construction described in Sect. 2. Then one can consider also the corresponding initial chains $\mathcal{C}(AB, AC, AD)$, $\mathcal{C}(AB, BC, BD)$, which contain point AB , and the resulting chains $\mathcal{C}(AC, BC, CD)$ and $\mathcal{C}(AD, BD, CD)$ containing the point CD . These new chains can be given only *after* construction of the point.

Finally we give our answer to Question 1 presenting the special type of quadrangular sets for which the construction of the sixth point follows from Moebius chain geometry principles, thus incorporating the conformal geometry interpretation of the discrete Schwarzian Kadomtsev–Petviashvili equation into the Desargues map approach.

Definition 5 By Moebius quadrangular set we mean the six ordinary points of the Moebius–Veblen configuration with the labeling as in Proposition 9.

It is well known (Clifford 1871; Konopelchenko and Schief 2002) that the Moebius–Veblen configuration can be supplemented by four chains $\mathcal{C}(AB, BC, AC)$, $\mathcal{C}(AB, BD, AD)$, $\mathcal{C}(AC, CD, AD)$ and $\mathcal{C}(BC, CD, BD)$ which then all intersect in the so called Clifford point. The new circle-point configuration of 8 points and 8 circles, with each point/circle incident with 4 circles/points, is called the Clifford configuration. Actually, as it was described in Doliwa (2007b), this result in an equivalent version was known already to Miquel (Miquel 1838).

5 Conclusion

The projective structure of the line and the notion of quadrangular sets can be used to provide geometric meaning to non-commutative discrete Schwarzian KP system. When the thinner structure of the underlying division ring is considered then the theory becomes more intriguing. Up to now the case of complex and quaternionic Moebius spaces (with the subfield of real numbers) have been thoroughly examined (Konopelchenko and Schief 2002, 2009). There are known works, see for

example (Bazhanov et al. 2008), where the Miquel condition has been used to study particular algebras of quantum integrable systems. We expect that the chain geometry may become useful platform to investigate other quantum algebras of mathematical physics.

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Complexity and Integrability in 4D Bi-rational Maps with Two Invariants



Giorgio Gubbiotti, Nalini Joshi, Dinh Thi Tran, and Claude-Michel Viallet

Abstract In this letter we give fourth-order autonomous recurrence relations with two invariants, whose degree growth is cubic or exponential. These examples contradict the common belief that maps with sufficiently many invariants can have at most quadratic growth. Cubic growth may reflect the existence of non-elliptic fibrations of invariants, whereas we conjecture that the exponentially growing cases lack the necessary conditions for the applicability of the discrete Liouville theorem.

Keywords Integrability · Complexity · Algebraic entropy · Growth of bi-rational maps · Higher-order difference equation

1 Introduction

Bi-rational maps in two dimensions have played a crucial role in the study of integrable discrete dynamical systems since the seminal paper of Penrose and Smith (1981) and the introduction of the QRT mappings in Quispel et al. (1988, 1989). Elliptic curves and rational elliptic surfaces proved to be one of the main tools in understanding the geometry behind this kind of integrability, see Sakai (2001), Duistermaat (2011), Tsuda (2004). In this letter we give examples of higher-order maps

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