Ata Allah Taleizadeh

Imperfect Inventory Systems

Inventory and Production Management

Springer

Imperfect Inventory Systems

Ata Allah Taleizadeh

Imperfect Inventory Systems

Inventory and Production Management

Springer

Ata Allah Taleizadeh
School of Industrial Engineering
University of Tehran
Tehran, Iran

ISBN 978-3-030-56973-0 ISBN 978-3-030-56974-7 (eBook)
https://doi.org/10.1007/978-3-030-56974-7

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Contents

Chapter 1
Introduction

In today's manufacturing environment, managing inventories is one of the basic concerns of enterprises dealing with materials according to their activities, because material as the principal inventories of enterprises specially production ones composes the large portion of their assets. As a result, managing inventories influences directly financial, production, and marketing segments of enterprises so that efficient management of inventories leads to improving their profits. In addition, the effect of managing inventories on the selling prices of finished products is undeniable because more than half of production systems' revenues are spent to buy materials or production components. On the other hand, customers expect to receive their orders at a lower price apace. So, an efficient managing inventories and production planning are key managerial and operational tools to achieve the main goals, which are satisfying the customers' demand and becoming lower-cost producer, in order to increase market share.

Economic production quantity (EPQ) model is a well-known economic lot size model used in production enterprises that internally produce products. However, traditional EPQ model is utilized for perfect production process to determine the optimal production lot size so that overall production/inventory costs are minimized. In reality, a perfect production run rarely exists. Breakdown is an inevitable issue in production processes. Indeed, after a production period, a production process often shifts to out-control state owing to machine wear or corrosion which leads to generating defective items with loss cost. In order to reimburse these costs, some production strategies including reworking and repairing defective items, quality control, and maintenance planning to reduce the defective or scrape item costs are employed. So, the main prophecy of this book is to introduce all mentioned production strategies which can lessen unexpected imperfect item costs. The main focus of this book is to introduce mathematical models of imperfect inventory control systems in which at least one of imperfect items, scraped item, rework

A. A. Taleizadeh, *Imperfect Inventory Systems*,
https://doi.org/10.1007/978-3-030-56974-7_1

policy, quality control or maintenance planning may be used. In the following a brief introduction about each chapter is presented.

1.1 Imperfect Items

Since the introduction of the economic order quantity (EOQ) model by Harris (1913), frequent contributions have been made in the literature toward the development of alternative models that overcome the unrealistic assumptions embedded in the EOQ formulation. For example, the assumption related to the perfect quality items is technologically unattainable in most supply chain applications (Cheng 1991). In contrast, products can be categorized as “good quality,” “good quality after reworking,” “imperfect quality,” and “scrap” (Chan et al. 2003). In practice, the presence of defective items in raw material or finished products inventories may deeply affect supply chain coordination, and, consequently, the product flows among supply chain levels may become unreliable (Roy et al. 2011). In response to this concern, the enhancement of currently available production and inventory order quantity models, which accounts for imperfect items in their mathematical formulation, has become an operational priority in supply chain management (Khan et al. 2011). This enhancement may also include the knowledge transfer between supply chain entities in order to reduce the percentage of defective items. In the second chapter of this book, the main focus is on introducing several mathematical models of EOQ inventory systems with imperfect items considering different kinds of shortages under different assumptions.

1.2 Scrap

The economic order quantity (EOQ) model was first introduced in 1913. Seeking to minimize the total cost, the model generated a balance between holding and ordering costs and determined the optimal order size. Later, the EPQ model considered items produced by machines inside a manufacturing system with a limited production rate, rather than items purchased from outside the factory. Despite their age, both models are still widely used in major industries. Their conditions and assumptions, however, rarely pertain to current real-world environments. To make the models more applicable, different assumptions have been proposed in recent years, including random machine breakdowns, generation of imperfect and scrap items, and discrete shipment orders. The assumption of discrete shipments using multiple batches can make the EPQ model more applicable to real-world problems. The EPQ inventory models assume that all the items are manufactured with high quality and defective items are not produced. However, in fact, defective items appear in the most of manufacturing systems; in this sense, researchers have been developing EPQ inventory models for defective production systems. In these production systems, defective items are of two types: scrapped items and reworkable items. Usually non-conforming products are scrapped and are removed from the systems' inventories. This strategy is employed for production enterprises in which either imperfect items cannot be repaired or both repair/reworking cost is more than their selling revenues. In turn, enterprises prefer to reject imperfect items instead of performing reworking/repair procedure. In the third chapter of this book, the EPQ model with scraped items under different kinds of shortages and both continuous and discrete delivery are introduced.

1.3 Rework

Rework is one of the key drivers of production designs applied in imperfect production systems in which their production lines face defectives. It helps producers reproduce the non-conforming items, which are detected within/after inspecting process, and sell them as healthy ones. Although a reworking process makes an additive cost for production companies, it causes the producers to profit from buying the reworked items more than their reworking costs, so they prefer to rework the imperfect items in order to reduce their unexpected expenses. In the fourth chapter of this book, rework process in imperfect EPQ model under different assumptions is introduced. Indeed, several mathematical models of EPQ problem with defective and rework process are presented.

1.4 Multi-product Single Machine

The economic production quantity (EPQ) is a commonly used production model that has been studied extensively in the past few decades. One of the considered constraints in the EPQ inventory models is producing all items by a single machine. Since all of the products are manufactured on a single machine with a limited capacity, a unique cycle length for all items is considered. It is assumed there is a real constant production capacity limitation on the single machine on which all products are produced. If the rework is placed, both the production and rework processes are accomplished using the same resource, the same cost, and the same speed. The first economic production quantity inventory model for a single-product single-stage manufacturing system was proposed by Taft (1918). Perhaps Eilon (1985) and Rogers (1958) were the first researchers that studied the multi-products single manufacturing system. Eilon (1985) proposed a multi-product lot-sizing problem classification for a system producing several items in a multi-product single-machine manufacturing system. In the fifth chapter of this book, multi-product single-machine EPQ model with defective and scraped items and also rework process under different assumptions are presented.

1.5 Quality Considerations

Traditional economic order quantity (EOQ) models offer a mathematical approach to determine the optimal number of items a buyer should order to a supplier each time. One major implicit assumption of these models is that all the items are of perfect quality (Rezaei and Salimi 2012). However, presence of defective products in manufacturing processes is inevitable. There is no production process which can guarantee that all its products would be perfect and free from defect. Hence, there is a yield for any production process. Basic and classical inventory control models usually ignore this fact. They assume all output products are perfect and with equal quality; however, due to the limitation of quality control procedures, among other factors, items of imperfect quality are often present. So it has given researchers the opportunity to relax this assumption and apply a yield to investigate and study its impact on several variables of inventory models such as order quantity and cycle time. In the sixth chapter of this book, several inventory control models under quality considerations such as sampling, inspections, return, etc. with different assumptions of inventory systems are presented.

1.6 Maintenance

The role of the equipment condition in controlling quality and quantity is well-known (Ben-Daya and Duffuaa 1995). Equipment must be maintained in top operating conditions through adequate maintenance programs. Despite the strong link between maintenance production and quality, these main aspects of any manufacturing system are traditionally modeled as separate problems. In the last chapter of this book, maintenance and inventory systems are considered together, and several mathematical models are presented.

References

Ben-Daya, M., & Duffuaa, S. O. (1995). Maintenance and quality: The missing link. *Journal of Quality in Maintenance Engineering, 1*(1), 20–26.

Chan, W. M., Ibrahim, R. N., & Lochert, P. B. (2003). A new EPQ model: Integrating lower pricing, rework and reject situations. *Production Planning & Control, 14*(7), 588–595.

Cheng, T. C. E. (1991). EPQ with process capability and quality assurance considerations. *Journal of the Operational Research Society, 42*(8), 713–720.

Eilon, S. (1985). Multi-product batch production on a single machine—A problem revisited. *Omega, 13*, 453–468.

Harris, F. W. (1913). What quantity to make at once. In *The library of factory management. Operation and costs* (The factory management series) (Vol. 5, pp. 47–52). Chicago, IL: A.W. Shaw Co..

Khan, M., Jaber, M. Y., & Bonney, M. (2011). An economic order quantity (EOQ) for items with imperfect quality and inspection errors. *International Journal of Production Economics, 133*(1), 113–118.

Rezaei, J., & Salimi, N. (2012). Economic order quantity and purchasing price for items with imperfect quality when inspection shifts from buyer to supplier. *International Journal of Production Economics, 137*(1), 11–18.

Rogers, J. (1958). A computational approach to the economic lot scheduling problem. *Management Science, 4*, 264–291.

Roy, M. D., Sana, S. S., & Chaudhuri, K. (2011). An economic order quantity model of imperfect quality items with partial backlogging. *International Journal of Systems Science, 42*(8), 1409–1419.

Taft, E. W. (1918). The most economical production lot. *Iron Age, 101*(18), 1410–1412.

Chapter 2
Imperfect EOQ System

2.1 Introduction

Since the introduction of the economic order quantity (EOQ) model by Harris (1913), frequent contributions have been made in the literature toward the development of alternative models that overcome the unrealistic assumptions embedded in the EOQ formulation. For example, the assumption related to the perfect-quality items is technologically unattainable in most supply chain applications. In contrast, products can be categorized as "good quality," "good quality after reworking," "imperfect quality," and "scrap" (Chan et al. 2003; Pal et al. 2013). In practice, the presence of defective items in raw material or finished product inventories may deeply affect supply chain coordination, and, consequently, the product flows among supply chain levels may become unreliable (Roy et al. 2015). In response to this concern, the enhancement of currently available production and inventory order quantity models, which accounts for imperfect items in their mathematical formulation, has become an operational priority in supply chain management (Khan et al. 2011). This enhancement may also include the knowledge transfer between supply chain entities in order to reduce the percentage of defective items (Adel et al. 2016).

Also some related works can be found in Hasanpour et al. (2019), Keshavarz et al. (2019), Taleizadeh et al. (2015, 2016a, 2018a, b), Taleizadeh and Zamani-Dehkordi (2017a, b), Salameh and Jaber (2000), Maddah and Jaber (2008), and Papachristos and Konstantaras (2006).

The EOQ models with imperfect-quality items in three categories are categorized and their subcategories are shown in Fig. 2.1.

The common notations of imperfect EOQ models are shown in Table 2.1.

A. A. Taleizadeh, *Imperfect Inventory Systems*,
https://doi.org/10.1007/978-3-030-56974-7_2

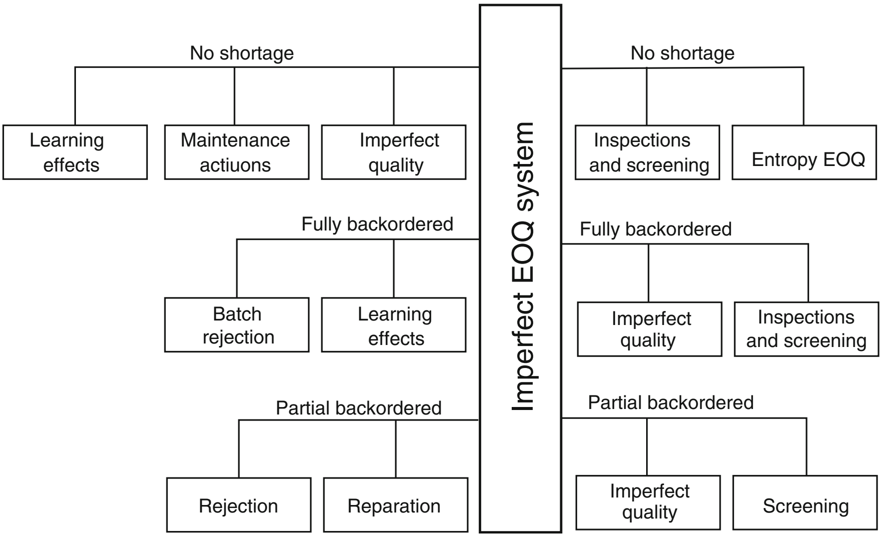

Fig. 2.1 Categories of EOQ model of imperfect-quality items

2.2 Literature Review

The academic literature related to inventory control for imperfect-quality items is multidisciplinary in nature and, for reviewing/presentation purposes in this chapter, is thematically organized around two main streams: (1) deterioration, perishability, and shelf lifetime constraints and (2) model formulations and related solution techniques that consider imperfect-quality items (Adel et al. 2016).

2.2.1 Deterioration, Perishability, and Lifetime Constraints

The terms "deterioration," "perishability," and "obsolescence" are used interchangeably in the literature and may often be perceived as ambiguous because they are linked to particular underlying assumptions regarding the physical state/fitness and behavior of items over time. Usually, deterioration refers to the process of decay, damage, or spoilage of a product, i.e., the product loses its value of characteristics and can no longer be sold/used for its original purpose (Wee 1993). In contrast, an item with a fixed lifetime perishes once it exceeds its maximum shelf lifetime and then must be discarded (Ferguson and Ketzenberg 2005). Obsolescence incurs a partial or a total loss of value of the on-hand inventory in such a way that the value for a product continuously decreases with its perceived utility (Song and Zipkin 1996; Also some related works can be found in works of Nobil, et al. (2019), Lashgary et al. (2016, 2018), Kalantary and Taleizadeh (2018), Diabat et al.

Table 2.1 Notations

P	Production rate (units per unit time)
D	Demand rate (units per unit time)
R	Repair rate (units per unit time)
s	Selling price for good-quality items ($/unit)
v	Selling price for imperfect or salvage value per items ($/unit)
C	Production/purchasing cost ($/unit)
C_R	Rework cost per unit ($/unit)
C_J	Reject cost per unit (including transportation, handling, and damage cost) ($/unit)
C_b	Backordering cost ($/unit/unit time)
C_T	Transportation cost per unit ($/unit)
C_d	Disposal cost per unit ($/unit)
g	Goodwill cost per unit ($/unit)
$\widehat{\pi}$	Lost sale cost per unit ($/unit)
K	Fixed setup/ordering cost ($/lot)
K_S	Fixed transportation or shipment cost ($/lot)
h	Holding cost per unit per unit time ($/unit/unit time)
h_1	The holding cost for defective items per unit per unit time ($/unit/unit time)
h_R	The holding cost for reworked items per unit per unit time ($/unit/unit time)
γ	Fraction of imperfect items (percent)
C_I	The unit screening or inspection cost ($/unit)
x	Inspection rate (units per unit time)
p	Imperfect rate (units per unit time)
$E[p]$	Expected imperfect rate
$f(p)$	Probability density function of p
T	Ordering cycle duration (time)
t	Screening time (time)
$f(\gamma)$	Probability density function of imperfect products (γ)
y	Production/ordering quantity (unit)
B	Backordered level (unit)
β	Partial backordering rate (%) $0 < \beta \leq 1$
$E[.]$	Expected value of a random variable

(2017), Mohammadi et al. (2015), Tat et al. (2015), Hasanpour et al. (2019), Taleizadeh (2014), Taleizadeh and Rasouli-Baghban, (2015, 2018), Taleizadeh et al. (2013a, b, 2015, 2016, 2019), Taleizadeh and Nemattolahi (2014) and Tavakkoli and Taleizadeh (2017), Bakker et al. (2012).

2.2.2 Imperfect-Quality Items

The classical EOQ has been a widely accepted model for inventory control purposes due to its simple and intuitively appealing mathematical formulation. However, it is true to say that the operation of the model is based on a number of explicitly or implicitly made unrealistic mathematical assumptions that are never actually met in practice (Jaber et al. 2004). Salameh and Jaber (2000) developed a mathematical model that permits some of the items to drop below the quality requirements, i.e., a random proportion of defective items are assumed for each lot size shipment, with a known probability distribution. The researchers assumed that each lot is subject to a 100% screening, where defective items are kept in the same warehouse until the end of the screening process and then can be sold at a price lower than that of perfect-quality items. Huang (2004) developed a model to determine an optimal integrated vendor–buyer inventory policy for flawed items in a just-in-time (JIT) manufacturing environment. Maddah and Jaber (2008) developed a new model that rectifies a flaw in the one presented by Salameh and Jaber (2000) using renewal theory. Jaber et al. (2008) extended it by assuming that the percentage defective per lot reduces according to a learning curve.

Jaggi and Mittal (2011) investigated the effect of deterioration on a retailer's EOQ when the items are of imperfect quality. In their research, defective items were assumed to be kept in the same warehouse until the end of the screening process. Jaggi et al. (2011) and Sana (2012) presented inventory models, which account for imperfect-quality items under the condition of permissible delay in payments. Moussawi-Haidar et al. (2014) extended the work of Jaggi and Mittal (2011) to allow for shortages.

In a real manufacturing environment, the defective items are not usually stored in the same warehouse where the good items are stored. As a result, the holding cost must be different for the good items and the defective ones (e.g., Paknejad et al. 2005). With this consideration in mind, Wahab and Jaber (2010) presented the case where different holding costs for the good and defective items are assumed. They showed that if the system is subject to learning, then the lot size with the same assumed holding costs for the good and defective items is less than the one with differing holding costs. When there is no learning in the system, the lot size with differing holding costs increases with the percentage of defective items. For more details about the extensions of a modified EOQ model for imperfect-quality items, see Khan et al. (2011).

Here are some main models in literature with their mathematical model, solution procedure, and numerical examples. In the next sections, these models starting from the basic to complicated ones are presented. First, EOQ models with imperfect quality items are studied considering no shortage, back-ordering shortage, and partial back-ordering.

2.3 EOQ Model with No Shortage

2.3.1 Imperfect Quality

In this section, two imperfect EOQ models developed by Salameh and Jaber (2000) and Maddah and Jaber (2008) are presented. Consider the EOQ model with a

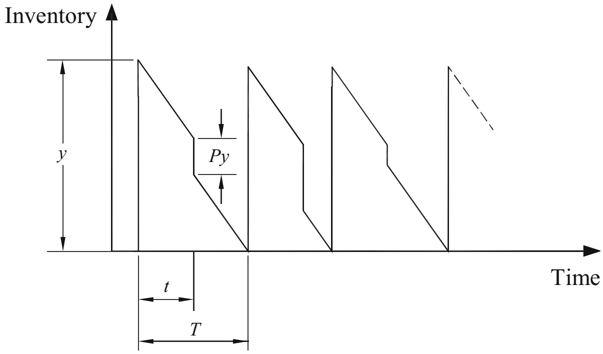

Fig. 2.2 Inventory level (Salameh and Jaber 2000; Maddah and Jaber 2008)

demand rate of D units per unit time. An order of size y is placed every time the inventory level reaches zero and is assumed to be delivered instantaneously. The fixed ordering cost is K, the fixed shipping of imperfect-quality items is K_S, the unit purchasing cost is C, and the inventory holding cost is h per unit per unit time. Each order contains a fraction P of defective items, a random variable with support in [0, 1]. Each order is subjected to a 100% inspection process at a rate of x units per unit time, $x{\cdot}D$. The screening cost is d per unit. Upon completion of the screening process, items of imperfect quality are sold as a single batch at a reduced price of v per unit. The price of a perfect-quality item is s per unit, $s < v$.

The behavior of the inventory level in an ordering cycle is shown in Fig. 2.2, where T is the ordering cycle duration ($T = (1 - p)y/D$, and $t = y/x$). Salameh and Jaber (2000) assumed that $(1 - p)\, y \geq D{\cdot}t$, or, equivalently, $p \leq 1 - D/x$, in order to avoid shortages. Under the above assumptions, the expected profit is presented as:

$$\mathrm{TP}(y) = \overbrace{sy(1-p) + vyp}^{\text{Revenue}} - \overbrace{K}^{\text{Fixed cost}} - \overbrace{Cy}^{\text{Purchasing cost}} - \overbrace{C_I y}^{\text{Inspection cost}} - \overbrace{h\left(\frac{[y(1-p)]^2}{2D} + \frac{py^2}{x}\right)}^{\text{Holding cost}} \tag{2.1}$$

Then the expected profit per unit time is derived as:

$$E[\mathrm{TPU}(y)] = E\left[\frac{\mathrm{TP}(y)}{T}\right] \tag{2.2}$$

After some simplifications,

$$E[\text{TPU}(y)] = D\left(s - v + h\frac{y}{x}\right) + D\left(v - \frac{hy}{x} - C - C_\text{I} - \frac{K}{y}\right) \times E\left[\frac{1}{1-p}\right] - \frac{hy(1 - E[p])}{2} \tag{2.3}$$

And the optimal order quantity is derived as:

$$y^{SJ} = \sqrt{\frac{2KDE[1/(1-p)]}{h[1 - E[p] - 2D(1 - E[1/(1-p)])/x]}} \tag{2.4}$$

Then, Maddah and Jaber (2008) corrected Eq. (2.2) as:

$$E[\text{TPU}(y)] = \frac{E[\text{TP}(y)]}{E[T]} \tag{2.5}$$

and derived a new expected profit function as:

$$E[\text{TP}(y)] = sy(1 - E[p]) + vyE[p] - K - Cy - C_\text{I}y - h\left(\frac{y^2E\left[(1-p)^2\right]}{2D} + \frac{E[p]y^2}{x}\right)$$

Since $E[T] = (1 - E[p])y/D$, then Eq. (2.5) is rearranged as:

$$E[\text{TPU}(y)] = \frac{[s(1-E[p]) + vE[p] - C - C_\text{I}]D - KD/y - hy(E\left[(1-p)^2\right]/2 + E[p]D/x)}{1 - E[p]} \tag{2.6}$$

After proofing the concavity of Eq. (2.6) with respect to y, the optimum order size is derived as:

$$y^* = \sqrt{\frac{2KD}{h\left(E\left[(1-p)^2\right] + 2E[p]D/x\right)}} \tag{2.7}$$

The expected profit in Eq. (2.6) has several terms independent of y. In subsequent analysis, these are dropped, and the objective function is redefined in terms of minimizing the expected "relevant" cost per unit time as (Maddah and Jaber 2008):

$$\mathrm{EC}(y) = \frac{1}{1 - E[p]}\left[KD/y + hy\left(E\left[(1-p)^2\right]/2 + E[p]D/x\right)\right] \tag{2.8}$$

Maddah and Jaber (2008) showed that for a large inspection rate, the optimum order size in Eq. (2.7) converges to:

$$y^* = \sqrt{\frac{2KD}{h\left(E\left[(1-p)^2\right]\right)}} \tag{2.9}$$

In real cases, it is not optimal to ship imperfect-quality items as a single batch in each ordering cycle (Maddah and Jaber 2008). So Maddah and Jaber assumed that shipping any number of imperfect-quality batches has a fixed cost of K_S and developed their previous model under multiple batches. Now the decision variables are order size (y) and ordering cycle number (n), and they derived the expected cost per unit time of perfect-quality items similar to Eq. (2.8) as below:

$$E[\mathrm{CP}(y)] = \left[KD/y + hy\left(E\left[(1-p)^2\right]/2\right)\right]/(1 - E[p])$$

Figure 2.3 corresponds to the case when $n = 3$, where the imperfect-quality inventory is held for two ordering cycles and then shipped upon completing the screening of the last order. According to Maddah and Jaber (2008), let T_i be the duration of ordering period i of a shipping cycle, $i = 1, \ldots, n$. Note that $T_i = (1 - P_i)$ y/D, where P_i is the fraction of imperfect-quality items in order i of a shipping cycle. Then, the expected holding cost per shipping cycle is:

$$\mathrm{ECI_h}(y,n) = hE\left[\overbrace{\sum_{i=1}^{n-1} P_i y(1-P_i)y/D}^{\substack{\text{Imperfect quality inventory}\\ \text{cost from an order}\\ \text{carried over the ordering}\\ \text{period of the order itself}}} + \underbrace{\sum_{i=1}^{n-2} P_i y \sum_{j=i+1}^{n-1} (1-P_i)y/D}_{\substack{\text{Imperfect inventory cost from an order}\\ \text{carried through subsequent ordering}\\ \text{periods during a shipping cycle}\\ \text{excluding thenth ordering period}}} + \overbrace{\sum_{i=1}^{n} \frac{P_i y^2}{x}}^{\substack{\text{Imperfect inventory cost accumulated}\\ \text{in thenth period, which is carried}\\ \text{for a duration of } y/x\\ \text{before being shipped}}}\right] \tag{2.10}$$

Assuming that $P_1, \ldots, P_n$ are independent and identically distributed, the expression for $\mathrm{ECI_h}(y, n)$ in Eq. (2.10) after some simplifications changes to:

$$\mathrm{ECI_h}(y,n) = \frac{hy^2}{D}\left[\frac{n(n-1)}{2}\times E[p](1-E[p]) + nE[p]D\frac{y}{x} - (n-1)\mathrm{var}[p]\right]$$

Knowing $E\left[\sum_{i=1}^{n}(1-P_i)y/D\right] = n(1-E[p])y/D$, the expected imperfect-quality item cost per unit time is:

$$E[\mathrm{CI}(y,n)] = \frac{1}{1-E[p]}\left\{\frac{K_\mathrm{S}}{n}\frac{D}{y} + hy\left[\frac{(n-1)}{2}E[p](1-E[p]) + E[p]\frac{D}{x} - \frac{n-1}{n}\mathrm{var}[p]\right]\right\}$$

And total cost will be:

$$E[\mathrm{TC}(y,n)] = \frac{1}{1-E[p]}\left\{\left(K+\frac{K_\mathrm{S}}{n}\right)\frac{D}{y} + \frac{hy}{2}\left[E\left[(1-p)^2\right] - \frac{2(n-1)}{n}\mathrm{var}[p] + (n-1)E[p](1-E[p]) + 2E[p]\frac{D}{x}\right]\right\} \tag{2.11}$$

Because of convexity of Eq. (2.11), one can easily derive that:

$$y^*(n) = \sqrt{\frac{2(K+(K_\mathrm{S}/n))D}{\underbrace{h\left[E\left[(1-p)^2\right] - (2(n-1)/n)\mathrm{var}[p] + (n-1)E[p](1-E[p]) + 2E[p](D/x)\right]}_{\gamma(n)}}} \tag{2.12}$$

Maddah and Jaber (2008) showed that optimal values of n can be found by optimizing the expected total cost presented in Eq. (2.13) which is presented in Eq. (2.14):

$$\mathrm{ECT_1}(n) = \mathrm{ECT}(n, y^*(n)) = \sqrt{2\kappa(n)\gamma(n)D} \tag{2.13}$$

$$\widetilde{n} = \sqrt{\frac{K_1\left[E\left[(1-p)^2\right] - 2(1-K/K_\mathrm{S})\mathrm{var}[p] - E[p](1-E[p]) + 2E[p](D/x)\right]}{KE[p](1-E[p])}} \tag{2.14}$$

Then, the optimal value of n is one of the two integers which is closest to $\widetilde{n}$, whichever leads to lower value of $\mathrm{ECT_1}(n)$. That is, $n^* = \mathrm{argmin}(\mathrm{ECT_1}(n))$ where $[x]$ is the largest integer $\leq x$ and $[x]$ is the smallest integer $\geq x$. Finally, the optimal order quantity is found from Eq. (2.12) as $y* = y*(n*)$ (Maddah and Jaber 2008).

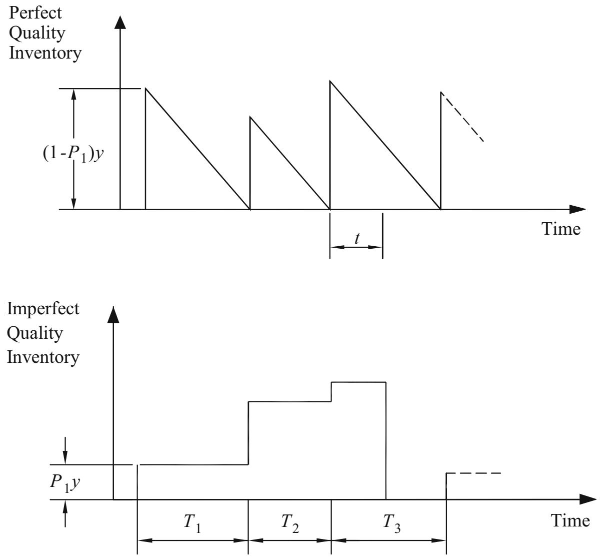

Fig. 2.3 Perfect and imperfect inventory levels when shipments are consolidated, $n = 3$ (Maddah and Jaber 2008)

Example 2.1 Maddah and Jaber (2008) developed numerical results similar to those in Salameh and Jaber (2000). This illustrates the application of their model and allows comparing their results with those of Salameh and Jaber. Consider a situation with the following parameters: demand rate, $D = 50{,}000$ units/year; ordering cost, $K = \$100$/cycle; holding cost, $h = \$5$/unit/year; screening rate, $x = 175{,}200$ units/year; screening cost, $C_I = \$0.5$/unit; purchasing cost, $C = \$25$/unit; selling price of good-quality items, $s = \$50$/unit; selling price of imperfect-quality items, $v = \$20$/unit; and the fraction of imperfect-quality item, p, uniformly distributed on (a, b), $0 < a < b < 1$, i.e., $P \sim (a, b)$. With $p \sim U(a, b)$, $E[p] = (a + b)/2$, $\mathrm{Var}[p] = (b - a)/12$ and:

$$E\left[(1-p)^2\right] = \frac{1}{b-a}\int_a^b (1-p)^2 dp = \frac{a^2 + ab + b^2}{3} + 1 - a - b \qquad (2.15)$$

Assuming $a = 0$ and $b = 0.04$, then the optimal order quantity using Eq. (2.7) becomes $y^* = 1434$ units, and the related cost from Eq. (2.6) is $E[\mathrm{TPU}(y^*)] = \$1{,}212{,}274$.

Assuming shipping of imperfect-quality items has a fixed cost of $K_S = \$50$ with same values for other parameters used in the previous example, in the following, the continuous value of n that minimizes $\mathrm{ECT}_1(n)$ is $\tilde{n} = 4.93$. So, n^* is either 4 or 5 where $\mathrm{ECT}_1(4) = 7614 > \mathrm{ECT}_1(5) = 7600$. So, $n^* = 5$. The optimal order quantity is then given from Eq. (2.11) as $y^*(5) = 1447$.

2.3.2 Maintenance Actions

The objective of the analysis in this section is to determine the optimal lot size $y*$ such that the expected total cost is minimized when maintenance and reworking actions are taken into account. For describing this section, some new notations are used as presented in Table 2.2 (Porteus 1986).

Before presenting the model, first we should take notice of the remark presented by Hou et al. (2015) in Eq. (2.16). He derived the expected number of unhealthy item as below:

$$E(N) = \theta\left(y - \sum_{j=1}^{y} q^{-j}\right) \tag{2.16}$$

and

$$\Pr\{X = j\} = \begin{cases} q^{-j}q, 0 \le j \le y \\ q^{-y}, j = y \end{cases} \tag{2.17}$$

Then, they showed that:

$$E(X) = q\sum_{j=1}^{y-1} jq^{-j} + yq^{-y} = \sum_{j=1}^{y} q^{-j} \tag{2.18}$$

Finally, the number of defective items in y is $N = \theta(y - X)$ and the $E(N)$ is what presented in Eq. (2.16). Now based on Eq. (2.18), the expected cyclic cost of rework process will be:

$$C_R E(N) = C_R\theta\left(y - \sum_{j=1}^{y} q^{-j}\right) \tag{2.19}$$

Since the related cost to maintenance should be considered when the manufacturing process is out-of-control at the end of a production uptime for a lot of size y, the expected cyclic-related cost is:

Table 2.2 Notations of a given problem

Q	The probability that the system from in-control state shifts to out-of-control state
$\overline{q}$	The probability that the system stays in-control state during the production of an item and $\overline{q} = 1 - q$
θ	The percentage of defective items when the process is in the out-of-control state
X	Random variable representing number of items produced in the in-control state
C_m	Maintenance cost per unit (\$/unit)

$$C_{\mathrm{m}}(1-q^{-y}) \tag{2.20}$$

So the cyclic total cost is (Hou et al. 2015):

$$\mathrm{TC}(y) = \overbrace{K}^{\text{Fixed cost}} + \underbrace{\frac{hy^2}{2D}}_{\text{Holding cost}} + \overbrace{C_{\mathrm{m}}(1-q^{-y})}^{\text{Maintenance cost}} + \underbrace{C_{\mathrm{R}}\theta\left(y - \sum_{j=1}^{y} q^{-j}\right)}_{\text{Rework cost}} \tag{2.21}$$

And the expected cost per unit of time becomes:

$$\begin{aligned} f(y) &= \mathrm{TC}(y)/T \\ &= \frac{DK}{y} + \frac{h}{2}y + C_{\mathrm{R}}D\theta + \frac{D}{y}\left[C_{\mathrm{m}}(1-q^{-y}) - C_{\mathrm{R}}\theta\sum_{j=1}^{y} q^{-j}\right] \end{aligned} \tag{2.22}$$

It should be noticed that for $q = 0$, the production system is always in the in-control state and the produced items are healthy, and Eq. (2.22) reverses to the traditional EOQ model with healthy item. But $C_{\mathrm{m}} = 0$ means all produced items are defective, and Eq. (2.22) will reduce to the approximated model (using Taylor series expansion) in Porteus (1986) as presented in Eq. (2.23):

$$f_{\mathrm{p}}(y) = \frac{DK}{y} + \frac{y}{2}(h + C_{\mathrm{R}}Dq) \tag{2.23}$$

and derive an approximately optimal lot size as follows (Hou et al. 2015):

$$y_{\mathrm{p}}^{*} = \sqrt{\frac{2DK}{h + C_{\mathrm{R}}Dq}} \tag{2.24}$$

Since Eq. (2.23) was not a good approximation, Hou et al. (2015) presented a comprehensive method to derive the optimal values. They provided the bounds for searching the optimal lot size $y*$ that minimizes $f(y)$ of Eq. (2.22) as $\beta = C_{\mathrm{m}} - \frac{C_{\mathrm{R}}\theta\bar{q}}{q}$ and using necessary condition for optimal points ($f'(y^*) = 0$) derived optimal values. They prove that $y*$ exists and is unique when q equals 0 or 1 such that $f'(y^*) = 0$ satisfies. But for $0 < q < 1$, let:

$$g(y) = y^2 f'(y^*) = -DK + \frac{h}{2}y^2 - D\beta(1 - q^{-y} + yq^{-y}\ln q^{-}) \tag{2.25}$$

since $g(y)$ is a continuous function with $\lim_{y\to 0^+} g(y) = -D$ and $\lim_{y\to\infty} g(y) = \infty > 0$. Furthermore, the first derivative of $g(y)$ is given by Hou et al. (2015):

$$g'(y) = y\left[h - D\beta(\ln q^{-})^2 q^{-y}\right] \tag{2.26}$$

After some algebra, Hou et al. (2015) proposed the optimal lot size $y*$ when $0 < q < 1$:

$$y_1 = \sqrt{\frac{2D(K + C_\mathrm{m})}{h}} \quad y_2 = \sqrt{\frac{2DK}{h}} \tag{2.27}$$

And proved that:

$$\begin{aligned} &\text{if} \quad \beta \le 0 \rightarrow 0 < y \le y_2^* \le y_1^* \\ &\text{if} \quad \beta > 0 \rightarrow 0 < y_2^* < y < y_1^* \end{aligned} \tag{2.28}$$

Using Eqs. (2.27) and (2.28), the following algorithm is proposed to find the optimal values.

Algorithm 2.1 *Step 1*: *Let* $\varepsilon > 0$, *and compute* β, y_2, *and* y_1.

Step 2: *If* $\beta \le 0$, *set* $y_L = 0$, $y_U = y_2$; *otherwise, set* $y_L = y_2$, $y_U = y_1$.

Step 3: *Set* $y_{opt} = \frac{y_L + y_U}{2}$.

Step 4: *If* $|g(y_{opt})| < \varepsilon$, *go to Step 6; otherwise, go to Step 5.*

Step 5: *If* $|g(y_{opt})| < 0$, *set* $y_L = y_{opt}$; *however, if* $|g(y_{opt})| > 0$, $y_U = y_{opt}$. *Then, go to Step 3.*

Step 6: *Set* $y* = y_{opt}$ *and compute* $f(y*)$.

Example 2.2 Consider $K = \$600$/cycle, $h = \$8$/unit/year, $D = 1000$ units/year, and $C_\mathrm{R} = \$5$/unit, $C_\mathrm{m} = \$200$/cycle, $\theta = 0.75$, and $\theta = 0.1$. Then, it can be verified that $\beta = 166.25$. Using Algorithm 2.1, $y* = 437.68$ units and $f(y*) = \$7251.43$ (Hou et al. 2015).

2.3.3 Screening Process

In this section, now consider a general EOQ model for items with imperfect quality under varying demand, defective items, screening process, and deterioration rates for an infinite planning horizon presented by Alamri et al. (2016). Assume that each lot is subject to a 100% screening where items that are not conforming to certain quality standards are stored in a different warehouse. Therefore, different holding costs for the good and defective items are considered. Items deteriorate while they are in storage, with demand, screening, and deterioration rates being arbitrary functions of time. The percentage of defective items per lot reduces according to a learning curve. After a 100% screening, imperfect-quality items may be sold at a discounted price as a single batch at the end of the screening process or incur a disposal penalty charge. Moreover, a general step-by-step solution procedure is provided for continuous intra-cycle periodic review applications.

Table 2.3 Notations of a given problem

$D(t)$	Demand rate (units per unit time)
$x(t)$	Screening rate (units per unit time)
$\delta(t)$	Deterioration rate (units per unit time)
p_j	The percentage defective per lot reduces according to a learning curve
j	Cycle index ($j = 1, 2, \ldots$)
Q_j	Lot of size delivered at the beginning of each cycle j (unit)

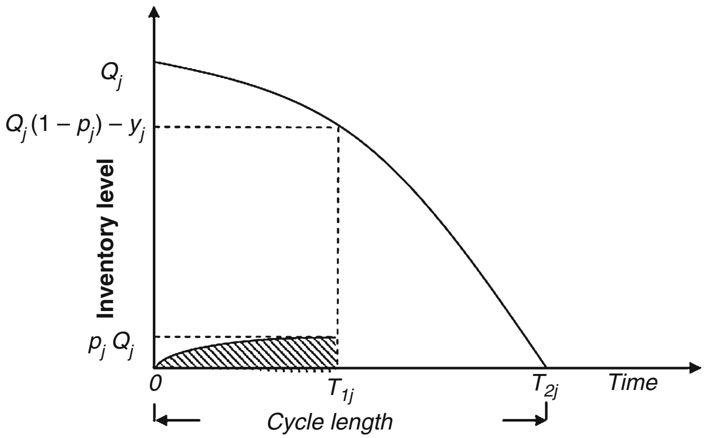

Fig. 2.4 Inventory variation of an economic order quantity (EOQ) model for one cycle (Alamri et al. 2016)

Some related notation for this problem is presented in Table 2.3.

Alamri et al. (2016) assumed that a single item held in stock lead time is zero and no restrictions exist. Moreover, any order arrives before the end of that same cycle.

In order to avoid the shortage, Alamri et al. (2016) assumed $(1 - p_j)x(t) \geq D(t)$, $\forall t \geq 0$. *Lot size* covers both deterioration and demand during both the first phase (screening) and the second phase (non-screening). Each lot is subjected to a 100% screening process that starts at the beginning of the cycle and ceases by time T_{1j}, by which point in time Q_j units have been screened and y_j units have been depleted, which is the summation of demand and deterioration. During this phase, items not conforming to certain quality standards are stored in a different warehouse. The variation in the inventory level during the first and second phase (please refer to Fig. 2.4) and the variation in the inventory level for the defective items (Alamri et al. 2016) are presented in Eq. (2.29):

$$\frac{dI_{gj}(t)}{dt} = -D(t) - p_j x(t) - \delta(t) I_{gj}(t), \quad 0 \le t \le T_{1j} \tag{2.29}$$

Using boundary condition $I_{gj}(0) = Q_j$:

$$Q_j = \int_0^{T_{1j}} x(u)du, \tag{2.30}$$

$$\frac{dI_{gj}(t)}{dt} = -D(t) - \delta(t) I_{gj}(t), \tag{2.31}$$

And with the boundary condition $I_{gj}(T_{2j}) = 0$:

$$\frac{dI_{dj}(t)}{dt} = p_j x(t), \quad 0 \le t \le T_{1j} \tag{2.32}$$

knowing $I_{dj}(0) = 0$.

After some complicated algebra, the solutions of the above differential equations are (Alamri et al. 2016):

$$I_{gj}(t) = \mathrm{e}^{-(g(t)-g(0))} \int_0^{T_{1j}} x(u)du - \mathrm{e}^{-g(t)} \int_0^t \left[D(u) + p_j x(u)\right] \mathrm{e}^{g(u)} du, \quad 0 \le t \le T_{1j} \tag{2.33}$$

$$I_{gj}(t) = \mathrm{e}^{-g(t)} \int_t^{T_{2j}} D(u) \mathrm{e}^{g(u)} du, \quad 0 \le t \le T_{1j} \tag{2.34}$$

$$I_{gj}(t) = \int_0^t p_j x(u)du, \quad 0 \le t \le T_{1j} \tag{2.35}$$

$$g(t) = \imath\zeta\delta(t)dt \tag{2.36}$$

The per cycle cost components for the given inventory system are as follows. The total purchasing cost during the cycle $= C\int_0^{T_{1j}} x(u)du$. Note that this cost includes the defective and deterioration costs. Holding cost = $h[I_{gj}(0, T_{1j}) + I_{gj}(T_{1j}, T_{2j})] + h_1 I_{dj}(0, T_{1j})$. Thus, the total cost per unit time of the underlying inventory system during the cycle $[0, T_{2j}]$, as a function of T_{1j} and T_{2j}, say $Z(T_{1j}, T_{2j})$ is given by:

$$G(t) = \zeta \mathrm{e}^{g(t)} dt \tag{2.37}$$

Our objective is to find T_{1j} and T_{2j} that minimize $Z(T_{1j}, T_{2j})$. However, the variables T_{1j} and T_{2j} are related to each other as follows:

$$0 < T_{1j} < T_{2j} \tag{2.38}$$

$$e^{g(0)} \int_0^{T_{1j}} x(u)du = \int_0^{T_{2j}} D(u)e^{g(u)}du + \int_0^{T_{1j}} p_j x(u)e^{g(u)}du \tag{2.39}$$

Thus, their goal is to solve the following optimization problem, which they shall call problem (m).

(m) = {minimize $Z(T_{1j}, T_{2j})$ given by Eq. (2.37) subject to Eq. (2.39) and $h_j = 0$:

$$h^j = e^{g(0)} \int_0^{T_{1j}} x(u)du - \int_0^{T_{1j}} p_j x(u)e^{g(u)}du - \int_0^{T_{2j}} D(u)e^{g(u)}du \tag{2.40}$$

It can be noted from Eq. (2.40) that $T_{1j} = 0 \Rightarrow T_{2j} = 0$ and $T_{1j} > 0 \Rightarrow T_{1j} < T_{2j}$. Thus Eq. (2.40) implies constraint (Eq. 2.38). Consequently, if they temporarily ignore the monotony constraint (Eq. 2.38) and call the resulting problem as (m_1), then it does satisfy any solution of (m_1). Hence, (m) and (m_1) are equivalent. Moreover, $T_{1j} > 0 \Rightarrow$ RHS of Eq. (2.33) > 0, i.e., Eq. (2.39) guarantees that the number of good items is at least equal to the demand during the first phase.

First, Alamri et al. (2016) noted from Eq. (2.30) that T_{1j} can be determined as a function of Q_j, say:

$$T_{1j} = f_{1j}(Q_j) \tag{2.41}$$

Taking also into account Eq. (2.40), they found that T_{2j} can be determined as a function of T_{1j}, and thus of Q_j, say:

$$T_{2j} = f_{2j}(Q_j) \tag{2.42}$$

Thus, if they substitute Eqs. (2.40)–(2.42) in Eq. (2.36), then problem (m) will be converted to the following unconstrained problem with the variable Q_j (which they shall call problem (m_2)):

$$W(Q_j) = \frac{1}{f_{2j}} \Bigg\{ (C + C_I) \int_0^{f_{1j}} x(u)du + h \left[-G(0)e^{g(0)} \int_0^{f_{1j}} x(u)du \right. \\ \left. + \int_0^{f_{1j}} p_j x(u)G(u)e^{g(u)}du + \int_0^{f_{2j}} D(u)G(u)e^{g(u)}du \right] + h_1 \left[\int_0^{f_{1j}} [f_{1j} - u] p_j x(u)du \right] + K \Bigg\} \tag{2.43}$$

Now, the necessary condition for having a minimum for problem (m_2) is:

$$\frac{dW}{dQ_j} = 0 \tag{2.44}$$

Letting $W = \frac{w}{f_{2j}}$, then:

$$\frac{dW}{dQ_j} = \frac{w'_{Q_j} f_{2j} - f'_{2j,Q_j} w}{f_{2j}^2} \tag{2.45}$$

where w'_{Q_j} and f'_{2j,Q_j} are the derivatives of w and f_{2j} w.r.t. Q_j, respectively. Hence, Eq. (2.45) is equivalent to (Alamri et al. 2016):

$$w'_{Q_j} f_{2j} = f'_{2j,Q_j} w \tag{2.46}$$

Also, taking the first derivative of both sides of Eq. (2.40) w.r.t. Q_j, one obtains:

$$e^{g(0)} - p_j e^{g(f_{1j})} = f'_{2j,Q_j} D(f_{2j}) e^{g(f_{2j})} \tag{2.47}$$

From which and Eqs. (2.38)–(2.40) it can be obtained:

$$w'_{Q_j} = (C + C_\mathrm{I}) + h\left[\left(G(f_{2j}) - G(0)\right)e^{g(0)} + \left(G(f_{1j})\right.\right.$$
$$\left.\left.\times - G(f_{2j})\right)p_j e^{g(f_{1j})}\right] + \frac{h_1}{x(f_{1j})}\int_0^{f_{1j}} p_j x(u)du. \tag{2.48}$$

$$W = \frac{w}{f_{2j}} = \frac{w'_{Q_j}}{f'_{2j,Q_j}} \tag{2.49}$$

where W is given by Eq. (2.40) and w'_{Q_j} is given by Eq. (2.48). Equation (2.49) can be used to determine the optimal value of Q_j and its corresponding total minimum cost and then the optimal values of T_{1j} and T_{2j} (Alamri et al. 2016).

Example 2.3 Alamri et al. (2016) presented an example to illustrate the efficiency of their mathematical model and solution procedures. They considered $x(t) = at + b$, $D(t) = at + r$, $p_j = \frac{\tau}{C_\mathrm{b} + e^{\gamma j}}$, and $\delta(t) = \frac{l}{z - \beta t}$where $b, d, l, \tau, C_\mathrm{b}, z > 0$; $a, r, \gamma, \beta, t \geq 0$; and $\beta t < z$.

Alamri et al. (2016) adopted the values considered in the study by Wahab and Jaber (2010), as presented in Table 2.4.

The optimal values of Q_j*, T_{1j}*, T_{2j}*, and ω_j*, the corresponding total minimum cost for ten successive cycles, are obtained, and the results are shown in Table 2.5.

Table 2.4 Input parameters (Alamri et al. 2016)

Parameter	Value	Parameter	Value
C	100 ($/unit)	α	500 (unit/year)
C_I	0.5 ($/unit)	r	50,000 (unit/year)
h	20 ($/unit/year)	l	1 (unit/year)
h_1	5 ($/unit/year)	z	20 (unit/year)
K	3000 ($/cycle)	β	25 (unit/year)
a	1000 (unit/year)	τ	70.067 (unit/year)
b	100,200 (unit/year)	C_b	819.76 (unit/year)
γ	0.7932 (unit/year)		

Table 2.5 Optimal results for varying demand, screening, and deterioration rates with p_j (Alamri et al. 2016)

j	p_j	T_{1j}	T_{2j}	Q_j*	$p_j{\cdot}Q_j$*	ω_j*	W_j*	w_j*
1	0.08524	0.035424	0.06482	3550	303	5.4	5,585,464	362,030
2	0.08497	0.035419	0.06483	3550	302	5.4	5,583,830	361,980
3	0.08436	0.035407	0.06485	3548	299	5.4	5,580,142	361,850
4	0.08305	0.035380	0.06489	3546	294	5.4	5,572,240	361,580
5	0.08030	0.035324	0.06498	3540	284	5.4	5,555,724	361,020
6	0.07482	0.035212	0.06516	3529	264	5.5	5,523,107	359,900
7	0.06502	0.035013	0.06548	3509	228	5.5	5,465,734	357,890
8	0.05042	0.034715	0.06594	3479	175	5.6	5,382,467	354,900
9	0.03369	0.034376	0.06644	3445	116	5.7	5,290,159	351,490
10	0.01944	0.034088	0.06686	3416	66	5.8	5,214,030	348,600

2.3.4 Learning Effects

2.3.4.1 Different Holding Costs

Salameh and Jaber (2000) developed a model to determine the economic lot size by maximizing the expected total profit per unit time. Each delivered lot has defective items with a known probability function and is screened completely. Then the defective items are sold as a single batch at a discounted price at the end of the screening period (Wahab and Jaber 2010).

In Salameh and Jaber's model, it is observed that they use the same holding cost for both good items and defective items. However, in the real manufacturing environment, the good items and the defective items are treated in a different way. So, the holding cost, $h = iC$, must be different for the good items and the defective items (e.g., Paknejad et al. 2005). With this consideration, they assigned holding costs h and h_1 (where $h > h_1$) for a unit of good item per period and a unit of defective item per period, respectively. In Fig. 2.5, inventory of defective items is depicted by the shaded area. In this section, the work of Wahab and Jaber (2010) based on Salameh and Jaber (2000), Maddah and Jaber (2008), and Jaber et al.