Applied and Numerical Harmonic Analysis

Paolo Boggiatto · Tommaso Bruno
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Landscapes of Time-Frequency Analysis

ATFA 2019

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Dedicated to our Families,
their continuous support is our strength

ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time-frequency and*
Numerical partial differential equations	*Time-scale analysis*
	Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptivemodeling inherent in time-frequency-scalemethods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

University of Maryland
College Park

John J. Benedetto
Series Editor

Preface

The second international conference entitled "*Aspects of Time-Frequency Analysis* (ATFA19)" took place in the period of 25–27 June 2019, at DISMA, Politecnico di Torino, see http://www.atfa19.polito.it/. The local organizing committee consisted of people from both university institutions of the city: Tommaso Bruno, Fabio Nicola, Anita Tabacco, and Maria Vallarino (Politecnico di Torino) and Paolo Boggiatto, Elena Cordero, and Alessandro Oliaro (Università di Torino).

The present volume collects ten contributions, mostly from invited speakers at the conference. These articles cover a good selection of topics of current interest in the field. They appear in alphabetic order of the first author, but let us review them following the topics they address in this short summary.

Let me start with those contributions that connect the area of time-frequency analysis with real-world applications. The first article to be mentioned here is the contribution by Leon Cohen, entitled "*Time-Frequency Analysis: What We Know and What We Don't*". It is a great opportunity to hear from the author of a well-known book in the area [2] about his current view on the field. In fact, there is a surprising little overlap of the list of references in his book, as well as in the book of B. Boashash [1] with the same title, with the standard references in the mathematical literature, specifically [4] and [3]. This indicates that there is more need to establish better connections and interactions between engineers and mathematicians.

The connection to physics plays an important role in the contributions of J.P. Gazeau and C. Habonimana, entitled "*Signal Analysis and Quantum Formalism: Quantizations with No Planck Constant*" and M. de Gosson's "*Generalized Anti-Wick Quantum States*". Both establish the connection to quantum theory, which from the very beginning was one of the motivations of time-frequency analysis, in terms of the well-known coherent states.

The article by S.I. Trapasso "*A Time-Frequency Analysis Perspective on Feynman Path Integrals*" shows how to make use of the modulation space theory to establish rigorous mathematical statements inspired by the ideas of R. Feynman concerning path integrals. However, one finds in the article written by D. Labate, B.R. Pahari, S. Hoteit, and M. Mecati "*Quantitative Methods in Ocular Fundus Imaging: Analysis of Retinal Microvasculature*" how time-frequency analysis methods find a concrete application in the medical sciences, via the example of the anatomic structure of the retina. This article also describes practical issues of data handling or the segmentation of retinal vessels.

A somewhat more abstract viewpoint, still related to data handling, is provided by the article "*Data Approximation with Time-Frequency Invariant Systems*" by D. Barbieri, C. Cabrelli, E. Hernández, and U. Molter. It is explained in the article how an optimal system (within a given TF framework) can be determined, for a given data set.

The contribution by F. Bartolucci, S. Pilipović, and N. Teofanov on "*The Shearlet Transform and Lizorkin Spaces*" describes specific aspects of the mathematical foundation of shearlet theory, namely the correct choice of the space of test functions to be used, connecting modern shearlet theory with a construction going back to Lizorkin (in his studies of inhomogeneous Besov spaces, if we use modern terminology), also related to the setting of Triebel–Lizorkin spaces in the terminology of H. Triebel's universe of the "*Theory of Function Spaces*" [6–8].

There is also a strong group theoretic connection shown in the opening article "*Radon Transform: Dual Pairs and Irreducible Representations*" by S. Alberti, F. Bartolucci, F. De Mari, and E. De Vito. This contribution connects in a very interesting way wavelet transform theory with group representations in the spirit of S. Helgason (see, e.g., [5]).

Finally, let us mention two further articles related to anti-Wick (respectively, Toeplitz) operators in the time-frequency context: F. Bastianoni reports in "*Time-Frequency Localization Operators: State of the Art*", while R. Corso and F. Tschinke report in "*Some Notes About Distribution Frame Multipliers*". They treat different aspects of operators, which are realized as multiplication operators on the transform domain. While the first one covers the case of the STFT (short-time Fourier transform) and results in anti-Wick-type operators, the second concentrates on a theme, again close to mathematical physics, where one has a continuous family of distributions forming a kind of continuous basis. Here, the family $(\delta_x)_{x\in\mathbb{R}}$ or the family of pure frequencies are prototypical examples.

All those who have contributed to this volume are grateful to the Proceedings team and to the publisher (Birkhäuser) for organizing this book in their prestigious *ANHA* (*Applied and Numerical Harmonic Analysis*) series. We acknowledge their patience and perseverance in keep reminding some of us, to finally come up with this interesting collection of articles.

Given the fact that this was already the second conference in this series, the participants (and those who missed the opportunity of joining us this time) may hope for another event like this in Torino in the years to come, even though Italy is hit by a pandemic crisis at the time of writing of this preface. Thus, we also wish Italy and especially our colleagues in Torino a good and prompt recovery. We are looking forward to visit Italy in the not too distant future and enjoy another topical conference there.

Wien, Austria Hans G. Feichtinger
27 March 2020

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2. Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy.

Due to the funds from the aforementioned institutions, the organizers were able to cover travel and living expenses of many participants, including numerous young researchers, graduate students, and postdocs.

They are also very grateful to the NuHAG (Numerical Harmonic Analysis Group), Faculty of Mathematics, University of Vienna, for the longstanding scientific collaboration and friendship and for contributing to make ATFA19 (Aspects of Time-Frequency Analysis) so stimulating and fruitful.

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Radon Transform: Dual Pairs and Irreducible Representations

Giovanni S. Alberti, Francesca Bartolucci, Filippo De Mari, and Ernesto De Vito

Abstract We illustrate the general point of view we developed in an earlier paper (SIAM J. Math. Anal., 2019) that can be described as a variation of Helgason's theory of dual G-homogeneous pairs (X, Ξ) and which allows us to prove intertwining properties and inversion formulae of many existing Radon transforms. Here we analyze in detail one of the important aspects in the theory of dual pairs, namely the injectivity of the map label-to-manifold $\xi \to \hat{\xi}$ and we prove that it is a necessary condition for the irreducibility of the quasi-regular representation of G on $L^2(\Xi)$. We further explain how our construction applies to the classical Radon and X-ray transforms in $\mathbb{R}^3$.

Keywords Homogeneous spaces · Radon transform · Dual pairs · Square-integrable representations · Inversion formula · Wavelets · Shearlets

1 Introduction

The circle of ideas and problems that may be collectively named "Radon transform theory" was born at least a century ago [17] but still abounds with questions and new perspectives that range from very concrete computation-oriented tasks to geometric or representation theoretic issues. We may describe the heart of the matter by paraphrasing Gelfand [8]:

"*Let X be some space and in it let there be given certain manifolds which we shall suppose to be analytic and dependent analytically on parameters $\xi_1, \ldots, \xi_k$, that is $\{\hat{\xi}(\xi) = \hat{\xi}(\xi_1, \ldots, \xi_k)\}$. With a function f on X we associate its integrals over these manifolds:*

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$$\mathscr{R}f(\xi) = \int_{\hat{\xi}} f(x)\mathrm{d}m_{\xi}(x).$$

We then ask whether it is possible to determine f knowing the integrals $\mathscr{R}f(\xi)$."

Among the many generalizations and theorems that may be subsumed in this basic, yet profound, mathematical sketch, it is certainly worth mentioning Helgason's contribution, inspired [12] by work of Fritz John's, in turn triggered by Radon's original result [17] dating back to 1917. In particular, Helgason developed the notion of dual pairs and double fibrations, whereby (Lie) groups and homogeneous spaces thereof stand at center stage. His basic observation comes by inspecting John's inversion formula for the integral transform–nowadays the prototypical Radon transform–defined by integration over planes in $\mathbb{R}^3$. The inversion takes the form

$$f(\mathbf{x}) = -\frac{1}{8\pi^2}\Delta_{\mathbf{x}}\Big(\int_{S^2} \mathscr{R}f(\mathbf{n}, \mathbf{n}\cdot\mathbf{x})\mathrm{d}\mathbf{n}\Big),$$

where $(\mathbf{n}, t) \mapsto \mathscr{R}f(\mathbf{n}, t)$ is the function on $S^2 \times \mathbb{R}$ given by the integral of f over the plane $\hat{\xi}(\mathbf{n}, t) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{n}\cdot\mathbf{x} = t\}$, $\Delta_{\mathbf{x}}$ is the Laplacian, and $\mathrm{d}\mathbf{n}$ is the Riemannian measure on the sphere S^2. This formula, observes Helgason [12], "*involves two dual integrations, $\mathscr{R}f$ is the integral over the set of points in a plane and then* d**n***, the integral over the set of planes through a point.*" Furthermore, the domain X on which the functions of interest are defined (here $X = \mathbb{R}^3$) and the set Ξ of relevant manifolds (here the two-dimensional planes) are homogeneous spaces of the same group G, namely the group of isometries of $\mathbb{R}^3$, and enjoy a sort of duality, well captured by the differential-geometric notion of incidence that was introduced by Chern [5].

Helgason proceeds on developing this duality in group-theoretic terms, emphasizing a remarkable formal symmetry, according to which the objects of interest come naturally in pairs, one living in X and its twin in Ξ. Most notably, each point $\xi \in \Xi$ (the pair $(\xi_1, \xi_2) = (\mathbf{n}, t)$ in our basic example) labels one of the actual submanifolds $\hat{\xi}$ of X on which the relevant integrals are to be taken (the plane $\hat{\xi}(\mathbf{n}, t)$). Conversely, with each point $x \in X$ it is natural to associate the "sheaf" of planes passing through it. In the example above, this is precisely the set $\check{\mathbf{x}} = \{\hat{\xi}(\mathbf{n}, \mathbf{x}\cdot\mathbf{n}) : \mathbf{n} \in S^2\}$ over which the integral of $\mathscr{R}f$ is taken.

In the abstract setting developed by Helgason, the whole construction enjoys natural properties as long as the mappings $\xi \mapsto \hat{\xi}$ and $x \mapsto \check{x}$ are both injective, requirement that is then built into the definition of dual pair and expressed algebraically. Note that in the above example, the map $(\mathbf{n}, t) \mapsto \hat{\xi}(\mathbf{n}, t)$ is two-to-one and this lack of injectivity is reflected by the fact that $\mathscr{R}f$ is an even function. The central object is of course the Radon transform

$$\mathscr{R}f(\xi) = \int_{\hat{\xi}} f(x)\,\mathrm{d}m_{\xi}(x)$$

for integrable functions on X, where m_{ξ} is a suitable measure on $\hat{\xi}$.

Utilizing a variation of this framework, which is recalled in full detail below, we have addressed [1] some issues that are naturally expressed in this language. Our main contribution (see Theorem 1) is a general result concerning the “unitarization” of $\mathscr{R}$ from $L^2(X, \mathrm{d}x)$ to $L^2(\Xi, \mathrm{d}\xi)$ and the fact that the resulting unitary operator intertwines the quasi-regular representations π and $\hat{\pi}$ of G on $L^2(X, \mathrm{d}x)$ and $L^2(\Xi, \mathrm{d}\xi)$, respectively. This unitarization really means first pre-composing the closure of $\mathscr{R}$ with a suitable pseudo-differential operator and then extending this composition to a unitary map, as is done in the existing and well-known predecessors of Theorem 1, such as those in [11] and in [21]. The representations π and $\hat{\pi}$ play of course a central role and are assumed to be irreducible, and π is assumed to be square-integrable (see Assumptions (A4) and (A5) below). The combination of unitary extension and intertwining leads to an interesting inversion formula for the true Radon transform, see Theorem 2.

Compared to [1], the present article adopts a slightly different, though fully compatible, formalism in the sense that we take here the point of view that seems most natural in applications. Indeed, the space X where the signals of interest are defined and the set of submanifolds of X where integrals are to be taken are both in the foreground, and the group G of geometric actions that one wants to consider comes next, tailored to the problem at hand. In this regard, it is important to observe that, in principle, there are many different realizations of X as homogeneous space, and the choice of G is tantamount to choosing the particular set of transformations (or symmetries) that one wants to focus on. In this context, it is of course important that there are sufficiently many of these transformations. As for the submanifolds, we observe that in most applications one has in mind a prototypical submanifold $\hat{\xi}_0$. We thus choose and fix $\hat{\xi}_0$, which we refer to as the *root* submanifold, as the image of the base point $x_0 \in X$ under the action of some closed subgroup H of G. Thus $\hat{\xi}_0 = H[x_0]$, and the other submanifolds are obtained by exploiting the fact that X is a transitive G-space. This entails that X is covered with all the shifted versions of $\hat{\xi}_0$ by means of the geometric transformations given by the elements of G. Incidentally, in this way one often achieves families of foliations, and in most cases this leads to a natural splitting of the parameters in Ξ, those that label the foliation and those that select the leaf in the foliation.

Although largely inspired by the work of Helgason, our approach is different in several ways that are discussed in detail in Sect. 2. His construction rests not only on the strict invariance of the measures on X, Ξ, and $\hat{\xi}_0$ (versus relative invariance as in our construction) but also on the fact that the correspondence $\xi \to \hat{\xi}$ between “labels” in the transitive G-space Ξ and submanifolds of X is assumed to be injective. In the present article we investigate this issue in detail and focus on the subgroup $\widetilde{H}$ of G that fixes $\hat{\xi}_0$, in principle larger than H. We find (Proposition 1) that the map $\xi \to \hat{\xi}$ is injective if and only if $\widetilde{H} = H$ and we further show in Theorem 3 that, under reasonable assumptions on $\widetilde{H}$, if this equality fails, then $\hat{\pi}$ cannot be irreducible. This implies that in order for Assumption (A5) to be fulfilled, one must choose H as large as possible among those subgroups of G that fill out $\hat{\xi}_0$

by acting on x_0. Our theory is then illustrated with the help of two examples, namely the classical Radon transform and the X-ray transform in $\mathbb{R}^3$, both analyzed with the group SIM(3) of rotations, dilations, and translations. Again, this is different from Helgason's standard choice, the isometry group M(3).

The paper is organized as follows. In Sect. 2 we set up the context and recall the main results of [1]. In Sect. 3 we present a rather detailed analysis of the relations existing between the objects naturally arising from an arbitrary choice of H and those that come from the maximal choice $\widetilde{H}$. This leads to the main contribution of this work, namely the fact that a gap between $\widetilde{H}$ and H implies that the quasi-regular representation $\hat{\pi}$ of G on $L^2(\Xi)$ cannot be irreducible. Section 4 illustrates our theory with two classical examples in three-dimensional Euclidean space.

2 The Framework

In this section we introduce the setting and the main result of [1].

2.1 Notation

We briefly introduce the notation. We set $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and $\mathbb{R}^+ = (0, +\infty)$. The Euclidean norm of a vector $v \in \mathbb{R}^d$ is denoted by $|v|$ and its scalar product with $w \in \mathbb{R}^d$ by $v \cdot w$. For any $p \in [1, +\infty]$ we denote by $L^p(\mathbb{R}^d)$ the Banach space of functions $f : \mathbb{R}^d \to \mathbb{C}$ that are p-integrable with respect to the Lebesgue measure $\mathrm{d}x$ and, if $p = 2$, the corresponding scalar product and norm are $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. If E is a Borel subset of $\mathbb{R}^d$, $|E|$ denotes its Lebesgue measure. The Fourier transform is denoted by $\mathscr{F}$ both on $L^2(\mathbb{R}^d)$ and on $L^1(\mathbb{R}^d)$, where it is defined by

$$\mathscr{F} f(\omega) = \int_{\mathbb{R}^d} f(x)\, \mathrm{e}^{-2\pi i\, \omega \cdot x}\, \mathrm{d}x, \qquad f \in L^1(\mathbb{R}^d).$$

If G is a locally compact second countable (lcsc) group, we denote by $L^2(G, \mu_G)$ the Hilbert space of square-integrable functions with respect to a left Haar measure μ_G on G. If X is a lcsc transitive G-space with origin x_0, we denote by $g[x]$ the action of G on X. A Borel measure ν on X is relatively invariant if there exists a positive character α of G such that for any measurable set $E \subseteq X$ and $g \in G$ it holds $\nu(g[E]) = \alpha(g)\nu(E)$, see, e.g., [19]. Furthermore, a Borel section is a measurable map $s : X \to G$ satisfying $s(x)[x_0] = x$ and $s(x_0) = e$, with e the neutral element of G; a Borel section always exists since G is second countable [22, Theorem 5.11]. We denote the (real) general linear group of size $d \times d$ by $\mathrm{GL}(d, \mathbb{R})$.