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Teresa W. Haynes
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Topics in Domination in Graphs

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Preface

While concepts related to domination in graphs can be traced back to the mid-1800s in connection with various chessboard problems, domination was first defined as a graph-theoretical concept in 1958. Domination in graphs experienced rapid growth since its introduction resulting in over 1200 papers published on domination in graphs by the late 1990s. Noting the need for a comprehensive survey of the literature on domination in graphs, in 1998 Haynes, Hedetniemi, and Slater published the first two books on domination, *Fundamentals of Domination in Graphs* and *Domination in Graphs: Advanced Topics*. We refer to these as Books I and II.

The explosive growth has continued and today more than 4000 papers have been published on domination in graphs, and the material in Books I and II is more than 20 years old. Thus, the authors think it is time for an update on the developments in domination theory since 1998. We also want to give a comprehensive treatment of the major topics in domination. This coverage of domination including both the fundamental major results and updates will be in the form of three books, which we shall call Books III, IV, and V.

Book III, *Domination in Graphs: Core Concepts*, is written by the authors and concentrates, as the title suggests, on the three main types of domination in graphs: domination, independent domination, and total domination. It contains major results on these basic domination numbers, including proofs of selected results that illustrate many of the proof techniques that are used in domination theory. For the companion books, Books IV and V, we invited leading researchers in domination to contribute chapters.

Book V has three parts, the first of which focuses on several domination-related concepts. The second part focuses on domination in (i) hypergraphs, (ii) chessboards, and (iii) digraphs and tournaments. The third part focuses on the development of algorithms and complexity of domination parameters.

The present volume, Book IV, concentrates on major domination parameters that were not covered in Book III. Although well over 70 types of dominating sets have been defined, Book IV focuses on the primary ones that have received the most attention in the literature. In particular, the chapters include such parameters as paired domination, connected domination, restrained domination, domination functions, Roman domination, and power domination.

The authors of Book IV provide a survey of known results with a sampling of proof techniques for each parameter. To avoid excessive repetition of definitions and notations, Chapter 1 provides a glossary of commonly used terms and Chapter gives an overview of models of domination from which the parameters are defined.

This book is intended as a reference resource for researchers and is written to reach the following audience: First, the audience includes the established researchers in the field of domination who want an updated comprehensive coverage on domination. Second are the researchers in graph theory and graduate students who wish to become acquainted with topics in domination including major accomplishments in the field and proof techniques used. We anticipate that it could also be used in a seminar course on domination in graphs.

We wish to thank the authors who contributed chapters to this book as well as the reviewers of the chapters.

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Contents

Glossary of Common Terms

Teresa W. Haynes, Stephen T. Hedetniemi, and Michael A. Henning

1 Introduction

It is difficult to say when the study of domination in graphs began, but for the sake of this glossary let us say that it began in 1962 with the publication by Oystein Ore's book *Theory of Graphs* [15]. In *Chapter 13 Dominating Sets, Covering Sets and Independent Sets* of [15], we see for the first time the name *dominating set*, defined as follows: "A subset D of V is a *dominating set* for G when every vertex not in D is the endpoint of some edge from a vertex in D." Ore then defines the *domination number*, denoted $\delta(G)$, of a graph G, as "the smallest number of vertices in any minimal dominating set." So, at this point, and for the first time, domination has a "name" and a "number."

Of course, prior to this Claude Berge [3], in his book *Theory of Graphs and its Applications*, which was first published in France in 1958 by Dunod, Paris, had previously defined the same concept, but had, in *Chapter 4 The Fundamental*

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Numbers of the Theory of Graphs of [3], given it the name "the coefficient of external stability."

Before Berge, Dénes König, in his 1936 book *Theorie der Endlichen und Unendlichen Graphen* [13], had defined essentially the same concept, but in *VII Kapitel, Basisproblem für gerichtete Graphen*, König gave it the name "punktbasis," which we would today say is an independent dominating set.

And even before König, in the books by Dudeney in 1908 [8] and W. W. Rouse Ball in 1905 [2], one can find the concepts of domination, independent domination, and total domination discussed in connection with various chessboard problems. And it was Ball who, in turn, credited such people as W. Ahrens in 1910 [1], C. F. de Jaenisch in 1862 [7], Franz Nauck in 1850 [14], and Max Bezzel in 1848 [4] for their contributions to these types of chessboard problems involving dominating sets of chess pieces.

But it was Ore who gave the name *domination* and this name took root. Not long thereafter, Cockayne and Hedetniemi [6] gave the notation $\gamma(G)$ for the domination number of a graph, and this also took root and is the notation adopted here.

Since the subsequent chapters in this book will deal with domination parameters, there will be much overlap in the terminology and notation used. One purpose of this chapter is to present definitions common to many of the chapters in order to prevent terms being defined repeatedly and to avoid other redundancy. Also, since graph theory terminology and notation sometimes vary, in this glossary we clarify the terminology that will be adopted in subsequent chapters.

We proceed as follows. In Section 2.1, we present basic graph theory definitions. We discuss common types of graphs in Section 2.2. Some fundamental graph constructions are given in Section 2.3. In Section 3.1 and Section 3.2, we present parameters related to connectivity and distance in graphs, respectively. The covering, packing, independence, and matching numbers are defined in Section 3.3. Finally in Section 3.4, we define selected domination-type parameters that will occur frequently throughout the book.

For more details and terminology, the reader is referred to the two books *Fundamentals of Domination in Graphs* [10] and *Domination in Graphs, Advanced Topics* [11] written and edited by Haynes, Hedetniemi, and Slater, and the book *Total Domination in Graphs* by Henning and Yeo [12]. An annotated glossary, from which many of the definitions in this chapter are taken, was produced by Gera, Haynes, Hedetniemi, and Henning in 2018 [9].

2 Basic Terminology

In this section, we give basic definitions, common types of graphs, and fundamental graph constructions.

2.1 Basic Graph Theory Definitions

Before we proceed with our glossary of parameters, we need to define a few basic terms, which are used in the definitions in the following subsections. For $k \geq 1$ an integer, we use the standard notation $[k] = \{1, \ldots, k\}$ and $[k]_0 = [k] \cup \{0\} = \{0, 1, \ldots, k\}$.

A (finite, undirected) *graph* $G = (V, E)$ consists of a finite nonempty set of *vertices* $V = V(G)$ together with a set $E = E(G)$ of unordered pairs of distinct vertices called *edges*. Each edge $e = \{u, v\}$ in E is denoted with any of e, uv, vu, and $\{u, v\}$. We say that a graph G has *order* $n = |V|$ and *size* $m = |E|$.

Two vertices u and v in G are *adjacent* if they are joined by an edge e, that is, u and v are adjacent if $e = uv \in E(G)$. In this case, we say that each of u and v is *incident* with the edge e. Further, we say that the edge e *joins* the vertices u and v. Two edges are *adjacent* if they share a common vertex. Two vertices in a graph G are *independent* if they are not adjacent. A set of pairwise independent vertices in G is an *independent set* of G. Similarly, two edges are *independent* if they are not adjacent.

A *neighbor* of a vertex v in G is a vertex u that is adjacent to v. The *open neighborhood* of a vertex v in G is the set of neighbors of v, denoted $N_G(v)$. Thus, $N_G(v) = \{u \in V \mid uv \in E(G)\}$. The *closed neighborhood of* v is the set $N_G[v] = \{v\} \cup N_G(v)$. For a set of vertices $S \subseteq V$, the *open neighborhood* of S is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$ and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we omit it in the above expressions. For example, we write $N(v)$, $N[v]$, $N(S)$, and $N[S]$ rather than $N_G(v)$, $N_G[v]$, $N_G(S)$, and $N_G[S]$, respectively.

For a set of vertices $S \subseteq V$ and a vertex v belonging to the set S, the *S-private neighborhood* of v is defined by $\text{pn}[v, S] = \{w \in V \mid N_G[w] \cap S = \{v\}\}$, while its *open S-private neighborhood* is defined by $\text{pn}(v, S) = \{w \in V \mid N_G(w) \cap S = \{v\}\}$. As remarked in [12], the sets $\text{pn}[v, S] \setminus S$ and $\text{pn}(v, S) \setminus S$ are equivalent and we define the *S-external private neighborhood* of v to be this set, abbreviated $\text{epn}[v, S]$ or $\text{epn}(v, S)$. The *S-internal private neighborhood* of v is defined by $\text{ipn}[v, S] = \text{pn}[v, S] \cap S$ and its *open S-internal private neighborhood* is defined by $\text{ipn}(v, S) = \text{pn}(v, S) \cap S$. We define an *$S$-external private neighbor* of v to be a vertex in $\text{epn}(v, S)$ and an *S-internal private neighbor* of v to be a vertex in $\text{ipn}(v, S)$.

The *degree* $d_G(v)$ of a vertex v is the number of neighbors v has in G, that is, $d_G(v) = |N_G(v)|$. Again if the graph G is clear from the context, we use $d(v)$ rather than $d_G(v)$. We remark that some books use $\deg(v)$ and $\deg v$ to denote the degree of v. We leave it to the authors to choose which of these notations to adopt in their chapters. For a subset of vertices $S \subseteq V$, the *degree of* v *in* S, denoted $d_S(v)$, is the number of vertices in S adjacent to the vertex v; that is, $d_S(v) = |N_G(v) \cap S|$. In particular, if $S = V$, then $d_S(v) = d_G(v)$. The *degree sequence* of a graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is the sequence $d_1, d_2, \ldots, d_n$, where $d_i = d(v_i)$

for $i \in [n]$. Often the degree sequence, $d_1, d_2, \ldots, d_n$, is written in non-increasing order, and so $d_1 \geq d_2 \geq \ldots \geq d_n$.

An *isolated vertex* is a vertex of degree 0 in G. A graph is *isolate-free* if it does not contain an isolated vertex. The minimum degree among the vertices of G is denoted by $\delta(G)$, and the maximum degree by $\Delta(G)$. A *leaf* is a vertex of degree 1, while its neighbor is a *support vertex*. A *strong support vertex* is a (support) vertex with at least two leaf neighbors.

For subsets X and Y of vertices of G, we denote the set of edges that join a vertex of X and a vertex of Y in G by $[X, Y]$.

Two graphs G and H are *isomorphic*, denoted $G \cong H$, if there exists a bijection $\phi\colon V(G) \to V(H)$ such that two vertices u and v are adjacent in G if and only if the two vertices $\phi(u)$ and $\phi(v)$ are adjacent in H. A *parameter* of a graph G is a numerical value (usually a non-negative integer) that can be associated with a graph such that whenever two graphs are isomorphic, they have the same associated numerical value.

By a *partition* of the vertex set V of a graph G, we mean a family $\pi = \{V_1, V_2, \ldots, V_k\}$ of nonempty pairwise disjoint sets whose union equals V, that is, for all $1 \leq i < j \leq k$, $V_i \cap V_j = \emptyset$ and

$$\bigcup_{i=1}^{k} V_i = V.$$

For such a partition π, we will say that π has *order* k.

A *walk* in a graph G from a vertex u to a vertex v is a finite, alternating sequence of vertices and edges, starting with the vertex u and ending with the vertex v, in which each edge of the sequence joins the vertex that precedes it in the sequence to the vertex that follows it in the sequence. A *trail* is a walk containing no repeated edges, and a *path* is a walk containing no repeated vertices. We will mainly be concerned with paths. A path joining two vertices u and v is called a (u, v)*-path* or a u-v *path* or a u, v*-path* in the literature. The *length* of a walk equals the number of edges in the walk. A graph G is *connected* if there is a path between every pair of vertices of G.

A *cycle* is a path in which the first and last vertices are the same and all other vertices are distinct. A *chord* of a cycle C is an edge between two nonconsecutive vertices of C.

The *distance* $d(u, v)$ between two vertices u and v, in a connected graph G, equals the minimum length of a (u, v)-path in G. A shortest, or minimum length, path between two vertices u and v is called a (u, v)*-geodesic*; a v*-geodesic* is any shortest path from v to another vertex; a *geodesic* is any shortest path in a graph.

A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A subgraph G' of a graph G is called a *spanning subgraph* of G if $V' = V$. If $G = (V, E)$ and $S \subseteq V$, then the *subgraph of* G *induced by* S is the graph $G[S]$, whose vertex set is S and whose edges are all the edges in E both of whose vertices are in S.

Let F be an arbitrary graph. A graph G is said to be *F-free* if G does not contain F as an induced subgraph.

If $G = (V, E)$ and $S \subseteq V$, the subgraph obtained from G by deleting all vertices in S and all edges incident with one or two vertices in S is denoted by $G - S$; that is, $G - S = G[V \setminus S]$. If $S = \{v\}$, we simply denote $G - \{v\}$ by $G - v$. The *contraction* of an edge $e = xy$ in a graph G is the graph obtained from G by deleting the vertices x and y and adding a new vertex and edges joining this new vertex to all vertices that were adjacent to x or y in G.

A *component* of a graph is a maximal connected subgraph. An *odd (even) component* is a component of odd (even) order. Let $\mathrm{oc}(G)$ equal the number of odd components of G. A vertex $v \in V$ is a *cut vertex* if the graph $G - v$ has more components than G. An edge $e = uv$ is a *bridge* if the graph $G - e$ obtained by deleting e from G has more components than G.

2.2 Common Types of Graphs

A graph of order $n = 1$ is called a *trivial graph*, while a graph with at least two vertices is called a *nontrivial graph*. A graph of size $m = 0$ is an *empty graph*, while a graph with at least one edge is a *nonempty graph*. Recall that a *connected graph* is a graph for which there is a path between every pair of its vertices.

A *k-regular graph* is a graph in which every vertex has degree k. A *regular graph* is a graph that is k-regular for some $k \geq 0$. A 3-regular graph is also called a *cubic graph*.

A graph of order n that is a cycle is denoted by C_n and a graph that is a path is denoted by P_n. Note that a cycle is a 2-regular graph.

A graph is *acyclic* if it does not contain a cycle. A *tree* is a connected acyclic graph. Equivalently, a tree is a connected graph having size one less than its order. Thus, if T is a tree of order n and size m, then T is connected and $m = n - 1$. A *forest* is an acyclic graph. Thus, a forest is a disjoint union of trees. A *linear forest* is a forest in which every component is a path.

If G is a vertex-disjoint union of k copies of a graph F, we write $G = kF$.

A *complete graph* is a graph in which every two vertices are adjacent. A complete graph of order n is denoted by K_n. A *triangle* is a subgraph isomorphic to K_3 or C_3, since $K_3 \cong C_3$.

A graph G is *bipartite* if its vertex set can be partitioned into two independent sets X and Y. The sets X and Y are called the *partite sets* of G. A *complete bipartite graph*, denoted $K_{r,s}$, is a bipartite graph with partite sets X and Y, where $|X| = r$, $|Y| = s$, and every vertex in X is adjacent to every vertex in Y. The graph $K_{r,s}$ has order $r + s$, size rs, $\delta(K_{r,s}) = \min\{r, s\}$, and $\Delta(K_{r,s}) = \max\{r, s\}$.

A *star* is a nontrivial tree with at most one vertex that is not a leaf. Thus, a star is a complete bipartite graph $K_{1,k}$ for some $k \geq 1$. A *claw* is an induced copy of the graph $K_{1,3}$. Thus, a *claw-free graph* is a $K_{1,3}$-free graph.

For $r, s \geq 1$, a *double star* $S(r, s)$ is a tree with exactly two (adjacent) vertices that are not leaves, one of which has r leaf neighbors and the other s leaf neighbors.

A *diamond* is an induced copy of the graph $K_4 - e$, which is obtained from a copy of the complete graph of order 4 by deleting an edge e.

A graph G can be *embedded* on a surface S if its vertices can be placed on S and all of its edges can be drawn between the vertices on S in such a way that no two edges intersect. A graph G is *planar* if it can be embedded in the plane; a *plane graph* is a graph that has been embedded in the plane.

A *rooted tree* T is a tree having a *distinguished vertex* labeled r, called the *root*. Let T be a rooted tree with root r. For each vertex v, let $P(v)$ be the unique (r, v)-path in T. The *parent* of a vertex v is its neighbor on $P(v)$, while the other neighbors of v are called its *children*. The set of children of v is denoted by $C(v)$. Note that the root r is the only vertex of T with no parent. A *descendant* of v is any vertex $u \neq v$ such that the $P(u)$ contains v, while an *ancestor* of v is a vertex $u \neq v$ that belongs $P(v)$ in T. In particular, every child of v is a descendant of v, while the parent of v is an ancestor of v. A *grandchild* of v is a descendant of v at distance 2 from v. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v, denoted T_v, is the subtree of T induced by $D[v]$. The *depth* of a vertex v in T equals $d(r, v)$, and the *height* of v, denoted ht(v), is the maximum distance from v to a descendant of v. Thus, ht(v) $= \max\{d(v, w) : w$ is a descendant of $v\}$.

For classes of graphs not defined here, we refer the reader to the definitive encyclopedia on graph classes, *Graph Classes: A Survey* [5] by Brandstädt, Le, and Spinrad.

2.3 Graph Constructions

Given a graph $G = (V, E)$, the *complement* of G is the graph $\overline{G} = (V, \overline{E})$, where $uv \in \overline{E}$ if and only if $uv \notin E$. Thus the complement, $\overline{G}$, of G, is formed by taking the vertex set of G and joining two vertices by an edge whenever they are not joined in G.

By a *graph product* $G \otimes H$ on graphs G and H, we mean a graph whose vertex set is the Cartesian product of the vertex sets of G and H (that is, $V(G \otimes H) = V(G) \times V(H)$) and whose edge set is determined entirely by the adjacency relations of G and H. Exactly how it is determined depends on what kind of graph product we are considering.

The *Cartesian product* $G \,\square\, H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

The *direct product* (also known as the *cross product*, *tensor product*, *categorical product*, and *conjunction*) $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ if and only if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

Given a graph $G = (V, E)$ and an edge $uv \in E$, the *subdivision* of edge uv consists of (i) deleting the edge uv from E, (ii) adding a new vertex w to V, and (iii) adding the new edges uw and wv to E. In this case we say that the edge uv has been *subdivided*. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing every edge of G exactly once.

Given a graph $G = (V, E)$, the *line graph* $L(G) = (E, E(L(G)))$ is the graph whose vertices correspond 1-to-1 with the edges in E, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges in G have a vertex in common, that is, if and only if the corresponding two edges are adjacent.

The *corona* $G \circ K_1$ of a graph G, also denoted cor(G) in the literature, is the graph obtained from G by adding for each vertex $v \in V$ a new vertex v' and the edge vv'. The edge vv' is called a *pendant edge*. The *k-corona* $G \circ P_k$ of G is the graph of order $(k+1)|V(G)|$ obtained from G by attaching a path of length k to each vertex of G so that the resulting paths are vertex-disjoint. In particular, the *2-corona* $G \circ P_2$ of G is the graph of order $3|V(G)|$ obtained from G by attaching a path of length 2 to each vertex of G so that the resulting paths are vertex-disjoint. The *generalized corona* $G \circ H$ is the graph obtained by adding a copy of H for each vertex v of G and joining v to every vertex of H. Thus, a generalized corona $G \circ H$, where $H = K_1$, is the ordinary corona $G \circ K_1$. We note that whether $G \circ P_k$ is intended to denote a k-corona or a generalized corona will be clear from context or specifically stated by the author.

3 Graph Parameters

In this section, we present common graph parameters that may appear in this book.

3.1 Connectivity and Subgraph Numbers

In this subsection, we present parameters related to connectivity in graphs.

(a) *blocks* bl(G), number of blocks in G. A *block* of a graph G is a maximal nonseparable subgraph of G, that is, a maximal subgraph having no cut vertices.
(b) *bridges* br(G), number of bridges in G.
(c) *circumference* cir(G), maximum length or order of a cycle in G.
(d) *clique number* $\omega(G)$, maximum order of a complete subgraph of G.
(e) *components* $c(G)$, number of maximal connected subgraphs of G.
(f) A *vertex cut* of a connected graph G is a subset S of the vertex set of G with the property that $G - S$ is disconnected (has more than one component). A vertex cut S is a *k-vertex cut* if $|S| = k$.

(g) *vertex connectivity* $\kappa(G)$, minimum cardinality of a vertex cut of G if G is not the complete graph and $\kappa(G) = n - 1$ if G is a complete graph K_n on $n \geq 2$ vertices. A graph G is *k-vertex-connected* (or *k-connected*) if $\kappa(G) \geq k$ for some integer $k \geq 0$. Thus, $\kappa(G)$ is the smallest number of vertices whose deletion from G produces a disconnected graph or the trivial graph K_1. A nontrivial graph has connectivity 0 if and only if it is disconnected.

(h) An *edge cut* of a nontrivial connected graph G is a nonempty subset F of the edge set of G with the property that $G - F$ is disconnected (has more than one component). Thus, the deletion of an edge cut from the connected graph G results in a disconnected graph. An edge cut F is a *k-edge cut* if $|F| = k$.

(i) *edge connectivity* $\lambda(G)$, minimum cardinality of an edge cut of G if G is nontrivial, while $\lambda(K_1) = 0$. A graph G is *k-edge-connected* if $\lambda(G) \geq k$ for some integer $k \geq 0$. Thus, $\lambda(G)$ is the smallest number of edges whose deletion from G produces a disconnected graph or the trivial graph K_1. Hence, $\lambda(G) = 0$ if and only if G is disconnected or trivial.

(j) *girth* of G, denoted *girth(G)* or $g(G)$ in the literature, the minimum length of a cycle in G.

3.2 Distance Numbers

This subsection contains the definitions of parameters which are defined in terms of the distances $d(u, v)$ between vertices u and v in a graph.

(a) *eccentricity* $\mathrm{ecc}(v)$, of a vertex v in a connected graph G, is the maximum of the distances from v to the other vertices of G.

(b) *diameter* $\mathrm{diam}(G)$, maximum distance among all pairs of vertices of G. Equivalently, the diameter of G is the maximum length of a geodesic in G. Thus, the diameter of G is the maximum eccentricity taken over all vertices of G. Two vertices u and v in G for which $d(u, v) = \mathrm{diam}(G)$ are called *antipodal* or *peripheral vertices* of G. A *diametral path* in G is a geodesic whose length equals the diameter of G.

(c) The *periphery* of a graph G is the subgraph of G induced by its peripheral vertices.

(d) *radius* $\mathrm{rad}(G)$, minimum eccentricity taken over all vertices of G.

(e) The *center* of a graph G, denoted $C(G)$, is the subgraph of G induced by the vertices in G whose eccentricity equals the radius of G. A vertex $v \in C(G)$ is called a *central vertex* of G.

3.3 Covering, Packing, Independence, and Matching Numbers

As previously defined, a set of pairwise independent vertices in G is an *independent set* of G. An independent set S is *maximal* if no superset of S is independent.

A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching*. Given a matching M, we denote by $V[M]$ the set of vertices in G incident with an edge in M. A *perfect matching* is a matching in which every vertex is incident with an edge of the matching. Thus, if G has a perfect matching M, then G has even order $n = 2k$ for some $k \geq 1$ and $|M| = k$.

A vertex and an edge are said to *cover* each other in a graph G if they are incident in G. A *vertex cover* in G is a set of vertices that covers all the edges of G, while an *edge cover* in G is a set of edges that covers all the vertices of G. Thus, a vertex cover in G is a set of vertices that contains at least one vertex of every edge in G.

A subset S of vertices in G is a *packing* if the closed neighborhoods of vertices in S are pairwise disjoint. Equivalently, S is a packing in G if the vertices in S are pairwise at distance at least 3 apart in G. Thus, if S is a packing in G, then $|N_G[v] \cap S| \leq 1$ for every vertex $v \in V(G)$. A packing is also called a 2-*packing* in the literature. More generally, for $k \geq 2$, a *k-packing* in G is a set of vertices in G that are pairwise at distance at least $k + 1$ apart in G. Thus, if S is a k-packing in G, then $d_G(u, v) > k$ for every two distinct vertices u and v that belong to S.

A subset S of vertices in G is an *open packing* if the open neighborhoods of vertices in S are pairwise disjoint. Thus, if S is an open packing in G, then $|N_G(v) \cap S| \leq 1$ for every vertex $v \in V(G)$.

All of the parameters in this subsection have to do with sets that are independent or cover other sets. These include some of the most basic of all parameters in graph theory.

(a) *vertex independence numbers* $i(G)$ and $\alpha(G)$, minimum and maximum cardinality of a maximal independent set in G. The lower vertex independence number, $i(G)$, is also called the *independent domination number* of G, while the upper vertex independence number, $\alpha(G)$, is also called the *independence number* of G. (While the notation $i(G)$ is fairly standard for the independent domination number, we remark that the independence number is also denoted by $\beta_0(G)$ in the literature.)

(b) *vertex covering numbers* $\beta(G)$ and $\beta^+(G)$, minimum and maximum cardinality of a minimal vertex cover in G. (We remark that the vertex covering number is also denoted by $\tau(G)$ or by $\alpha(G)$ in the literature.)

(c) *edge covering numbers* $\beta'(G)$ and $\beta'^+(G)$, minimum and maximum cardinality of a minimal edge cover in G.

(d) *k-packing number* $\rho_k(G)$, maximum cardinality of a k-packing in G for $k \geq 2$. When $k = 2$, the k-packing number $\rho_k(G)$ is called the *packing number* of G, denoted by $\rho(G)$. Thus, $\rho(G)$ is the maximum cardinality of a packing in G.

(e) *open packing number* $\rho^o(G)$, maximum cardinality of an open packing in G.

(f) *matching numbers* $\alpha'^{-}(G)$ and $\alpha'(G)$, minimum and maximum cardinality of a maximal matching in G. The upper matching number, $\alpha'(G)$, is also called the *matching number* of G. Recall that a *perfect matching* is a matching in which every vertex is incident with an edge of the matching. Thus, if a graph G of order n has a perfect matching, then $\alpha'(G) = \frac{1}{2}n$. It should be noted that by a well-known theorem of Gallai, that if G is a graph of order n with no isolated vertices, then $\alpha(G) + \beta(G) = n = \alpha'(G) + \beta'(G)$. (The matching number is also denoted by $\beta_1(G)$ in the literature.)

3.4 Domination Numbers

A *dominating set* in a graph $G = (V, E)$ is a set S of vertices of G such that every vertex in $V \setminus S$ has a neighbor in S. Thus, if S is a dominating set of G, then $N_G[S] = V$ and every vertex in $V \setminus S$ is therefore adjacent to at least one vertex in S. For subsets X and Y of vertices of G, if $Y \subseteq N_G[X]$, then the set X *dominates* the set Y in G. In particular, if X dominates $V(G)$, then X is a dominating set of G. If no proper subset of a dominating set S is a dominating set of G, then S is a *minimal dominating set* of G.

The many variations of dominating sets in a graph G are based on (i) conditions which are placed on the subgraph $G[S]$ induced by a dominating set S, (ii) conditions which are placed on the vertices in $V \setminus S$, or (iii) conditions which are placed on the edges between vertices in S and vertices in $V \setminus S$. We mention only the major domination numbers here.

A *total dominating set*, abbreviated TD-set, in a graph G with no isolated vertices is a set S of vertices of G such that every vertex in V is adjacent to at least one vertex in S. Thus, a subset $S \subseteq V$ is a TD-set in G if $N_G(S) = V$. If no proper subset of S is a TD-set of G, then S is a *minimal TD-set* of G. Every graph without isolated vertices has a TD-set, since $S = V$ is such a set. If X and Y are subsets of vertices in G, then the set X *totally dominates* the set Y in G if $Y \subseteq N_G(X)$. In particular, if X totally dominates $V(G)$, then X is a TD-set in G.

A *paired dominating set*, abbreviated PD-set, of G is a set S of vertices of G such that every vertex is adjacent to some vertex in S and the subgraph $G[S]$ induced by S contains a perfect matching M. Two vertices joined by an edge of M are said to be *paired* with respect to a perfect matching M and are also called *partners* in S. A PD-set S in a graph G is *minimal* if no proper subset of S is a PD-set of G.

A *connected dominating set*, abbreviated CD-set, in a graph G is a dominating set S of vertices of G such that $G[S]$ is connected. A CD-set S in a graph G is *minimal* if no proper subset of S is a CD-set of G.

(a) *domination numbers* $\gamma(G)$ and $\Gamma(G)$, minimum and maximum cardinalities of a minimal dominating set in G. The parameters $\gamma(G)$ and $\Gamma(G)$ are called the *domination number* and *upper domination number* of G, respectively. A dominating set of G of cardinality $\gamma(G)$ is called a γ*-set* of G, while a minimal dominating set of cardinality $\Gamma(G)$ is called a Γ-set of G.

(b) *independent domination* $i(G)$, minimum cardinality of a dominating set in G that is also independent. An independent dominating set of G of cardinality $i(G)$ is called an i-set of G. We note that the maximum order of a minimal independent dominating set equals the vertex independence number $\alpha(G)$.

(c) *total domination numbers* $\gamma_t(G)$ and $\Gamma_t(G)$, minimum and maximum cardinalities of a minimal total dominating set of G. The parameters $\gamma_t(G)$ and $\Gamma_t(G)$ are called the *total domination number* and *upper total domination number* of G, respectively. A TD-set of G of cardinality $\gamma_t(G)$ is called a γ_t*-set* of G, while a minimal TD-set of cardinality $\Gamma_t(G)$ is called a Γ_t-set of G.

(d) *paired domination numbers* $\gamma_{\text{pr}}(G)$ and $\Gamma_{\text{pr}}(G)$, minimum and maximum cardinalities of a minimal PD-set of G. The parameters $\gamma_{\text{pr}}(G)$ and $\Gamma_{\text{pr}}(G)$ are called the *paired domination number* and *upper paired domination number* of G, respectively. A PD-set of G of cardinality $\gamma_{\text{pr}}(G)$ is called a γ_{pr}*-set* of G, while a minimal PD-set of cardinality $\Gamma_{\text{pr}}(G)$ is called a Γ_{pr}-set of G.

(e) *connected domination numbers* $\gamma_c(G)$ and $\Gamma_c(G)$, minimum and maximum cardinalities of a minimal CD-set of G. The parameters $\gamma_c(G)$ and $\Gamma_c(G)$ are called the *connected domination number* and *upper connected domination number* of G, respectively. A CD-set of G of cardinality $\gamma_c(G)$ is called a γ_c*-set* of G, while a minimal CD-set of cardinality $\Gamma_c(G)$ is called a Γ_c-set of G.

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Models of Domination in Graphs

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1 Introduction

As we have said before, a set $S \subseteq V$ is a *dominating set* of a graph $G = (V, E)$ if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. In this chapter, we will take a look at dominating sets from a variety of different perspectives. Each perspective suggests a variation in the domination theme and different types or aspects of dominating sets.

We will not attempt to be comprehensive here, only to provide a sufficient number of different models to reveal domination in a much broader view. Chapter 11 in the book *Fundamentals of Domination in Graphs* [5] presents ten logical structures or frameworks where the concept of domination naturally arises. The suggested frameworks range from integer programming to hypergraphs. We repeat a few of these frameworks in this chapter.

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In the most general sense, we are interested in sets of vertices in a graph having some property $\mathcal{P}$, called $\mathcal{P}$-sets. We are interested in finding $\mathcal{P}$-sets of minimum and maximum cardinalities, using a notational system of the form $a(G)$ for the minimum cardinality of a $\mathcal{P}$-set, and upper case $A(G)$ for the maximum cardinality of a $\mathcal{P}$-set. These parameters are sometimes referred to as lower and upper parameters.

But we will make a further distinction. As an example, recall that a set $S \subseteq V$ is *independent* if no two vertices in S are adjacent. Since independence is an *hereditary property* (henceforth denoted H), meaning that every subset of an independent set is also independent, it would not make any sense to seek a minimum cardinality independent set, and so instead we seek a minimum cardinality maximal independent set (denoted *minimax*). Similarly, the property of being a dominating set is *superhereditary* (henceforth denoted SH), meaning that any superset of a dominating set is also a dominating set. Thus, it would not make any sense to seek a maximum cardinality dominating set, since the entire vertex set of any graph is a dominating set. So, instead, we seek the maximum cardinality of a minimal dominating set (denoted *maximin*). For concepts that are neither hereditary nor superhereditary, we generally seek a minimum cardinality $\mathcal{P}$-set, denoted *min*, and sometimes a maximum cardinality $\mathcal{P}$-set, denoted *max*.

To illustrate these examples, consider the double star $S(r, s)$ for $1 \le r \le s$ with two adjacent vertices u and v, where u is adjacent to r leaves and v is adjacent to s leaves. Let $L(u)$ and $L(v)$ denote the set of leaves adjacent to u and v, respectively. We note that $S(r, s)$ has exactly four minimal dominating sets, namely $S_1 = \{u, v\}$, $S_2 = L(u) \cup \{v\}$, $S_3 = L(v) \cup \{u\}$, and $S_4 = L(u) \cup L(v)$. It follows that S_1 is a minimum dominating set and S_4 is a maximin dominating set. Thus, the domination number $\gamma(S(r, s)) = |S_1| = 2$ and the upper domination number $\Gamma(S(r, s)) = |S_4| = r + s$. Further, the maximal independent sets of $S(r, s)$ are precisely the sets S_2, S_3, and S_4. Hence, S_4 is a maximum independent set and S_2 is a minimax independent set, and so the independent domination number $i(S(r, s)) = |S_2| = r + 1$ and the independence number $\alpha(S(r, s)) = |S_4| = r + s$.

In the remaining sections of this chapter, we will use the following notation, where recall for a vertex v in G, the set $N_G(v)$ denotes the set of neighbors of v in G, and the degree of v in G is denoted by $d_G(v) = |N_G(v)|$. Further recall that for a subset of vertices $S \subseteq V$, the degree of v in S, denoted $d_S(v)$, is the number of vertices in S adjacent to the vertex v; that is, $d_S(v) = |N_G(v) \cap S|$. In particular, if $S = V$, then $d_S(v) = d_G(v)$. If the graph G is clear from context, we simply write $N(v)$ and $d(v)$ rather than $N_G(v)$ and $d_G(v)$, respectively. For $k \ge 1$ an integer, we use the standard notation $[k] = \{1, \dots, k\}$ and $[k]_0 = [k] \cup \{0\} = \{0, 1, \dots, k\}$. At a glance,

$\overline{S} = V \setminus S$, that is, $\overline{S}$ denotes the vertices in V but not in S, called the *complement* of S in G.
$d_S(v) = |N(v) \cap S|$.
$d_S[v] = |N[v] \cap S|$.
$d_{\overline{S}}(v) = |N(v) \cap \overline{S}|$.
$d_{\overline{S}}[v] = |N[v] \cap \overline{S}|$.

$G[S]$, the subgraph of G *induced by* S.
$\delta(G) = \min\{d(v) \mid v \in V\}$.
$\Delta(G) = \max\{d(v) \mid v \in V\}$.

Also, to avoid excessive repetition, we will frequently list domination parameters in the following abbreviated format:

parameter name: concept definition; designation of being hereditary H, or superhereditary SH; if neither hereditary nor superhereditary no designation is given; notation for lower parameter and type of $\mathcal{P}$-set (min or minimax), notation for upper parameter and type $\mathcal{P}$-set (max or maximin).

For example, domination, where $\gamma(G)$ is the domination number and $\Gamma(G)$ is the upper domination number, is listed as:

domination: $N[S] = V$, that is, for every $v \in \overline{S}$, $d_S(v) \geq 1$; SH; $\gamma(G)$ (min), $\Gamma(G)$ (maximin).

2 Fundamental Domination Parameters

In this section, we present what are arguably the most basic of all parameters related to domination in graphs. From these basic parameters all others are derived in one way or another. We begin with a list of five fundamental domination parameters. Thereafter, we list seven related parameters. The designations H hereditary and SH superhereditary are given whenever they apply. Unless otherwise stated, S always denotes a subset of V and F always denotes a subset of E.

2.1 Domination Parameters

(a) *domination*: $N[S] = V$, that is, for every $v \in \overline{S}$, $d_S(v) \geq 1$; SH; $\gamma(G)$ (min), $\Gamma(G)$ (maximin).
(b) *independent domination*: $N[S] = V$ and S is independent; $i(G)$ (min), $\alpha(G)$ (max).
(c) *total domination*: $N(S) = V$, that is, for every $v \in V$, $d_S(v) \geq 1$; $\gamma_t(G)$ (min), SH; $\Gamma_t(G)$ (maximin);
(d) *paired domination*: $N[S] = V$ and $G[S]$ has a *perfect matching*, that is, an independent set of edges of cardinality $\frac{1}{2}|S|$; $\gamma_{pr}(G)$ (min), $\Gamma_{pr}(G)$ (maximin).
(e) *connected domination*: $N[S] = V$ and $G[S]$ is connected; $\gamma_c(G)$ (min), SH; $\Gamma_c(G)$ (maximin).

It is perhaps worth commenting why total domination and connected domination are both superhereditary properties (SH). Whereas a superset S^* of a connected set S might not be a connected set, if S is also a dominating set, then every vertex in

$\overline{S}$ is adjacent to a vertex in S implying that S^* is also a connected dominating set. Similarly, total domination is superhereditary.

2.2 Related Parameters

(a) *vertex covering*: every edge $e \in E$ is incident to a vertex in S; SH; $\beta(G)$ (min), $\beta^+(G)$ (maximin). Note that for any graph G of order $n = |V|$,

$$\alpha(G) + \beta(G) = n$$

and

$$i(G) + \beta^+(G) = n.$$

(b) *irredundance*: for every vertex $v \in S$, $N[v] \setminus N[S \setminus \{v\}] \neq \emptyset$; H; $ir(G)$ (minimax), $IR(G)$ (max).

(c) *enclaveless*: S does not contain an *enclave*, that is, a vertex $v \in S$, such that $N[v] \subseteq S$; H; $\psi(G)$ (minimax), $\Psi(G)$ (max). Note that for every graph G of order n,

$$\gamma(G) + \Psi(G) = n$$

and

$$\Gamma(G) + \psi(G) = n.$$

(d) *packing*: for every $u, v \in S$, $d(u, v) > 2$; H; $p_2(G)$ (minimax), $P_2(G)$ (max). The packing number $P_2(G)$ is also denoted $\rho(G)$ in the literature. Note that the packing number is a standard lower bound on the domination number for any graph G, that is, $P_2(G) \leq \gamma(G)$.

(e) *edge domination*: $F \subseteq E$ and every edge not in F is adjacent to some edge in F; SH; $\gamma'(G)$ (min), $\Gamma'(G)$ (maximin).

(f) *matching*: $F \subseteq E$ and F is an independent set of edges; H; $\alpha'^-(G)$ (minimax), $\alpha'(G)$ (max).

(g) *edge covering*: every vertex $v \in V$ is incident to an edge in $F \subseteq E$; SH; $\beta'(G)$ (min), $\beta'^+(G)$ (maximin). Note, it has been shown in [4] and [6], respectively, that for every graph G of order n,

$$\alpha'(G) + \beta'(G) = n,$$

and

$$\alpha'^-(G) + \beta'^+(G) = n.$$

3 Conditions on the Dominating Set

Many domination parameters are formed by combining domination with another graph theoretical property P. In this section, we consider the parameters defined by imposing an additional constraint on the dominating set. In the next section, we will see that a condition may also be placed on the dominated set or on the method of dominating.

We list samples of types of dominating sets S defined either by imposing a condition on the subgraph $G[S]$ induced by S or requiring that every vertex in S satisfy some added condition. Clearly, some of the basic types are defined within this framework. For example, if $G[S]$ has no edges, then the set S is an independent dominating set, if $G[S]$ has no isolated vertices, then S is a total dominating set, and if $G[S]$ is connected, then S is a connected dominating set. Since all of the pairs of parameters in this section consist of the smaller as a minimum and the larger as a maximum of minimal, the designations (min) and (maximin) are omitted.

(a) *acyclic domination*: $N[S] = V$ and $G[S]$ is acyclic (contains no cycles); $\gamma_a(G)$, $\Gamma_a(G)$.
(b) *bipartite domination*: $N[S] = V$ and $G[S]$ is bipartite; $\gamma_{\rm bip}(G)$, $\Gamma_{\rm bip}(G)$.
(c) *clique domination*: $N[S] = V$ and $G[S]$ is a complete graph: $\gamma_{cl}(G)$, $\Gamma_{cl}(G)$.
(d) *private domination*: $N[S] = V$ and for every $u \in S$ there exists a vertex $v \in \overline{S}$ such that $N(v) \cap S = \{u\}$; $\gamma_{\rm pvt}(G)$, $\Gamma_{\rm pvt}(G)$. Note that a well-known theorem of Bollobás and Cockayne [1] shows that for every graph G with no isolated vertices, $\gamma(G) = \gamma_{\rm pvt}(G)$, that is, G has a γ-set S such that for each vertex $u \in S$, there is a vertex $v \in \overline{S}$ with $N(v) \cap S = \{u\}$.
(e) *semitotal domination*: $N[S] = V$ and for every vertex $u \in S$, there exists a vertex $v \in S$ with $d(u, v) \leq 2$; SH; $\gamma_{t2}(G)$, $\Gamma_{t2}(G)$.
(f) *weakly connected domination*: $N[S] = V$ and $G' = (V, E_S)$ is connected, where E_S is the set of edges of G incident to at least one vertex of S; SH; $\gamma_w(G)$, $\Gamma_w(G)$.
(g) *semipaired domination*: $N[S] = V$ and the vertices in S can be partitioned into $|S|/2$ pairs $\{u, v\}$ such that $d(u, v) \leq 2$; $\gamma_{\rm pr2}(G)$, $\Gamma_{\rm pr2}(G)$.
(h) *convex domination*: $N[S] = V$ and for any two vertices $u, v \in S$, the vertices contained in all shortest paths between u and v, called $u - v$ geodesics, belong to S; $\gamma_{\rm conv}(G)$, $\Gamma_{\rm conv}(G)$.
(i) *weakly convex domination*: $N[S] = V$ and for any two vertices $u, v \in S$, there exists at least one $u - v$ geodesic, all of whose vertices belong to S; $\gamma_{\rm wconv}(G)$, $\Gamma_{\rm wconv}(G)$.
(j) *cycle domination*: $N[S] = V$ and $G[S]$ has a Hamilton cycle; $\gamma_{\rm cy}(G)$, $\Gamma_{\rm cy}(G)$.
(k) *equivalence domination*: $N[S] = V$ and $G[S]$ is disjoint union of complete subgraphs; $\gamma_e(G)$, $\Gamma_e(G)$.
(l) *k-dependent domination*: $N[S] = V$ and $\Delta(G[S]) \leq k$; $\gamma_{[k]}(G)$, $\Gamma_{[k]}(G)$.

4 Conditions on $\overline{S} = V \setminus S$

The framework considered in this section encompasses dominating sets S for which some condition is imposed on the vertices in the set $\overline{S}$ or the subgraph $G[\overline{S}]$ induced by $\overline{S}$. As before, we list a sampling of types of dominating sets in this framework. In many of them, we do not mention the upper parameters, which indicates that in general they have not been studied. As before, the designations H hereditary and SH superhereditary are given whenever they apply.

(a) *distance k-domination*: for every $v \in \overline{S}$, there exists a vertex $u \in S$ with $d(u, v) \leq k$; SH; $\gamma_{\leq k}(G)$, $\Gamma_{\leq k}(G)$.
(b) *k-step domination*: for every $v \in \overline{S}$, there exists a vertex $u \in S$ and a (u, v)-path of length equal to k; SH; $\gamma_{=k}(G)$.
(c) *k-domination*: for every vertex $v \in \overline{S}$, $d_S(v) \geq k$; SH; $\gamma_k(G)$.
(d) *restrained domination*: $N[S] = V$ and for every $v \in \overline{S}$, $d_{\overline{S}}(v) \geq 1$; $\gamma_r(G)$.
(e) *geodetic domination*: every vertex in $\overline{S}$ lies on a shortest path between two vertices in S; SH; $\gamma_g(G)$.
(f) *locating domination*: $N[S] = V$ and for every $v, w \in \overline{S}$, $N(v) \cap S \neq N(w) \cap S$; SH; $\gamma_L(G)$.
(g) *secondary domination*: every vertex $w \in \overline{S}$ is adjacent to at least one vertex $u \in S$ and is distance at most k to a second vertex in S; SH; $\gamma_{(1,k)}(G)$. Note that for any nontrivial graph without isolated vertices, $\gamma(G) = \gamma_{(1,4)}(G)$ and $\gamma_2(G) = \gamma_{(1,1)}(G)$.
(h) *downhill domination*: for every vertex $v \in \overline{S}$, there exists a vertex $u \in S$ and a (downhill) path $u = v_1, v_2, \ldots v_k = v$ from u to v, such that $d(v_i) \geq d(v_{i+1})$ for all $i \in [k-1]$; SH; $\gamma_{\text{down}}(G)$.
(i) *uphill domination*: for every vertex $v \in \overline{S}$, there exists a vertex $u \in S$ and an (uphill) path $u = v_1, v_2, \ldots v_k = v$ from u to v, such that $d(v_i) \leq d(v_{i+1})$ for all $i \in [k-1]$; SH; $\gamma_{\text{up}}(G)$.
(j) *exponential domination*: for every vertex $v \in \overline{S}$, $\mathrm{w}_s(v) \geq 1$, where

$$\mathrm{w}_s(v) = \sum_{u \in S} \frac{1}{2^{\overline{d}(u,v)-1}},$$

and $\overline{d}(u, v)$ equals the length of a shortest (u, v)-path in $V \setminus (S \setminus \{u\})$ if such a path exists, and ∞ otherwise; SH; $\gamma_{\exp}(G)$.
(k) *fair domination*: $N[S] = V$ and every two vertices $u, v \in \overline{S}$ have the same number of neighbors in S; $\text{fdom}(G)$.
(l) *H-forming domination*: every vertex $v \in \overline{S}$ is contained in a copy of a graph H (not necessarily induced) with a subset of vertices in S; SH; $\gamma_H(G)$.
(m) *outer-connected domination*: $N[S] = V$ and $G[\overline{S}]$ is connected; $\gamma_{\overline{c}}(G)$.
(n) *b-disjunctive domination*: for every $v \in \overline{S}$, either v is adjacent to a vertex $u \in S$ or there exist at least b vertices in S at distance 2 from v; SH; $\gamma_b^d(G)$.

(o) *secure domination*: $N[S] = V$ and for every vertex $u \in \overline{S}$, there is an adjacent vertex $v \in S$ such that the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set; SH; $\gamma_s(G)$.

5 Conditions on *V*

In this section, we consider a framework where the dominating set is defined by an added condition that is imposed on every vertex of G.

(a) *total domination*: $N(S) = V$, that is, for every vertex $v \in V$, $N(v) \cap S \neq \emptyset$; SH; $\gamma_t(G)$, $\Gamma_t(G)$.
(b) *odd domination*: $N[S] = V$, and for every $v \in V$, $|N[v] \cap S|$ is odd; $\gamma_{\text{odd}}(G)$. It is noteworthy that Sutner [7] was the first to observe that every graph G has an odd dominating set.
(c) *even domination*: $N[S] = V$, and for every $v \in V$, $|N[v] \cap S|$ is even; $\gamma_{\text{even}}(G)$.
(d) *identifying code number*: $N[S] = V$, and for every $v \in V$, $N[v] \cap S$ is unique; SH; $\gamma_{\text{id}}(G)$.
(e) *total distance k-dominating*: for every vertex $v \in V$, there exists a vertex $u \in S$, $u \neq v$, such that $d(u, v) \leq k$; SH; $\gamma_t^k(G)$.
(f) *k-tuple domination*: for every $v \in V$, $|N[v] \cap S| \geq k$; SH; $\gamma_{\times k}(G)$.

6 Conditions on Vertex Degrees

As we will see in this section, many types of dominating sets can be defined in terms of how many neighbors a vertex must have in either S or $\overline{S}$. These constraints are often perceived as requirements of access to the resources provided by members of a dominating set.

6.1 Degree Conditions on S and $\overline{S}$

Degree conditions as a framework of domination was first suggested by Telle [8]. We present a slightly different form of his framework here. There are four possible values under consideration, namely, $d_S(v)$ and $d_{\overline{S}}(v)$ for $v \in S$, and $d_S(v)$ and $d_{\overline{S}}(v)$ for $v \in \overline{S}$. Table 1 illustrates how with using combinations of these four values, different domination parameters are defined. We only include a few of the many parameters which can be defined by various combinations of the four degree values. A blank entry in Table 1 implies that this condition is not relevant to the definition. Let D-set, TD-set, ID-set, and RD-set denote dominating set, total dominating set, independent dominating set and restrained dominating set, respectively.

Table 1 Degree Conditions

S is	$v \in S, d_S(v)$	$v \in S, d_{\overline{S}}(v)$	$v \in \overline{S}, d_S(v)$	$v \in \overline{S}, d_{\overline{S}}(v)$
a D-set			≥ 1	
an ID-set	$= 0$		≥ 1	
a TD-set	≥ 1		≥ 1	
a perfect dominating set			$= 1$	
an RD-set			≥ 1	≥ 1
a k-dominating set			$\geq k$	
a D-set and $\overline{S}$ is a D-set		≥ 1	≥ 1	
a $[1, k]$-dominating set			≥ 1 and $\leq k$	
an odd D-set	even		odd	
an open odd D-set	odd		odd	
an efficient D-set	=0		$= 1$	
a 1-dependent D-set	≤ 1		≥ 1	

6.2 Degree Conditions Per Vertex

As in the previous section, the framework here is defined in terms of the minimum cardinality of a nonempty set S satisfying the stated conditions based on degree. The difference is that the constraints now depend on comparative comparative values of degrees. Recall that the boundary of a set S is $\partial(S) = N[S] \setminus S$.

(a) *alpha domination*: for every $v \in \overline{S}$, $d_S(v)/d(v) \geq \alpha$ where $0 < \alpha \leq 1$; SH; $\gamma_\alpha(G)$.
(b) *defensive alliance*: for every $v \in S$, $d_S[v] \geq d_{\overline{S}}(v)$; $a(G)$.
(c) *defensive k-alliance*: for every $v \in S$, $d_S(v) \geq d_{\overline{S}}(v) + k$; $a_k(G)$. Note that for $k = -1$, a defensive k-alliance is the standard defensive alliance, that is, $a_{-1}(G) = a(G)$.
(d) *global defensive alliance*: $N[S] = V$ and for every $v \in S$, $d_S[v] \geq d_{\overline{S}}(v)$; $\gamma_a(G)$.
(e) *offensive alliance*: for every $v \in \partial(S)$, $d_S(v) \geq d_{\overline{S}}[v]$; $a_o(G)$.
(f) *offensive k-alliance*: for every $v \in \partial(S)$, $d_S(v) \geq d_{\overline{S}}(v) + k$; $a_{ok}(G)$. Note that for $k = 1$, a k-offensive alliance is the normal offensive alliance.
(g) *global offensive alliance*: for every $v \in \overline{S}$, $d_S(v) \geq d_{\overline{S}}[v]$; $\gamma_{a_o}(G)$.
(h) *powerful alliance*: for every $u \in S$, $d_S[u] \geq d_{\overline{S}}(u)$ and for every $v \in \partial(S)$, $d_S(v) \geq d_{\overline{S}}[v]$; $a_p(G)$.
(i) *(static) monopoly*: for every vertex $v \in \overline{S}$, $d_S(v) \geq d_{\overline{S}}(v)$, that is, every vertex not in S has at least $\lceil d(v)/2 \rceil$ neighbors in S, or equivalently, every vertex in $\overline{S}$ has at least as many neighbors in S as it has in $\overline{S}$; SH; $m(G)$.
(j) *open, or total, monopoly*: for every vertex $v \in V$, $d_S(v) \geq d_{\overline{S}}(v)$, that is, every vertex in V has at least as many neighbors in S as it has in $\overline{S}$; SH; $m_t(G)$.
(k) *weak domination*: for every $v \in \overline{S}$, there exists a neighbor $u \in S$, $d(u) \leq d(v)$; SH; $\gamma_w(G)$.

(l) *strong domination*: for every $v \in \overline{S}$, there exists a neighbor $u \in S$, $d(u) \geq d(v)$; SH; $\gamma_s(G)$.
(m) *cost effective domination*: $N[S] = V$ and for every $v \in S$, $d_S(v) \leq d_{\overline{S}}(v)$; $\gamma_{\text{ce}}(G)$.
(n) *very cost effective domination*: $N[S] = V$ and for every $v \in S$, $d_S(v) < d_{\overline{S}}(v)$; $\gamma_{\text{vce}}(G)$.
(o) 1-*equitable domination*: $N[S] = V$ and for all $u, v \in S$, $|d_{\overline{S}}(u) - d_{\overline{S}}(v)| \leq 1$; $\gamma_{1eq}(G)$.
(p) 2-*equitable domination*: $N[S] = V$ and for all $u, v \in \overline{S}$, $|d_S(u) - d_S(v)| \leq 1$; $\gamma_{2eq}(G)$
(q) *equitable domination*: $N[S] = V$ and for all $u, v \in S$, $|d_{\overline{S}}(u) - d_{\overline{S}}(v)| \leq 1$, and for all $u, v \in \overline{S}$, $|d_S(u) - d_S(v)| \leq 1$; $\gamma_{eq}(G)$.
(r) *global distribution center*: $N[S] = V$ and for all $v \in \overline{S}$, there exists a vertex $u \in S$ such that $d_S[u] \geq d_{\overline{S}}[v]$; SH; $\text{gdc}(G)$.

7 Functions $f : V \to \mathbb{N}$

For every set $S \subseteq V$, there is a corresponding *characteristic function* $f_S : V \to \{0, 1\}$, such that $f(v) = 1$ if $v \in S$, and $f(v) = 0$ if $v \in \overline{S}$. This suggests a variety of options for the range $\mathbb{N}$ of a function $f : V \to \mathbb{N}$, in terms of domination. In this section, we present a sample of the functions that have been considered under this framework. The value of each of the following parameters equals the minimum weight of a function of the given type, where the *weight* $\text{w}(f)$ of such a function f is the sum of all assigned values,

$$\text{w}(f) = \sum_{v \in V} f(v).$$

7.1 Dominating Functions

(a) *domination*: $f : V \to \{0, 1\}$, for every vertex $v \in V$, $f(N[v]) \geq 1$; $\gamma(G)$.
(b) *fractional domination*: $f : V \to [0, 1]$, for every vertex $v \in V$, $f(N[v]) \geq 1$; $\gamma_f(G)$.
(c) *signed domination*: $f : V \to \{-1, 1\}$, for every vertex $v \in V$, $f(N[v]) \geq 1$; $\gamma_s(G)$.
(d) *minus domination*: $f : V \to \{-1, 0, 1\}$, for every vertex $v \in V$, $f(N[v]) \geq 1$; $\gamma_m(G)$.
(e) $\{k\}$-*domination*: $f : V \to \{0, 1, \ldots, k\}$, for every vertex $v \in V$, $f(N[v]) \geq k$; $\gamma_{\{k\}}(G)$.

(f) *k-rainbow domination*: $f : V \rightarrow \mathcal{P}\{1, 2, \ldots, k\}$, every vertex $v \in V$ is assigned a subset of $\{1, 2, \ldots, k\}$ such that for every vertex $v \in V$ with $f(v) = \emptyset$, the union of the sets assigned to the closed neighborhood $N[v]$ equals $\{1, 2, \ldots, k\}$; $\gamma_{rk}(G)$.

7.2 Roman Dominating Functions

The types of domination in this section are models of a military defense strategy instituted by Emperor Constantine, between 306 and 337 AD, in which the regions in the Roman Empire were defended by armies stationed at key locations. A region was secured by armies stationed there, and a region without an army was protected by sending mobile armies from neighboring regions. But Constantine decreed that a mobile field army could not be sent to defend a region, if doing so left its original region unsecured. This defense strategy gave rise to what is called *Roman domination*, given below. As in the previous section, the value of each of the following domination parameters equals the minimum weight of a function of the given type.

Definition 1 Roman domination*:* $f : V \rightarrow \{0, 1, 2\}$*, for every vertex* v *with* $f(v) = 0$*, there is a vertex* $u \in N(v)$ *with* $f(u) = 2$*;* $\gamma_R(G)$*.*

It is easy to see, for example, that for every graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$. From the initial definition of Roman domination as a framework, many varieties of domination can clearly be defined, and indeed, many have been defined. We only provide a sample here.

(a) *weak Roman domination*: $f : V \rightarrow \{0, 1, 2\}$, for every v with $f(v) = 0$, there is a vertex $u \in N(v)$ with $f(u) > 0$ such that the function f' with $f'(v) = 1$, $f'(u) = f(u) - 1$, and $f'(w) = f(w)$ otherwise, has no *undefended vertex*, meaning a vertex with $f'(N[w]) = 0$; $\gamma_r(G)$.
(b) *double Roman domination*: $f : V \rightarrow \{0, 1, 2, 3\}$, every vertex w with $f(w) = 0$ either has a neighbor u with $f(u) = 3$ or two neighbors u, v with $f(u) = f(v) = 2$, and if $f(w) = 1$, then w has at least one neighbor u with $2 \leq f(u) \leq 3$; $\gamma_{dR}(G)$.
(c) *Roman {2}-domination, also called Italian domination*: $f : V \rightarrow \{0, 1, 2\}$, every vertex v with $f(v) = 0$ has $f(N(v)) \geq 2$; $\gamma_{R2}(G)$ (also $\gamma_I(G)$).
(d) *Roman k-domination*: $f : V \rightarrow \{0, 1, 2\}$, every vertex v with $f(v) = 0$ is adjacent to at least k vertices u with $f(u) = 2$; $\gamma_{kR}(G)$.
(e) *independent Roman domination*: $f : V \rightarrow \{0, 1, 2\}$, every vertex v with $f(v) = 0$ has at least one neighbor u with $f(u) = 2$ and the set of vertices w with $f(w) > 0$ is an independent set; $i_R(G)$.
(f) *signed Roman domination*: $f : V \rightarrow \{-1, 1, 2\}$, for every vertex $v \in V$, $f(N[v]) \geq 1$, and every vertex v with $f(v) = -1$ has at least one neighbor u with $f(u) = 2$; $\gamma_{sR}(G)$.