# Studies in Choice and Welfare



For further volumes:
http://www.springer.com/series/6869

William V. Gehrlein • Dominique Lepelley

# Voting Paradoxes and Group Coherence

## The Condorcet Efficiency of Voting Rules

Springer

Professor William V. Gehrlein
Department of Business Administration
Alfred Lerner College
of Business & Economics
University of Delaware
303 Alfred Lerner Hall
Newark Delaware 19716
USA
wvg@udel.edu

Professor Dominique Lepelley
Faculté de Droit et d'Economie
Université de La Réunion
15, avenue René Cassin
BP 7151
97715 Saint-Denis Messag Cedex 9
Ile de La Réunion
FRANCE
dominique.lepelley@univ-reunion.fr

Studies in Choice and Welfare ISSN 1614-0311
ISBN 978-3-642-03106-9 e-ISBN 978-3-642-03107-6
DOI 10.1007/978-3-642-03107-6
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2010938117



*Cover design:* WMXDesign GmbH, Heidelberg, Germany

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

*To three people who did a great deal for their little brother:*
*Gordon (Pete), Linda and Bob*

*A mes parents:*
*Françoise et René Lepelley*

# Preface

An extraordinary amount of research has been conducted on the general topic of Voting Paradoxes. It has been studied for over two centuries by philosophers, mathematicians, economists, political scientists and other interested people from many different backgrounds. It has fascinated numerous people to think about the very strange and counterintuitive outcomes that might possibly be observed when a group of decision makers, or voters, takes on the task of selecting a winning candidate from a set of available candidates. Books have been written to describe many of these paradoxical outcomes and to categorize them according to the types of unusual behaviors that they display.

The most famous of these paradoxical outcomes is Condorcet's Paradox, or the Condorcet Effect, which is named after the renowned eighteenth century French mathematician-philosopher who formally described the phenomenon. Condorcet wrote at length about the possibility that cyclical majorities on pairs of candidates might occur, and he made some attempts to assess the likelihood that such an outcome might happen. Condorcet was also adamant in his assertion that if some candidate, that we call a Pairwise Majority Rule Winner (PMRW), would be capable of defeating each of the other candidates on the basis of paired comparisons by majority rule, then that candidate should be selected as representing the best choice according to the voters' preferences. As a result, this principle has become known as the Condorcet Criterion.

Much effort has been expended since Condorcet's early work to obtain probability representations for the likelihood that voting paradoxes will be observed in election settings. The basic motivation has been to determine if these possible paradoxical events might actually pose real threats to elections. The level of sophistication of the techniques that have been used to assess the probability that voting paradoxes will be observed has advanced at a very significant rate in recent years. These advances have allowed for the introduction of new dimensions into the formal probability representations that can be obtained. These new dimensions specifically allow for the consideration of the degree to which a group of decision makers, or voters, displays various measures of group mutual coherence. This led to the ultimate conclusion that while Condorcet's Paradox is a fascinating concept to think about, it should actually be a rare event in actual election settings with a small number of candidates, whenever a group of voters displays any significant level of group mutual coherence for any of a number of possible measures of such coherence.

Given that as a starting point, we began this study with two objectives in mind. First, it was of interest to investigate other voting paradoxes to determine if they too would suffer the same fate of being shown to be interesting phenomena to study, while having very little chance of ever being observed in reality. The second objective resulted from the fact that since Condorcet's Paradox should be a relatively rare event, there is a high probability that a PMRW will exist, to make the Condorcet Criterion very relevant. We therefore wanted to investigate the propensity of common voting rules to elect the PMRW, with an emphasis on an analysis of the impact that various levels of group mutual coherence might have on that outcome.

Our goal throughout was to integrate the theoretical results that we were obtaining from formal probability representations with empirical results from other studies. Some voting paradoxes are definitely more paradoxical than others, and it obviously can not be shown that all voting paradoxes should be very rare events. However, the more extreme paradoxes are generally found to pose very little threat to actual elections, in agreement with empirical findings. The study of the propensities of common voting rules to meet the Condorcet Criterion produces mixed results. Most voting rules can perform very well, depending upon the model that describes the mechanism with which group mutual coherence is attained. However, it is found that while Borda Rule is not always the most effective voting rule for selecting the PMRW in all scenarios, it is resistant to the potential problem of performing very poorly. Moreover, scenarios do exist for all other common voting rules in which the possible outcome of very poor performance is a significant issue. Borda Rule is also found to have a number of very interesting additional properties, to make it a very good choice as a voting rule. This all leads us to suggest the *Borda Compromise* position, to avoid the possibility of poor performance with other voting rules, when nothing is known a priori about the general structure of preferences for a group of voters.

A significant effort was made in our literature search to include references to all work that is directly related to the specific topic of interest. Apologies are extended in advance if we accidently overlooked some relevant related studies. On a personal note, Gehrlein wishes to extend sincere gratitude to the many people who have been supportive and encouraging through the long course of this project. This particularly includes his wife Barbara Eller, who has been the most supportive and encouraging of all. Lepelley is very grateful to Maurice Salles for introducing him to the wonderful world of Voting Theory, to Bill Gehrlein for his trust and to his wife Françoise for her constant support and patience throughout these last 35 years.

Newark, DE, USA William V. Gehrlein
La Réunion Island, FRANCE Dominique Lepelley
September 2010

# Contents

# Chapter 1
# Voting Paradoxes and Their Probabilities

## 1.1 Introduction

An extraordinary amount of research effort has been dedicated to the application of formal mathematical modeling techniques to the analysis of the question: "How should a group of individual decision-makers go about the process of selecting some alternative that can be viewed as being the best among a set of available alternatives?" Any group decision-making situation of this type can be viewed in the context of an election in which the available alternatives correspond to the candidates in the election, and where the alternative that is selected as the overall best corresponds to the winning candidate in the election. The individual decision-makers within the group are consequently acting as the voters in the election scenario. The first scholars to analyze such voting situations with formal mathematical modeling techniques were the eighteenth century contemporary French mathematician-philosophers Jean Charles de Borda and Marie Jean Antoine Nicolas Caritat, the Marquis de Condorcet. Thus, the mathematical analysis of this problem has a very long history.

The process of determining how groups of individual decision makers might go about selecting an overall best alternative in different situations is consistently discussed in the context of election procedures throughout this study. However, it is noted that a strong link between elections and general decision-making situations can be forged by observing that the same election procedures that we shall discuss later are used in many actual group decision-making studies, including: forestry management (Kangas et al. 2006; Palander and Laukkanen 2006), land use management (McDaniels and Thomas 1999), water resource management (Rosen and Sexton 1993; D'Angelo et al. 1998) and the evaluation of engineering designs (Dyer and Miles 1976; Dym et al. 2002).

Our attention is typically restricted to elections in which each voter has the same level of impact on the voting process, so that no subgroup of voters has more influence on the outcome of the voting process than does any other subgroup with the same number of voters once individual voter's preferences on candidates have

W.V. Gehrlein and D. Lepelley, *Voting Paradoxes and Group Coherence*, Studies in Choice and Welfare, DOI 10.1007/978-3-642-03107-6_1,

been formed. This does not prohibit the possibility that some individuals might be more persuasive than others in arguing for their particular viewpoint during preliminary debate before voting, but once the individual voters have determined their particular preferences on the candidates, each voter will then have the same degree of influence on the election outcome.

The determination of the winner of an election is a very simple task for any situation in which all voters have the same most preferred candidate, and all voters will get their most preferred outcome if that candidate is selected as the winner. However, it should be expected that there will almost always be some disagreement among the voters in an election as to which candidate is the best for selection as the winner, so that it will not be possible for each of the individual voters to get their most preferred outcome. The determination of the particular candidate that best represents the overall most preferred candidate of the group becomes a much more difficult problem to address in that case, since many different criteria can be used to measure the degree of how well each candidate represents the position of being the overall most preferred candidate of the group.

Any group of voters will almost certainly arrive at the conclusion of applying the notion of majority rule when there are only two candidates, so that the candidate that is more preferred by the greater number of voters will be selected as the winner. A sense of fairness suggests that the group should select that candidate, in order to provide the better outcome for the most voters with our assumption of equal voter influence in the process. An extensive analysis of the issue of the fairness of majority rule voting was developed by Rousseau (1762), and Rousseau's arguments are summarized in Young (1988).

Some writers have presented arguments that oppose the notion of the basic fairness that results from implementing majority rule, and these opposing arguments are typically centered on the fact that majority rule ignores the intensity of preferences of voters. Don Joseph Isadore Morales of Spain wrote a paper after reading about some work of Jean Charles de Borda (Borda 1784) that ignored intensity of preference in voting procedures, and the content of Morales' work is discussed in Daunou (1803). One of Morales' arguments was that situations could exist in which there is a minority group of voters who have a very strong preference that an issue should be adopted, while the majority of voters are marginally opposed to having it adopted. If the sizes of the two voting groups were nearly equal in such a case, Morales argues that the strong preference of the minority should outweigh the majority opinion. This leads to the conclusion that voting procedures should ask the individual voters to report some measure of their degree of preference for candidates, as opposed to asking for simple approve or disapprove responses.

We follow the same direction as most other researchers in this area and ignore the issue of intensity of preference. Vickery (1960) summarizes the logic behind this decision by noting that most voters have significant problems simply in correctly determining any actual differences that exist between candidates, without even considering the additional complexity that would result for voting systems that attempt to evaluate the strength of preference of individual voters. However, the argument about the appropriateness of ignoring intensity of individual voter's preferences is still not fully resolved (Tullock 1959; Ward 1961; Downs 1961; Bordley 1986; Saari 1995a; Baharad and Nitzan 2002).

By ignoring the issue of intensity of preference, we are in complete agreement with ideas that are originally proposed by Condorcet (Condorcet 1788a), where it is stressed that any election procedure must be kept as simple as possible, with only a series of simple ‘yes’ or ‘no’ responses being required from voters. Condorcet’s ideas in this particular area were a definite precursor to the notions that were expressed above from Vickery (1960).

## 1.2 The Case of More than Two Candidates

The basic concept of majority rule can take on different interpretations when more than two candidates are being considered, making the problem of selecting the winner much more complicated. Borda and Condorcet found that very counterintuitive election outcomes could be observed when these different interpretations of majority rule are used for elections with more than two candidates, and these possible unusual occurrences in voting events are referred to as *voting paradoxes*.

To develop formal definitions of these different interpretations of majority rule with more than two candidates, we start by defining some restrictions on the preferences that rational individual voters might have on candidates. Suppose that three candidates, $\{A, B, C\}$, are available for consideration in an election, and let $A \succ B$ denote the outcome that a given individual voter prefers Candidate $A$ to Candidate $B$. A voter’s preferences on pairs of candidates from a set of candidates are *complete preferences* if such a preference relation exists on each of the possible pairs of candidates. Since either $A \succ B$ or $B \succ A$ for all pairs of candidates like $A$ and $B$ when an individual voter’s preferences are complete, no voter indifference is allowed to exist between any two candidates. We assume that individual voter preferences are complete for now, but this assumption will be relaxed later to allow for some voter indifference between candidates.

It is also assumed that each of the individual voters has *transitive preferences* on the candidates. Transitivity is a very commonly used requirement in the definition of rational behavior in the context of individual voter’s preferences. If a given voter has preferences on pairs of candidates with $A \succ B$ and $B \succ C$, then transitivity requires that this voter must also have $A \succ C$. Transitivity prevents the existence of a situation in which any voter might respond in a cyclic fashion, such as $A \succ B$, $B \succ C$ and $C \succ A$. Condorcet (1785a) makes reference to the possibility that such cyclic preferences might exist as a “contradiction of terms”, and Condorcet (1788a) later stresses the importance of developing voting models to “make such absurdities impossible.” The use of the assumption of transitivity as one of the standards for rationality for individual voter’s preferences has nearly universal acceptance. However, just as in the earlier discussion of the general belief that intensity of preferences should play no role in majority rule voting, some studies have been conducted to focus on the development of individual preference models to explain why it might occasionally be reasonable to expect intransitive individual preferences. Gehrlein (1990a, 1994) presents surveys of this work.

Individual voter preferences on candidates that are both complete and transitive are *linear preference rankings*, and Fig. 1.1 shows each of the six possible linear preference rankings that each voter might have in a three-candidate election.

Here, $n_i$ denotes the number of voters that have the associated linear preference ranking on the three candidates, so that $n_1$ voters all have individual preferences with $A \succ B \succ C$, along with $A \succ C$ from the assumption of transitivity. Let $n$ define the total number of voters, with $n = \sum_{i=1}^{6} n_i$. A *voting situation*, $\boldsymbol{n}$, denotes any particular combination of $n_i$'s that sum to $n$. Voting situations just report the $n_i$ values that are associated with each possible individual preference ranking for a given election, without specifying the preferences of any individual voter. A *voter preference profile*, or *voter profile*, gives a complete list that shows the specific linear preference order that is held by each individual voter. A voting situation can be obtained directly from a voter profile simply by determining the number of voters within the profile that have each of the possible linear preference rankings. As a result, voters' preferences are not anonymous in the case of a voter profile, but they are in a voting situation.

There are two different ways that we use to extend the notion of majority rule to the case of more than two candidates. The most obvious of these extensions is widely known as *Plurality Rule* (*PR*). Each voter casts a vote for his or her most preferred candidate with PR, and the election winner is the candidate who receives the greatest number of votes. Let ***APB*** denote the event that $A$ beats $B$ by PR. Assuming that all of the voters will cast votes in agreement with their true preferences, $A$ will be the PR winner of in a three-candidate election if both ***APB*** [ $n_1 + n_2 > n_3 + n_5$] and ***APC*** [ $n_1 + n_2 > n_4 + n_6$]. Voters will always be assumed to vote in accordance with their true preferences throughout this study.

Borda (1784) considers the second extension of majority rule to the case of three-candidate elections by looking at the basic majority rule relation as it is applied to pairs of candidates. Let ***AMB*** denote the event that $A$ defeats $B$ by *Pairwise Majority Rule* (*PMR*) when only the preferences on the pair of candidates $A$ and $B$ are considered in voters' preference rankings, with the relative position of $C$ being completely ignored. Using the possible preference rankings on three candidates that are given in Fig. 1.1, it follows directly that ***AMB*** if $n_1 + n_2 + n_4 > n_3 + n_5 + n_6$, ***AMC*** if $n_1 + n_2 + n_3 > n_4 + n_5 + n_6$, and ***BMC*** if $n_1 + n_3 + n_5 > n_2 + n_4 + n_6$. Then, Candidate $A$ will be the winner by PMR, or the *Pairwise Majority Rule Winner* (*PMRW*), for the three-candidate case when both ***AMB*** and ***AMC***. The PMRW is commonly referred to as the *Condorcet Winner* in the literature, since Condorcet was a very strong advocate of the argument that the PMRW should always be selected as the winner of an election. If voters'

| | | | | | |
|---|---|---|---|---|---|
| $A$ | $A$ | $B$ | $C$ | $B$ | $C$ |
| $B$ | $C$ | $A$ | $A$ | $C$ | $B$ |
| $C$ | $B$ | $C$ | $B$ | $A$ | $A$ |
| $n_1$ | $n_2$ | $n_3$ | $n_4$ | $n_5$ | $n_6$ |

**Fig. 1.1** The six possible linear preference rankings on three candidates

preferences in a voting situation are such that both $AMC$ and $BMC$, then $C$ is the *Pairwise Majority Rule Loser* (*PMRL*) for the three-candidate case. The definitions of PMR, PMRW and PMRL are extended in the obvious fashion when there are more than three candidates in an election. It is possible that a PMR tie can exist on a pair of candidates when $n$ is even, and such ties are not considered with the definition of PMR that is given above, to make this definition refer to a *Strict PMR*, *Strict PMRW* and *Strict PMRL*.

Both Borda and Condorcet made some fascinating mathematically based observations about some of the possible paradoxical results that can be observed when more than two candidates are being considered in voting situations with these definitions of PR and PMR. These paradoxical results are discussed in detail in the next section as part of a general overview of the many different types of voting paradoxes that can be observed.

## 1.3 Voting Paradoxes

Many surveys of voting paradoxes exist in the literature (Fishburn 1974a; Brams 1976; Niemi and Riker 1976; Petit and Térouanne 1987; Nurmi 1998). Nurmi (1999) categorizes voting paradoxes into four groups: Incompatibility Paradoxes, Monotonicity Paradoxes, Choice Set Paradoxes and Representation Paradoxes. These results are summarized in the context of earlier discussion, following Gehrlein and Lepelley (2004), with some additional results. Representation Paradoxes that are presented in Nurmi (1999) are not directly related to the topic of the current study, so they are not discussed. Most of the paradoxes that are mentioned below will be discussed in detail later in this study. For now, we only give a brief overview of the types of voting paradoxes that can be observed.

### *1.3.1 Incompatibility Paradoxes*

*Incompatibility Paradoxes* represent voting situations in which there are multiple reasonable definitions as to which candidate should be viewed as being the ‘best’ possible candidate among the set of available candidates, and where these definitions cannot be satisfied simultaneously by a voting rule. When we apply this notion with the two reasonable definitions of having the ‘best’ candidate being determined by the use of PMR to obtain the PMRW and the use of PR to determine the winner, three classic incompatibility paradoxes can be observed.

#### 1.3.1.1 Condorcet’s Paradox

*Condorcet’s Paradox* is developed in Condorcet (1785b) with a famous example of a voting situation with 60 voters on three candidates, as shown in Fig. 1.2.

**Fig. 1.2** A voting situation showing a PMR cycle from Condorcet (1785b)

| | | | | |
|---|---|---|---|---|
| $A$ | $B$ | $B$ | $C$ | $C$ |
| $B$ | $A$ | $C$ | $A$ | $B$ |
| $C$ | $C$ | $A$ | $B$ | $A$ |
| $n_1 = 23$ | $n_3 = 2$ | $n_4 = 17$ | $n_5 = 10$ | $n_6 = 8$ |

Condorcet notes a very strange outcome, which he referred to as a "*contradictory system*", when PMR is used with this voting situation. In particular, we find that PMR comparisons lead to: ***AMB*** (33–27), ***BMC*** (42–18), and ***CMA*** (35–25). So, there is an intransitive cycle on the PMR relation on the three candidates, with no candidate emerging as being superior to each of the remaining candidates. Given Condorcet's strong arguments that the PMRW should always be selected as the winner of an election, we are left with a difficult question in this case: 'Which candidate should be selected as the winner, when a majority of voters would prefer that another candidate should be selected as the winner, regardless of which candidate you select?'

Condorcet (1785c) continues with his analysis of intransitive PMR voting situations, to show that there might be a PMRW with more than three candidates, while a PMR cycle might exist among some subset of the remaining candidates. Thus, a distinction is made between the possibility that there is a PMRW and the possibility that the PMR is completely transitive over all candidates. With only three candidates, the existence of a PMRW ensures that the PMR ranking is transitive for odd $n$. Condorcet notes that the possible existence of this situation on more than three candidates is of no consequence to the superiority of the PMRW among the candidates, as long as only one candidate is being elected.

It was noted earlier that Condorcet was quite adamant in his argument that a lack of transitivity of preference for individual voters was so contradictory, that a system must be used to eliminate "such absurdities". However, after eliminating intransitivity from the preferences of individual voters, we find that collective choice of voters with PMR still might produce intransitive results, suggesting that an irrational response can exist in the collective choice of a set of rational voters. An exhaustive survey of research on Condorcet's Paradox is presented in Gehrlein (2006a) and much of what we present on that particular topic in the current study is taken from that source.

#### 1.3.1.2 Borda's Paradox

*Borda's Paradox* results from a very interesting observation regarding possible conflicts between the outcomes of using PMR and PR to determine the winner of an election in Borda (1784). Borda's original example of this phenomenon uses the voting situation in Fig. 1.3 for 21 voters with linear preferences in a three-candidate election.

If PR is used with the voting situation in Fig. 1.3, ***APB*** (8–7), ***APC*** (8–6) and ***BPC*** (7–6) to give a linear ranking by PR, with ***APBPC***. A very different result is

**Fig. 1.3** An example voting situation displaying Borda's Paradox from Borda (1784)

| | | | |
|---|---|---|---|
| $A$ | $A$ | $B$ | $C$ |
| $B$ | $C$ | $C$ | $B$ |
| $C$ | $B$ | $A$ | $A$ |
| $n_1 = 1$ | $n_2 = 7$ | $n_5 = 7$ | $n_6 = 6$ |

observed using PMR. Here, $BMA$ (13–8), $CMA$ (13–8) and $CMB$ (13–8) to give a linear PMR ranking, with $CMBMA$. With this particular voting situation, PR and PMR reverse the election rankings on the three candidates. We refer to this specific phenomenon as representing an occurrence of a *Strict Borda Paradox*.

Borda was particularly distressed by the fact that the PMRL could be chosen as the winner by PR, leading to his suggestion that PR should never be used. Borda (1784) also suggests that Candidate $C$, the PMRW, "is really the favourite" for the voting situation in Fig. 1.3, in agreement with the arguments of Condorcet. However, the primary concern that is expressed in Borda's work is the possibility of the negative outcome that the PMRL could be selected as the winner by PR. We define a *Strong Borda Paradox* as a situation in which PR elects the PMRL, without necessarily having a complete reversal in PR and PMR rankings. The least stringent form of this general paradox is a *Weak Borda Paradox*, in which PR reverses the rankings by PMR on some pair of candidates, without necessarily electing the PMRL as the overall PR winner.

Borda (1784) proposed an election procedure to be used in order to deal with the possibility that various forms of Borda's Paradox might occur. The procedure that he referred to as "*election by order of merit*" has come to be widely known as *Borda Rule* (*BR*). Each voter starts the implementation of BR by listing their respective preference ranking on the candidates, where a rank of one refers to a voter's least preferred candidate and a rank of $m$ refers to the voter's most preferred candidate in an $m$-candidate election. Then, each voter's most preferred candidate is given $a + (m - 1)b$ points, the second most preferred candidate is given $a + (m - 2)b$ points, and so on until the least preferred candidate is given $a + (m - m)b$ points. The election winner is determined by summing the points that each candidate receives from all of the voters, and declaring the candidate with the most points as the winner. Borda suggests using the particular weighting scheme with $a = b = 1$, so that the number of points that are awarded to a candidate by a given voter is equivalent to the rank that the candidate has in that voter's preference ranking on the candidates.

For a general voting situation, as described in Fig. 1.1, with $n$ voters and three candidates, the *Borda Score* for $A$, $B$ and $C$ under BR with a weighting scheme with $a = b = 1$ would respectively be $BS(A)$, $BS(B)$ and $BS(C)$ with:

$$\begin{aligned}
BS(A) &= 3(n_1 + n_2) + 2(n_3 + n_4) + 1(n_5 + n_6) \\
BS(B) &= 3(n_3 + n_5) + 2(n_1 + n_6) + 1(n_2 + n_4) \\
BS(C) &= 3(n_4 + n_6) + 2(n_2 + n_5) + 1(n_1 + n_3). \qquad (1.1)
\end{aligned}$$

For the particular example that is taken from Borda (1784) in Fig. 1.3, we obtain $BS(C) = 47$, $BS(B) = 42$, and $BS(A) = 37$. If we let $A\boldsymbol{B}B$ denote the event that $A$ beats $B$ by BR, we get a linear ranking on the candidates, with $C\boldsymbol{B}B\boldsymbol{B}A$. This ranking of candidates by BR is now in the reverse order of the ranking with PR, and it is in perfect agreement with the ranking that was obtained by PMR. Borda (1784) never clearly stated that the ranking with BR would always be the same as the ranking with PMR, and this is not true for all voting situations. It is also obvious that the definition of the Borda Score for any candidate in (1.1) is effectively equivalent to using a procedure that simply counts the total number of instances in which this given candidate is preferred to other candidates in voter preference rankings.

#### 1.3.1.3 Condorcet's Other Paradox

Condorcet (1785c) develops the general notion of a *Weighted Scoring Rule* (*WSR*), and BR is a special case of this type of rule. A general WSR gives some number of points to candidates according to their relative position within each individual voter's preference ranking. BR with $a = b = 1$ is a form of a WSR that assigns weights of 3, 2 and 1 respectively for each first, second and third place ranking in voters' preferences. The winner is then determined as the candidate who receives the most total points. For three-candidate elections, we consistently define a WSR as one that assigns weights of 1, $\lambda$ and 0 for each first, second and third place ranking in voters' preferences. We restrict $0 \leq \lambda \leq 1$ since it would not make sense to award more points to the middle ranked candidate in a voter's preference ranking than to the most preferred candidate in the ranking, or to award fewer points to the middle ranked candidate than to the least preferred candidate. It is very simple to show that BR is completely equivalent to our definition of a WSR with $\lambda = 1/2$.

Condorcet (1785c) gives the example voting situation in Fig. 1.4 to show a phenomenon that Fishburn (1974a) refers to as *Condorcet's Other Paradox*.

Condorcet notes that $A\boldsymbol{M}B$ (41-40) and $A\boldsymbol{M}C$ (61-20) in this voting situation, so that Candidate $A$ is the PMRW, and then goes on to compute $Score(A, \lambda)$ and $Score(B, \lambda)$ for the WSR with weights 1, $\lambda$ and 0, with:

$$\begin{aligned} Score(A, \lambda) &= 1^*31 + \lambda^*39 + 0^*11 \\ Score(B, \lambda) &= 1^*39 + \lambda^*31 + 0^*11. \end{aligned} \tag{1.2}$$

| | | | | | |
|---|---|---|---|---|---|
| $A$ | $A$ | $B$ | $C$ | $B$ | $C$ |
| $B$ | $C$ | $A$ | $A$ | $C$ | $B$ |
| $C$ | $B$ | $C$ | $B$ | $A$ | $A$ |
| $n_1 = 30$ | $n_2 = 1$ | $n_3 = 29$ | $n_4 = 10$ | $n_5 = 10$ | $n_6 = 1$ |

**Fig. 1.4** A voting situation showing Condorcet's Other Paradox from Condorcet (1785c)

In order for Candidate $A$ to be elected by this WSR, we must have:

$$\begin{aligned}
&Score(A,\lambda) > Score(B,\lambda)\\
&31 + 39\lambda > 39 + 31\lambda\\
&8\lambda > 8\\
&\lambda > 1.
\end{aligned} \tag{1.3}$$

This contradicts our definition of a WSR, so that no WSR, including BR, can elect the PMRW in this example, which is Condorcet's Other Paradox.

### 1.3.2 Monotonicity Paradoxes

*Monotonicity Paradoxes* represent situations in which some reasonable definition has been established to determine which candidate should be viewed as being the 'best' available candidate, and where a voting rule has been selected and that voting rule is not monotonic. *Monotonicity* of a voting rule requires consistency of election outcomes as voters' preferences change. That is, increased support (decreased support) for a candidate in voters' preferences should not be detrimental (beneficial) to that candidate in the election outcome.

#### 1.3.2.1 No Show Paradox

The *No Show Paradox* is developed in Brams and Fishburn (1983a), with an example in which some subset of voters chooses not to participate in an election, and then prefers the resulting winner to the winner that would have been selected if they had actually participated in the election. The winner of an election is determined by *Negative Plurality Elimination Rule* (*NPER*) in a three-candidate election in this example. A *two-stage election procedure* is needed to implement NPER. In the first stage, voters cast votes for their two more preferred candidates. The candidate that receives the fewest number of votes is then eliminated, and the ultimate winner is selected in the second stage by using PMR on the remaining two candidates. The voting rule that is used in the first stage is referred to as *Negative Plurality Rule* (*NPR*) since it is equivalent to having each voter cast a negative vote against their least preferred candidate, with the candidate who receives the most negative votes being eliminated.

Consider a voting situation with 21 voters and three candidates $\{A, B, C\}$, as shown in Fig. 1.5 from Brams and Fishburn (1983a).

In the first stage of voting with NPR, Candidates $A$, $B$, and $C$ receive 15, 14 and 13 votes respectively. Candidate $C$ is therefore eliminated in the first stage and then $BMA$ by a vote of 11 to 10 in the second stage, to select $B$ as the overall winner.

**Fig. 1.5** An example voting situation from Brams and Fishburn (1983a)

| | | | | | |
|---|---|---|---|---|---|
| $A$ | $A$ | $B$ | $C$ | $B$ | $C$ |
| $B$ | $C$ | $A$ | $A$ | $C$ | $B$ |
| $C$ | $B$ | $C$ | $B$ | $A$ | $A$ |
| $n_1 = 3$ | $n_2 = 5$ | $n_3 = 5$ | $n_4 = 2$ | $n_5 = 3$ | $n_6 = 3$ |

**Fig. 1.6** The modified example voting situation from Brams and Fishburn (1983a)

| | | | | | |
|---|---|---|---|---|---|
| $A$ | $A$ | $B$ | $C$ | $B$ | $C$ |
| $B$ | $C$ | $A$ | $A$ | $C$ | $B$ |
| $C$ | $B$ | $C$ | $B$ | $A$ | $A$ |
| $n_1 = 1$ | $n_2 = 5$ | $n_3 = 5$ | $n_4 = 2$ | $n_5 = 3$ | $n_6 = 3$ |

Voters in this voting situation with the linear preference ranking $A \succ B \succ C$ would not get their most preferred candidate, since $B$ is the election winner. Suppose that two of these particular voters had not participated in this election for some reason. The voting situation that would have resulted is shown in Fig. 1.6.

In the first stage of voting on this modified voting situation with NPR with 19 voters, Candidates $A$, $B$, and $C$ receive 13, 12 and 13 votes respectively. Candidate $B$ is eliminated in the first stage and then $AMC$ by a vote of eleven to eight in the second stage. Since the winner in this modified voting situation is $A$, the two voters with linear preferences $A \succ B \succ C$ who did not participate will now have their most preferred candidate chosen as the winner. These two voters have therefore obtained a more preferred outcome from the election with NPER by not participating in the election, which violates the definition of monotonicity.

NPER does not necessarily elect the PMRW. However, Moulin (1988a) proved that any election procedure that does meet the condition that it must select the PMRW, when one exists on four or more candidates, must be subject to the possibility that the No Show Paradox can be observed. Pérez (2001) produces the same general observation as Moulin (1988) while considering two variations of this paradox. Ray (1986) had previously developed the notion of the No Show Paradox in the context of a less commonly used voting rule known as Single Transferable Vote.

#### 1.3.2.2 Additional Support Paradox

The *Additional Support Paradox* reverses the scenario of the No Show Paradox. In this case, a specified candidate will win with some voting rule for a given voting situation. Then, a new voting situation is created from the original voting situation in which some voters increase their support for the winning candidate by improving the position of that candidate in their preference rankings, with all other things remaining the same. Then, the voting rule winner in the modified voting situation is no longer the original winning candidate. This situation violates the notion of monotonicity, and this paradox is discussed in Richelson (1979), Straffin (1980), Fishburn (1982), and Nurmi (1987).

### 1.3.3 Choice Set Variance Paradoxes

*Choice Set Variance Paradoxes* represent situations in which a series of propositions are put before voters, where each individual issue will be approved or disapproved by majority rule voting. A paradoxical result then arises when the overall final election outcome on the propositions represents a result that is somehow inconsistent with the underlying preferences of the voters.

#### 1.3.3.1 Ostrogorski's Paradox

Suppose that there are $m$ independent issues that are to be presented to $n$ voters and that each individual issue will be approved or disapproved by majority rule voting. There are two parties, $R$ and $L$, that have opposing positions on each of the issues. Each voter therefore has a position that is in agreement with either Party $R$ or Party $L$ on each individual issue, but each voter does not necessarily agree with the position of the same party on every issue. A voter is considered to be a member of Party $R$ (Party $L$) if their individual position on issues is in agreement with Party $R$ (Party $L$) over a majority of the issues that are being considered. The outcome of voting on each issue will be determined to be in agreement Party $R$, or Party $L$, based on the majority rule outcome of voting on that issue. A *Strict Ostrogorski Paradox* occurs if a majority of voters have preferences that make them members of Party $R$ (Party $L$), while Party $L$ (Party $R$) has an election outcome on every issue that is in agreement with its position. A *Weak Ostrogorski Paradox* occurs if a majority of voters have preferences to make them members of Party $R$ (Party $L$), while Party $L$ (Party $R$) has a majority of election outcomes on issues that are in agreement with its position. This paradox was first presented in Ostrogorski (1902) and it will be discussed in detail in Chap. 4.

#### 1.3.3.2 Majority Paradox

As in the description of Ostrogorski's Paradox, there are $m$ issues that will be presented to $n$ voters, and each issue will be approved or disapproved by majority rule voting. Parties $R$ and $L$ have opposing positions on each issue, and each voter has a position on each issue that is in agreement with either Party $R$ or Party $L$. Each voter does not necessarily agree with the position of the same party on every issue. The outcome of voting on each issue will be in agreement Party $R$ or Party $L$, based on the majority rule voting. Party $R$ (Party $L$) is the *Overall Majority Party* (*OMP*) if there are more $R$ ($L$) entries than $L$ ($R$) entries in the $mn$ different party position associations for preferences of the voters over all of the issues.

The *Majority Paradox* occurs if the OMP is selected as the winner in a minority of elections on issues. There can not be a Strict Majority Paradox, as in the case of Ostrogorski's Paradox, since if any party is the winner by majority rule for every issue, then that same party must also be the OMP. The Majority Paradox was presented in Feix et al. (2004) and it will be discussed in detail in Chap. 4.

#### 1.3.3.3 Paradox of Multiple Elections

The *Paradox of Multiple Elections* was first presented in Brams et al. (1998), where there are $m$ independent issues that are to be presented to $n$ voters in a series of elections. Parties $R$ and $L$ have opposing positions on each of the issues, and each voter has a position on each issue that is in agreement with either Party $R$ or Party $L$. Each voter does not necessarily agree with the position of the same party on every issue. The outcome of voting on each issue is determined to be in agreement Party $R$ or Party $L$, based on majority rule voting. The Paradox of Multiple Elections occurs if there is not at least one voter who has preferences that are in agreement with Party $L$ – Party $R$ positions on each of the individual issues that are in agreement with final Party $R$ – Party $L$ position association of the majority rule vote outcomes on issues.

#### 1.3.3.4 Consistency Condition Paradox

The *Consistency Paradox* occurs when the winner by some voting rule for a given voting situation is not the same as the winner by the same rule on a subset of candidates that includes the original winner. The voters' preference rankings on the subset of candidates in the modified voting situation are assumed to remain the same as their relative ranking in the original voting situation. Variations of this paradox will be considered in detail in Chap. 7.

## 1.4 Empirical Evidence of the Existence of Voting Paradoxes

The voting paradox descriptions that are summarized above indicate that there is a distinct possibility that very counterintuitive election outcomes might be observed and create disruptions to elections. It is only natural that many empirical studies have been conducted to determine if any of these voting paradoxes pose a realistic threat to real election procedures.

### *1.4.1 Empirical Evidence of Condorcet's Paradox*

Condorcet's Paradox has received the great majority of attention in this line of investigation, since it results in the arguably most counterintuitive election outcome. Table 1.1 summarizes the results of numerous empirical studies that were discussed in detail in Gehrlein (2006a), along with some more recent results.

**Table 1.1** A summary of empirical studies looking for Condorcet's Paradox

| Source | Number of Elections | Candidates $m$ | Voters $n$ | Strict PMRW | Transitive PMR |
|---|---|---|---|---|---|
| Flood (1955) | 1 | 16 | 21 | Yes | No (1) |
| Riker (1958) | 1 | 4 | 255 | No (1) | No (1) |
| Riker (1965) | 1 | 3 | 426 | No (1) | No (1) |
| Niemi (1970) | 18 | 3–6 | 81–463 | No (4) | No (4) |
| Blydenburgh (1971) | 2 | 3 | 386 | No (1) | No (1) |
| Fishburn (1973a) | 1 | 5 | 175 | Yes | Yes |
| Bjurulf and Niemi (1978) | 1 | 3 | 87 | No (1) | No (1) |
| Dyer and Miles (1976) | 1 | 36 | 10 | Yes | No (1) |
| Riker (1982) | 2 | 3–4 | 172+ | No (2) | No (2) |
| Toda et al. (1982) | 1 | 6 | 5281 | Yes | Yes |
| Dobra (1983) | 32 | 3–37 | 4–27 | No (4) | No (?) |
| Chamberlin et al. (1984) | 5 | 5 | 11000+ | Yes | Yes |
| Dietz and Goodman (1987) | 1 | 4 | Large | Yes | Yes |
| Fishburn and Little (1988) | 3 | 3–5 | 1500+ | Yes | Yes |
| Rosen and Sexton (1993) | 1 | 4 | 31 | Yes | Yes |
| Radcliff (1994) | 4 | 3 | Large | Yes | Yes |
| Abramson et al. (1995) | 4 | 3 | Large | Yes | Yes |
| Gaubatz (1995) | 1 | 4 | Large | No (1) | No (1) |
| Browne and Hamm (1996) | 1 | 3 | 621 | No (1) | No (1) |
| Lagerspetz (1997) | 10 | 3–4 | 300 | No (2) | No (2) |
| Beck (1997) | 3 | 4–8 | 20 | No (1) | No (1) |
| Flanagan (1997) | 1 | 3 | 224 | No (1) | No (1) |
| Morse (1997) | 1 | 4 | 52 | No (1) | No (1) |
| Taylor (1997) | 1 | 3 | Large | Yes | Yes |
| Hsieh et al. (1997) | 1 | 3 | 450 | Yes | Yes |
| Taplin (1997) | 1 | 4 | 12 | Yes | Yes |
| Regenwetter and Grofman (1998) | 7 | 3 | Large | No (1) | No (1) |
| Truchon (1998) | 24 | 5–9 | 5–23 | Yes | No (15) |
| Van Deemen and Vergunst (1998) | 4 | 9–13 | 1500 | Yes | Yes |
| Stensholt (1999a) | 1 | 3 | 165 | No | No |
| Kurrild-Klitgaard (2001) | 1 | 3 | Large | No | No |
| Regenwetter et al. (2002a) | 8 | 3 | Large | Yes | Yes |
| Regenwetter et al. (2002b) | 3 | 3 | Large | Yes | Yes |
| Wilson (2003) | 1 | 3 | Large | Yes | Yes |
| Gehrlein (2004a) | 2 | 12–18 | 5 | Yes | No (1) |
| Kurrild-Klitgaard (2008) | 8 | 9–11 | 1000+ | Yes | Yes |
| Smith (2009) | 1 | 4 | Large | No (1) | No (1) |

The results in this table require that Strict PMR relations hold for the existence of a PMRW and for PMR transitivity.

To interpret the results in Table 1.1, we note for example that the study by Chamberlin et al. (1984) considers five different elections with five candidates in each election. There were at least 11000 voters in each election and the results showed that a PMRW existed in each case and that PMR was completely transitive in each case. The study in Niemi (1970) consisted of an examination of 18 elections, with the number of candidates ranging from 3 to 6 and the number of voters ranging from 81 to 463. A PMRW did not exist in four of the elections and PMR was not completely transitive in those same four elections. Obviously, if a PMRW does not exist for a voting situation then PMR can not be transitive for that voting situation. The study by Truchon (1998) found that a PMRW existed for all 24 elections that were considered, but that PMR was not transitive in 15 of these elections.

The results of Table 1.1 indicate that there is a possibility that Condorcet's Paradox might be observed, but that it probably is not a widespread phenomenon. This notion is further reinforced by two factors. First, it is much more likely that an observer would make the effort to write about examples in which they believed that this very interesting paradox might have occurred than they are to do so when it is not believed that such a paradox occurred.

The second major factor that has an impact on the relevance of these empirical studies was primarily promoted by Riker (1982), who presents many historical examples in which Condorcet's Paradox seems to have been present. Riker argues strongly that the existence of PMR cycles have typically been created artificially by the introduction of amendments, by the introduction of campaign issues, or by the misrepresentation of voters' preferences to manipulate the outcome of an election. Bjurulf and Niemi (1978), Chamberlin (1986), Levmore (1999), and Tullock (2000) all agree with Riker, to varying degrees, that PMR cycles are typically contrived. However, the ability of individuals to create artificial PMR cycles to the degree that Riker suggests is disputed. For example, Maske and Durden (2003) present a survey of some opposing viewpoints to Riker's arguments. Other studies, such as Browne and Hamm (1996), clearly state that no evidence was found of strategic misrepresentation of any kind in the actual situations for which a PMR cycle was found. An analysis of the studies in Table 1.1 also indicates that PMR cycles were found to exist in situations in which there is no plausible reason to expect that any type of manipulation to create PMR cycles would have been taking place. In conclusion, these empirical studies provide very strong evidence that Condorcet's Paradox has been observed in some voting situations. However, it should not be expected to be an event that occurs frequently, and the likelihood of its occurrence is quite possibly overstated in the results of Table 1.1, for the reasons that are noted above.

Tideman (1992) performs the most thorough study of empirical data to determine if PMR cycles ever actually exist. The results of 84 different elections that were overseen by the Electoral Reform Society of Great Britain and Ireland are examined in that study along with the results of three additional elections. Voters were requested to rank all of the candidates in all cases, but they did not always do so. Candidates that were not reported in a voter's ranking were all listed as being indifferent to each other, and they were all ranked at the bottom of the voter's preferences. The number of candidates ranged from 3 to 29 and the number of voters ranged from nine to 3,500. There was complete transitivity, allowing for tied PMR voting, in 61 of the 87 elections.

Tideman makes a number of very interesting general observations for the 26 remaining elections in the study for which strict PMR was not completely transitive. Moreover, all of these observations are totally consistent with the results of all of the empirical studies that are summarized in Table 1.1:

- Elections with a few candidates almost always have transitive PMR orderings.
- Pairs of candidates that are ranked by a small number of voters are more likely to be involved in a PMR cycle than pairs that are ranked by many voters.
- The size of majorities on pairs that are involved in PMR cycles tends to be small, even after accounting for the fact that these typically involve a small number of voters.
- Candidates that are involved in PMR cycles tend to be located near the center of the overall PMR ranking. So, candidates that are most preferred, or most disliked, by the electorate are not likely to be involved in PMR cycles.
- PMR cycles typically contain pairs that are ranked relatively close together in the overall PMR ranking.

### 1.4.2 Empirical Evidence of Borda's Paradox

Borda's Paradox also is certainly counterintuitive, but the possibility of its existence is not as striking as the possibility that Condorcet's Paradox might occur. As a result, fewer empirical studies have been conducted in attempts to discover if any of the forms of Borda's Paradox that were discussed above have occurred in practice. Table 1.2 summarizes the results of these studies, and there were a large number of voters in all cases.

A total of 270 elections were analyzed in the studies from Table 1.2, and the results for the first study are obtained from combined information from Weber (1978a) and Riker (1982). There was only one observation of a Strict Borda Paradox, and only five of the studies that only looked at a single election showed evidence that a Strong Borda Paradox occurred. There is much more evidence that a Weak Borda Paradox might occur. These findings are consistent with the general

**Table 1.2** Summary of empirical studies looking for Borda's Paradox

| Source | Number of Elections | Candidates $m$ | Strict Borda Paradox | Strong Borda Paradox | Weak Borda Paradox |
|---|---|---|---|---|---|
| Weber (1978a), Riker (1982) | 1 | 3 | No | Yes | Yes |
| Riker (1982) | 1 | 3 | No | Yes | Yes |
| Van Newenhizen (1992) | 1 | 3 | No | Yes | Yes |
| Taylor (1997) | 1 | 3 | No | Yes | Yes |
| Colman and Poutney (1978) | 261 | 3 | No | No | Yes (14) |
| Bezembinder (1996) | 4 | 7 | No | No | Yes (?) |
| Niou (2001) | 1 | 3 | Yes | Yes | Yes |

conclusion in Fishburn (1974a), where a survey of different voting paradoxes is given. Monte-Carlo computer simulation estimates were obtained for the likelihood that each paradox might occur, and it was concluded that the most extreme forms of voting paradoxes are probably very rare in practice.

All of these findings lead to the ultimate conclusion that Borda's Paradox can exist, although it might not be a regularly observed phenomenon. As in the case of the empirical evidence of Condorcet's Paradox, it is much more likely that an observer would make the effort to write about examples in which they believed that an interesting voting paradox might have occurred than they are to do so when it is not believed that such a paradox occurred. So, the results in Table 1.2 are likely to overestimate the probability that various forms of Borda's Paradox might be observed in practice.

## 1.5 Probability Representations for Voting Paradoxes

Many studies have been conducted to develop formal mathematical representations for the probability that various voting paradoxes might be observed, and this particular approach to the problem is the primary focus of the current study. The history of studies of this type goes back to the work of Condorcet, who wrote the following statement while discussing his extensive work on the analysis of election procedures and voting paradoxes (Condorcet 1793, p. 7):

> But after considering the facts, the average values or the results, we still need to determine their probability.

The degree of sophistication that has been used in the methods that have been developed to obtain these probability representations has evolved significantly over the more recent decades since the work of Guilbaud (1952), which will be discussed later. This increased degree of sophistication has largely resulted from efforts that were being made to reconcile the predicted likelihoods of voting outcomes from these mathematical models with the observed likelihoods of their outcomes from empirical studies. In each of these models, different assumptions are made about the relative likelihood that a randomly selected voter profile or voting situation will be observed, so that various measurable characteristics of the resulting voter profiles or voting situations that are generated by these models will change. As an ultimate result of these studies, much has been learned about the relationship between these measurable characteristics of voter profiles or voting situations and the probability that different voting paradoxes will be observed.

We discuss these various mathematical modeling procedures here as they apply to the development of probability representations for the likelihood that Condorcet's Paradox will be observed, since Condorcet's Paradox has received the most attention in the literature. Surveys of much of this work are given in Gehrlein (1983, 1997). The same models will then be brought back later in the process of developing representations for the probability of observing other voting paradoxes.

Once the notions behind these models are presented, we shall go on to consider the distinctions between these different models and to assess what can be discerned from the results that are obtained from each. The first studies in this area considered the likelihood that various voter profiles will be observed, so that is where we begin.

### *1.5.1 Multinomial Probability Models for Voter Profiles*

The probability that any given voter preference profile will be observed can be considered to be the result of the random selection of $n$ individual voter's preference rankings on the candidates. In this situation, we let $\boldsymbol{p}$ denote a six-dimensional vector for the three-candidate case, where $p_i$ denotes the probability that a randomly selected voter from the population of potential voters will have the corresponding possible linear preference ranking on candidates that is shown in Fig. 1.7. That is, a randomly selected voter will have the linear preference ranking $A \succ B \succ C$ with probability $p_1$. We also make a critical assumption here that each voter's preference ranking on candidates is arrived at independently of the other voters' preferences.

Following the standard methods that are used in a classical analysis of this type of problem with probability modeling, we start with an urn that contains some total number of balls, with each ball being one of six different colors. Each color corresponds to one of the six possible linear preference rankings on the three candidates. The proportions of the total number of balls of each color in the urn are equal to their associated probabilities for the population that are specified in $\boldsymbol{p}$. Then, balls are sequentially drawn at random from the urn $n$ different times, with the selected ball being returned to the urn after its color is noted on each draw. The random selection of balls is being done with replacement during the experiment so that the probability of observing any particular possible preference ranking for an individual voter does not change from draw to draw. The color of the ball that is drawn during the $i$th step of this sequential drawing is used to assign the associated linear preference ranking to the $i$th voter before the ball is placed back in the urn. Following previous discussion, this procedure is used to obtain voter preference profiles in which the preferences of each individual voter are identifiable, so that the voter's preferences are not anonymous.

A multinomial probability model is appropriate for use in developing representations for observing any particular given event under such an experiment. As noted previously, the voting situation, $\boldsymbol{n}$, that results from any given voter preference profile with its identifiable voters can be obtained simply by determining the

| | | | | | |
|---|---|---|---|---|---|
| $A$ | $A$ | $B$ | $C$ | $B$ | $C$ |
| $B$ | $C$ | $A$ | $A$ | $C$ | $B$ |
| $C$ | $B$ | $C$ | $B$ | $A$ | $A$ |
| $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ |

**Fig. 1.7** Probabilities for the six linear preference rankings on three candidates

number of voters in the voter preference profile that have each of the six possible linear preference rankings. The probability that any given $\boldsymbol{n}$ will be observed from the identifiable voters in a randomly generated voter preference profile is then given directly by the multinomial probability $n! \prod_{i=1}^{6} \frac{p_i^{n_i}}{n_i!}$.

The probability that any particular voting paradox will be observed can be obtained quite simply by enumerating all of the possible voting situations that lead to the existence of the given paradox, and summing the associated probabilities that each of these voting situations will be observed. For now, we restrict attention to the probability that Condorcet's Paradox will be observed, by considering the probability that a Strict PMRW will exist on three candidates.

The general restrictions that are necessary for a voting situation to have Candidate $A$ as the strict PMRW for the case of odd $n$ follows from earlier discussion as:

$$\begin{aligned} n_3 + n_5 + n_6 &\leq \frac{n-1}{2} \Rightarrow A\boldsymbol{M}B \\ n_4 + n_5 + n_6 &\leq \frac{n-1}{2} \Rightarrow A\boldsymbol{M}C. \end{aligned} \tag{1.4}$$

The restrictions that are needed for the individual $n_i$ terms to result in the conditions in (1.4) are given by:

$$\begin{aligned} &0 \leq n_6 \leq \frac{n-1}{2} \\ &0 \leq n_5 \leq \frac{n-1}{2} - n_6 \\ &0 \leq n_4 \leq \frac{n-1}{2} - n_6 - n_5 \\ &0 \leq n_3 \leq \frac{n-1}{2} - n_6 - n_5 \\ &0 \leq n_2 \leq n - n_6 - n_5 - n_4 - n_3 \\ &n_1 = n - n_6 - n_5 - n_4 - n_3 - n_2. \end{aligned} \tag{1.5}$$

A representation for the probability, $P_{PMRW}^{\{A\}}(3, n, \boldsymbol{p})$, that Candidate $A$ is the strict PMRW for odd $n$ for any given $\boldsymbol{p}$ follows directly as

$$P_{PMRW}^{\{A\}}(3, n, \boldsymbol{p}) = \sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} n! \prod_{i=1}^{6} \frac{p_i^{n_i}}{n_i!}, \tag{1.6}$$

where $n_1 = n - n_6 - n_5 - n_4 - n_3 - n_2$. Similar logic can then be used to find representations for the probability that each of $B$ and $C$ is the PMRW. The probability, $P_{PMRW}^{S}(3, n, \boldsymbol{p})$, that a Strict PMRW exists for a given $\boldsymbol{p}$ with $n$ voters