Reinhard Kahle
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The Legacy of Kurt Schütte

Springer

The Legacy of Kurt Schütte

Kurt Schütte, 1909–1998

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Editors

The Legacy of Kurt Schütte

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Kurt Schütte, 1994

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Preface

This book is dedicated to Kurt Schütte, one of the greatest proof theorists of the 20th century. He was born in 1909, in the same year as Gerhard Gentzen, and died in 1998. Schütte studied in Berlin and Göttingen. He was David Hilbert's last doctoral student with a dissertation on the decision problem in mathematical logic, in 1933. He subsequently had to spend the war years up to 1945 as a meteorologist. After the war, gradually reestablishing himself in the academic world, Schütte became a pioneer in infinitary proof theory where one considers proof systems—dubbed "halbformale Systeme" by him—accommodating inference rules with infinitely many premises. The complexity of these derivations is measured via several ordinal magnitudes, and the cost of their proof-theoretic transformations, such as cut elimination, is calibrated in terms of ordinal functions, notably ones developed by G. Cantor and O. Veblen. In the 1950s the proof theory of systems with the ω-rule, a rule which had already been proposed by Hilbert [4], flourished in the hands of Schütte. He extended his approach to systems of ramified analysis, bringing this technique to perfection in his 1960 monograph "Beweistheorie" [11].

In the 1950s Schütte also greatly advanced techniques for proving completeness of proof systems. In his 1956 paper [9] (submitted 1954) he introduced the method of search trees (Suchbäume) where one associates a canonical tree (aka the universal tree), $\mathbb{B}_F$, with a formula F. With it he not only gave a very transparent proof of Gödel's completeness theorem for first order logic but also for infinitary ω-logic.[1] Moreover, in his 1956 paper Schütte also proved that infinitary ω-logic is already complete for cut-free computable derivations in which the premises of the ω-rule are always given by a recursive function (see [9, Theorem 6]).[2]

Schütte was a great minimalist. He saw deeply into Gentzen's calculus, discerning its potential for concentration, and developed the concepts of negative and positive

[1] The technique of search trees is related to Beth's semantic tableaux method [1] and Hintikka's sets of formulas method (nowadays called Hintikka sets) [5, 6] that were developed independently roughly at the same time.

[2] The latter result is sometimes referred to as Shoenfield's completeness theorem as Shoenfield established the completeness of the recursive ω-rule in 1959 in [17] via a different method (no canonical tree); but Schütte clearly has the priority here.

part that generalize Gentzen's of antecedent and succedent.[3] Schütte was rather fond of his calculus, but to his great regret even his former doctoral students later abandoned it in favor of sequent-type calculi. In a manuscript from 1991 (not meant for publication), Schütte recast the ordinal analysis of the theory *KPM* of [8] in his favorite framework, making several technical changes and stressing that the detour via a Tait-style calculus in [8] was superfluous as the essential distinctions are much more transparently captured via positive and negative parts: "*Der überflüssige Umweg über ein Tait-artiges System wurde vermieden, da sich ja mit den äußerst elementaren Begriffen der Positiv- und Negativteile in einfachster Weise genau das ausdrücken läßt, worauf es in der Beweistheorie ankommt.*" [16].

At the end of the 1950s, Schütte made an important contribution to the solution of Takeuti's Fundamental Conjecture (TFC), a problem that was at the center of attention of proof theorists at the time. TFC asserts the eliminability of all cuts in the simple theory of types. Some special cases studied by Takeuti himself via syntactic methods were encouraging. Schütte's main contribution to TFC was a reformulation in equivalent semantic terms, namely, that suitable partial valuations could be extended to total ones [10]. The Fundamental Conjecture was solved eventually along the lines of Schütte's reformulation, first by Bill Tait for second-order type theory and later by Dag Prawitz [7] and Moto-o Takahashi [19] in full and independently.

Perhaps Schütte's most famous contribution to logic is the determination of the limit of predicativity in the guise of the ordinal Γ_0 that he and Solomon Feferman achieved via different methods [2, 12, 13]. However, Schütte's proof-theoretic work of the 1950s and in his book Beweistheorie was crucial for the treatment of predicative analysis in both cases. Γ_0 is known as the Feferman-Schütte ordinal.

The type of predicativity delineated by this ordinal is one described in terms of autonomous progressions of theories, the autonomy condition being due to Kreisel who combined ideas of Poincaré and Russell on predicativity. Without any autonomy conditions, progressions of theories had been studied under the name of ordinal logics by Alan Turing in his 1939 Princeton doctoral thesis [20].

Schütte also worked on non-classical logics. Shortly after Kripke had introduced his semantics for modal and intuitionistic first-order logics, Schütte presented compactness, completeness and soundness proofs for these logic in his usual elegant way in a short monograph [14].[4]

In the year 1977, Springer published Schütte's third monograph with the title "Proof Theory" [15]. Originally it was planed as a translation of his "Beweistheorie" into English but it evolved into a completely different book. It is a masterly economical book that nobody can blame for being too talkative. As a student, the second editor of the present volume was intrigued by it and studied it from cover to cover. Despite having previously read monographs on model theory, recursion theory and set theory, nothing struck him as mysterious and difficult as "Proof Theory". It

[3] These parts constitute the fulcrum of the proof system in [9] and also of the proof system of the 1960 monograph [11] and everything that came after that. The related notion of signed formula was subsequently used in [18], but without reference to Schütte. See also p. 221, p. 223 this volume.

[4] For more on this book's significance see p. 235 this volume.

certainly lived up to Schütte's maxim: "The master builder removes the scaffolding when the building is completed".

The year 1988 saw the publication of "Proof Theory of Impredicative Subsystems of Analysis" by Wilfried Buchholz and Kurt Schütte. The ordinal analyses of theories in this book are based on Buchholz' Ω_σ-rules and the framework of Schütte's proof calculus of positive and negative forms. It is a difficult yet fascinating book that furnishes an amazingly compact and complete treatment of many subsystems of analysis within the span of just 119 pages.

After his retirement Schütte remained very active, keenly following the latest developments in ordinal analysis. By that time he was almost blind. When the second editor visited Schütte in his flat in the middle of the 1990s he helped him lighting his cigarettes as he could only feel their tip. It is quite a miraculous feat how under these circumstances Schütte could not only penetrate the technically intricate and most advanced ordinal analyses of *KPM* and $KP + \Pi_3$-Reflection but was able to recast them in his favorite calculus, modifying and simplifying the treatment in a very elegant way and thereby furnishing alternative approaches. The last part of this book makes some of these very late papers available for the first time. Hermann Weyl, in one of his last papers [21][5] wrote about aging as a mathematician, recalling a well-known passage from Hardy's *A Mathematician's Apology* [3]:

> *"The mood which Hardy's words reflect with such obvious sincerity is not alien to me who long ago passed sixty, and I agree wholeheartedly with him that 'mathematics is a young man's game.' "*

Fortunately, Schütte gave us a proof that this need not be one's fate.

This book is divided into four parts. The first, titled *History and Memories*, is devoted to Schütte's ways as a proof theorist and as a person. It also contains two articles on the history of logic and proof theory that Schütte (one jointly with Helmut Schwichtenberg) wrote in German. The second part, *Proof Theory at Work*, is concerned with current developments in proof theory that are very close to Schütte's own work in proof theory. The third part, *Further Legacy*, contains contributions by authors who were either in close contact with him, collaborated with him or look at ordinal analysis and constructivity from a different viewpoint. The last part, *Kurt Schüttes Spätwerk*, is comprised of four papers (three of which were previously unpublished) that Schütte wrote in the last decade of his life.

During the preparation of this volume two great logicians whose work was closely intertwined with Schütte's died: Solomon Feferman (1928–2016) and Gaisi Takeuti (1926–2017).

Feferman's contribution to this volume consists of a talk he gave at the Schütte memorial colloquium in Munich on 14 November 1999. It contains his own recollections of Schütte together with a list of challenges for proof theory on the eve of the year 2000. We are happy to include this paper in Chapter 2 of this volume.

Takeuti remembered Schütte in his "Memoirs of a Proof Theorist" [22], and we like to cite the corresponding passages:

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[5] Written after 1953; Weyl died in 1955 one month after his 70th birthday.

The reader may not believe it, but I think that hardly anybody in the world except my students, Schütte and his disciples read my articles on this subject seriously. Notably, Schütte's interest in my work was the result of Gödel's influence, as I shall explain. Gödel thought that, for the progress of my fundamental conjecture, it would be useful to put Schütte and myself together, and so he invited Schütte to the Institute for Advanced Study. When I dropped by at the Institute one late summer day, a stranger came directly to me, and asked "Would you know Takeuti?" I replied that I was the person himself and then he introduced himself as Schütte and told me: "I have just talked with Gödel, and found that Gödel is interested in your fundamental conjecture" and so on and so forth. I imagine Gödel told him that there might be some kind of relationship between what Schütte had worked on and my fundamental conjecture, and he suggested some research directions to him.

Schütte was a person other than Gödel who became interested in my work. I think Schütte's interest was aroused by Gödel. Looking back, we can say that Gödel's foresight was correct, considering that the results on my fundamental conjecture by Motoo Takahashi and Prawitz were based on Schütte's work done at the Institute at that time.

Thanks to Gödel, during the two years of my stay at the Institute, many logicians such as Bernays, Schütte, and Feferman were there. Smullyan and Putnam were at the University as well. They held a logic seminar every week, and the logic group was very lively. In particular, there were two proof-theorists in the rare Gentzen style together (Schütte and Takeuti), and so we were high-spirited. Smullyan would make me laugh by referring to us in a joking manner: "Is your name TakéSchütte?"

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Tübingen and Leeds, *Reinhard Kahle*
June 2020 *Michael Rathjen*

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Part IV Kurt Schüttes Spätwerk

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Part I
History and Memories

Chapter 1
"Sehr geehrter Herr Professor!"
Proof Theory in 1949 in a Letter from Schütte to Bernays

Reinhard Kahle

Abstract We present a letter which Kurt Schütte sent in 1949 to his former *de-facto* PhD supervisor Paul Bernays. This letter contains an outline of the proof-theoretic methods which became standard in infinitary proof theory.

1.1 Hilbert's Programme after Gödel and Gentzen

In the 1920s, David Hilbert had conceived a foundational programme in Mathematical Logic, which aimed to provide formal consistency proofs, carried out by "weak means", of formalized mathematics. One rationale behind it was to rebut the criticisms leveled against classical mathematics by intuitionism, and to beat Brouwer at his own game: the metamathematical tools were supposed to be acceptable from an intuitionistic point of view, and if one could prove the consistency of stronger theories by such tools, intuitionists would have to accept them. This was somehow the mathematical strategy of Hilbert, even if the full story was rather more involved, and, at some point, overshadowed by personal quarrels between Hilbert and Brouwer.

Initially, Hilbert had proposed *finitist mathematics* as the metamathematical framework to carry out the intended consistency proofs. Even without a clear specification of finitist mathematics, it is an immediate consequence of Gödel's results that Hilbert's original ideas cannot be carried out. According to testimonies of Bernays and Ackermann, Hilbert immediately adopted a "new meta-mathematical standpoint";[1] today this shift manifests itself by the replacement of finitist mathematics with *constructive mathematics*, as a framework which should still be acceptable

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[1] "Besonders interessiert hat mich der neue meta-mathematische Standpunkt, den Sie jetzt einnehmen und der durch die Gödelsche Arbeit veranlaßt worden ist." [1, p.1f].

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from an intuitionistic point of view. Soon after Gödel, Gentzen was able to provide a consistency proof of Arithmetic in terms of transfinite induction up to ε_0. While obviously no longer finitist, it is a consistency proof in terms of Hilbert's new methodological standpoint.[2] As much as this result has to be appreciated, one may observe that the development of proof theory in terms of consistency proofs for stronger systems than Arithmetic got somehow stuck. There were several important methodological advances in proof theory and, most notably, Bernays compiled the state-of-the-art of proof theory in the seminal two volume monograph *Grundlagen der Mathematik*, published together with Hilbert [11, 12]. But Gentzen was well aware of the fact that the consistency proof for Arithmetic could, at best, only be a first step towards a consistency proof for Analysis. He wrote in 1938, [6, p. 235f.]:

> Indeed, it seems not entirely unreasonable to me to suppose that *contradictions* might possibly be concealed even in classical *analysis*.
>
> . . . the most important [consistency] proof of all in practice, that for *analysis*, is still outstanding.

It is reported that Gentzen worked, up to the end of his life in 1945 in a prison in Prague, on a consistency proof for Analysis.[3] But no concrete result was published.

Schütte was originally attracted to mathematical logic by reading Oskar Becker's *Mathematische Existenz* [2] while studying in Berlin in 1930, [23, p. 93 in this volume]. Upon his return to Göttingen, he found the perfect environment for studying mathematical logic. Eventually, he finished his PhD under the *de-facto* supervision of Paul Bernays in 1933 with a thesis on the decision problem, [16]. David Hilbert was only his formal supervisor, and Schütte actually met him only twice personally, [23, p. 95 in this volume]. One may note that the topic of his thesis was not really about proof theory in the line of Hilbert's Programme. Apparently, by that time he was not particularly involved in Hilbert's Programme, neither in the early, finitist version, nor in the revised, constructive one. Concerning Gentzen—who finished his PhD only a couple of months after him, but already under the formal supervision of Hermann Weyl—he said that they had no personal contact, but only met occasionally in the hallways of the Mathematical Institute in Göttingen.[4] In any case, after his PhD Schütte first pursued the qualifications needed to be a secondary school teacher and switched, in 1937, to meteorology. He, thus, served during World War II as meteorologist in the German army.

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[2] For a more detailed discussion of Gentzen's results in this context, see [13].

[3] See, in particular, the last paragraph on page 3 of Schütte's letter, below.

[4] Personal communication by H. Schwichtenberg.

1.2 Schütte's Return to Logic

After the war, Schütte returned to Göttingen and finalized his school teacher education. Having been, for more than a decade, outside of University research, he met Arnold Schmidt by chance in Göttingen. Arnold Schmidt reports about this encounter to Paul Bernays in a letter from May 23rd, 1948:[5]

> Probably it might interest you that Mr. Schütte surfaced here again. During the Nazi time he was at the weather service and returned to school teaching after the end of the war. I met him him—after I had lost sight of him since the time I left Göttingen—some months ago here in Göttingen; and when I asked how his relation to logic had developed, he reacted entirely negatively. He kept himself busy with mathematics, in particular with algebra, once in a while in his leisure hours; but mathematical logic and foundational research doesn't interest him at all any longer; after all, there would be "absolutely nothing to do any longer"; it is totally thought through to the end and in a state of stagnation. I strongly pointed out to him that this would not be correct; and I told him that, if he would come to me, I could tell him immediately a dozen of interesting problems. Several months later he recently arrived; he had just finished his teacher exam and was keen on filling the ensuing compulsory leisure time with mathematics. I informed him about the state of the art of logic and foundational research and tried to give him a number of suggestions; to my joy he dedicates himself with increasing enthusiasm to the recommended reading and suggested lines of thought; maybe in a while it will lead to something interesting.

Bernays's replied, on July 31st, 1948, that he was very much interested and happy to hear about Schütte and that Schmidt should convey his regards.[6]

[5] German original:

> Übrigens wird es Sie vielleicht interessieren, dass Herr Schütte hier wieder aufgetaucht ist. Er war während der Nazizeit im Wetterdienst und ist nach Kriegsende zum Schuldienst zurückgekehrt. Ich traf ihn, nachdem ich ihn seit meinem damaligen Weggang von Göttingen ganz aus den Augen verloren hatte, vor einigen Monaten zufällig hier in Göttingen, und als ich fragte, wie sein Verhältnis zur Logik sich entwickelt habe, verhielt er sich vollkommen ablehnend. Er habe sich zwar mit Mathematik, vor allem mit Algebra, noch ab und zu in seinen Mussestunden etwas befasst, aber die mathematische Logik und Grundlagenforschung interessiere ihn garnicht mehr, da sei doch "garnichts mehr zu machen", sie sei doch völlig ausgedacht und stagniert. Ich habe ihn sehr eindringlich darauf hingewiesen, dass das nicht richtig sei, und habe ihm gesagt, wenn er zu mir komme, könne ich ihm gleich ein Dutzend interessanter Probleme sagen. Nach mehreren Monaten kam er vor kurzem an; er hatte sein Referendarexamen gerade gemacht und wollte die nun eintretende erzwungene Mussezeit mathematisch ausfüllen. Ich habe ihn über den Stand der Logik und Grundlagenforschung informiert und ihm eine Reihe von Anregungen zu geben versucht, und ich sehe zu meiner Freude, dass er sich mit wachsendem Eifer den von mir vorgeschlagenen Lektüren und Überlegungen widmet; möglicherweise kommt da in einiger Zeit etwas dabei heraus.

[6] "Was Sie mir über Herrn Schütte schreiben, hat mich lebhaft interessiert; es war mir sehr erfreulich, nach langer Zeit wieder Nachricht über ihn zu haben. Wenn Sie ihn wieder sprechen, möchten Sie ihn von mir grüssen."

On August 9th, 1948, Schmidt can report the first "fruit" of Schütte's renewed interest in Logic:[7]

> I will give your regards to Mr. Schütte. In my last letter I described to you, how he was initially totally opposed to mathematical logic and foundational research and as I succeeded very slowly to interest him again in the important problems. These efforts yielded in the meantime an extraordinary fruit. Mr. Schütte concerned himself with shortenings of Gentzen's proof, however, I could point out to him that these were defective. As a result, he fully committed himself with all his working power to the consistency problem; and after he had completely mastered the topic, as I could clearly see, he extended the method of Gentzen to a consistency proof of analysis, using the formalism of analysis which you presented at the end of the last appendix of the second volume of your book. After a first, rough perusal of the proof, it seems to me that this time the matter is fully correct; I hope to be able to carry out the more detailed checking in the coming week[.]
>
> For the time being, the crucial snag thereby is that Schütte's extension of Gentzen's procedure seems to be formalizable in analysis itself; Mr Schütte is intensely searching for the cause of the ensuing contradiction with Gödel's theorems, whereby he considers a mistake in his proof to be the least likely outcome; he rather believes that hidden side conditions of Gödel's theorem or his own procedure might be the culprit. Therefore, the issue is apparently in a state where one should not talk about it, however, I wanted to let especially you know, very informally, about the embryonic state of developments that take place here.

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[7] German original:

> Herrn Schütte werde ich Ihre Grüsse ausrichten. Ich schilderte Ihnen in meinem letzten Brief, wie er zunächst ganz gegen die mathematische Logik und Grundlagenforschung eingestellt war und wie es mir dann ganz allmählich gelang, ihn wieder für die wichtigen Probleme zu interessieren. Dieses Bemühen hat nun inzwischen eine ganz erstaunliche Frucht getragen. Herr Schütte befasste sich zunächst mit Abkürzungen des Gentzen-Beweises, die ich ihm allerdings als fehlerhaft nachweisen konnte. Daraufhin hat er sich nun wirklich mit vollem Ernst und ganzer Arbeitskraft auf das Widerspruchsfreiheitsproblem gestürzt, und nachdem er, wie ich deutlich feststellen konnte, ganz in der Sache firm war, hat er die Gentzensche Methode zu einem Widerspruchsfreiheitsbeweis der Analysis erweitert, wobei jener Formalismus der Analysis zugrundegelegt ist, den Sie im letzten Anhang des 2. Bandes Ihres Buches als letzten angaben. Bei erster, grober Durchsicht des Beweises scheint mir die Sache diesmal ganz intakt zu sein; die genauere Prüfung hoffe ich in der kommenden Woche vornehmen zu können[.]
>
> Der entscheidende Haken dabei ist vorläufig der, dass die von Schütte vorgenommene Erweiterung des Gentzenschen Verfahrens in der Analysis formalisierbar zu sein scheint; Herr Schütte ist eifrig dabei, die Ursache des hieraus entfliessenden scheinbaren Widerspruchs zum Gödelschen Satze nachzuspüren, wobei er einen Fehler in seinem Beweis für am wenigsten wahrscheinlich hält; eher glaubt er an versteckte Nebenvoraussetzungen des Gödelschen Satzes oder seines eigenen Verfahrens. Insofern ist die Angelegenheit offenbar noch in einem Zustande, in dem man nicht von ihr reden sollte, immerhin wollte ich gerade Ihnen ganz inoffiziell doch eben schon einmal von diesen neuen hiesigen embryonalen Entwicklungen Mitteilung machen.

1.3 Schütte to Bernays, August 26th, 1949

In Bernays's *Nachlass* one can find two letters of Schütte to him from 1934. Afterwards the correspondence apparently stopped and only on September 16th, 1948 Schütte resumed the correspondence. In this letter he gives his former supervisor a brief report about his life since 1935 (without even mentioning his new family) and then turning straight to logical matters in line with the style of Arnold Schmidt's letters to him. A letter of November 4th, 1948 reports on some further progress in his work, enclosing some manuscripts (which are not preserved). On April 16th, 1949 he sends Bernays a copy of what became his publication [17]. Four months later, on August 26th, 1949 he sends him a letter which we will reproduce in the following. It contains an outline of Schütte's strategy for the ordinal analysis of (weak systems of) analysis by use of the ω-rule, a strategy which one can consider to be the blueprint for *Schütte-style proof theory* as it is popular in proof theory up to this day.

Fig. 1.1 Photography from the meeting *Kolloquium zur Logistik und der mathematischen Grundlagenforschung „unter der geistigen Leitung von Paul Bernays"* in Oberwolfach in autumn 1949.[9]
From left to the right: Irmgard Süß, Hans-Heinrich Ostmann, Paul Bernays, Gisbert Hasenjäger, Arnold Schmidt, Herbert von Kaven, Kurt Schütte.
Sources: Archives of the Mathematisches Forschungsinstitut Oberwolfach and Universitätsarchiv Freiburg. Reprinted with permission.

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[9] Documents concerning this meeting can be found in the digital archive of the *Mathematisches Forschungsinstitut, Oberwolfach*: https://oda.mfo.de/handle/mfo/1998.

(20b) Göttingen, 26. August 1949
Hospitalstr. 4b.

Sehr geehrter Herr Professor!

Ich möchte mir erlauben, Ihnen von einer Abänderung meines Widerspruchsfreiheitsbeweises für die gestufte Analysis zu berichten. Gegenüber dem ursprünglichen Beweis, den ich Ihnen zugesandt hatte, ziehe ich jetzt nämlich eine etwas andere Beweisführung vor, die zu denselben und teilweise noch umfassenderen Ergebnissen führt, aber wesentlich durchsichtiger und kürzer ist. Ich glaube, hiermit eine Form gefunden zu haben, in welcher die Zusammenhänge am klarsten hervortreten. Verwertet werden

1. die von Gentzen entdeckte Möglichkeit, einen logischen Kalkul mit „umweglosen" Herleitungen zu entwickeln (Gentzenscher Hauptsatz),

2. die von Lorenzen entdeckte Möglichkeit, den Gentzenschen Hauptsatz auch unter Einbeziehung der vollständigen Induktion zu retten, und zwar durch Heranziehung von Schlüssen mit unendlich vielen Prämissen,

3. die von Gentzen vorgenommene Zuordnung von Ordnungszahlen zu den Herleitungen und die metamathematisch angewandte transfinite Induktion über ein Anfangsstück der II. Zahlklasse.

Diese Grundgedanken wende ich an:

1. mit meinem aufbauenden logischen Kalkul (ohne Sequenzen und mit logischen Grundzeichen für „oder", „nicht" und „alle"),

2. mit dem Schluß der „unendlichen Induktion"

$$\frac{\mathfrak{A}(z) \vee \mathfrak{N} \quad \text{für alle Ziffern } z}{(x)\,\mathfrak{A}(x) \vee \mathfrak{N}}$$

wie er entsprechend bereits von Hilbert empfohlen wurde,

3. mit einer Zuordnung von Ordnungszahlen zu den Herleitungsformeln, die nur folgenden Bedingungen unterliegt:

a) Bei einem umformenden Schluß haben Ober- und Unterformel gleiche Ordnung,

b) Bei einem aufbauenden Schluß und bei einem Schnitt hat die Unterformel eine größere Ordnung als jede Oberformel.

Dabei sind die Herleitungen finit beschreibbare unendliche Figuren, die noch gewissen Beschränkungen unterliegen. Der Widerspruchsfreiheitsbeweis wird in Form eines Beweises der Schnitt-Eliminierbarkeit geführt. Dieser Beweisgedanke entspricht vollkommen dem von Lorenzen angewandten, die Durchführung ist eine etwas andere. Ich glaube, daß meine Untersuchungen neben denen von Lorenzen deshalb nicht überflüssig sind, weil die

Fig. 1.2 Page 1 of Schütte's letter to Bernays.

Göttingen, August 26th, 1949
Hospitalstr. 4b

Dear Professor!

I would like to take the liberty to report to you on a modification of my consistency proof of stratified analysis.[10] Namely, in contrast to the original proof which I sent to you, I now prefer a slightly different proof strategy, which leads to the same but partly even more comprehensive results, however, which is substantially more transparent and shorter. I think that hereby I found a form in which the connections attain their clearest expression. Herein the following are utilized:

1. the possibility, discovered by Gentzen, of developing a logical calculus with "detour-free" derivations (Gentzen's *Hauptsatz*),
2. the possibility, discovered by Lorenzen, of preserving Gentzen's *Hauptsatz* also under inclusion of formal induction, namely by means of inferences with an infinite number of premises,
3. Gentzen's assignment of ordinals to derivations combined with transfinite induction applied metamathematically over an initial segment of the second number class.

The key ideas I apply together:

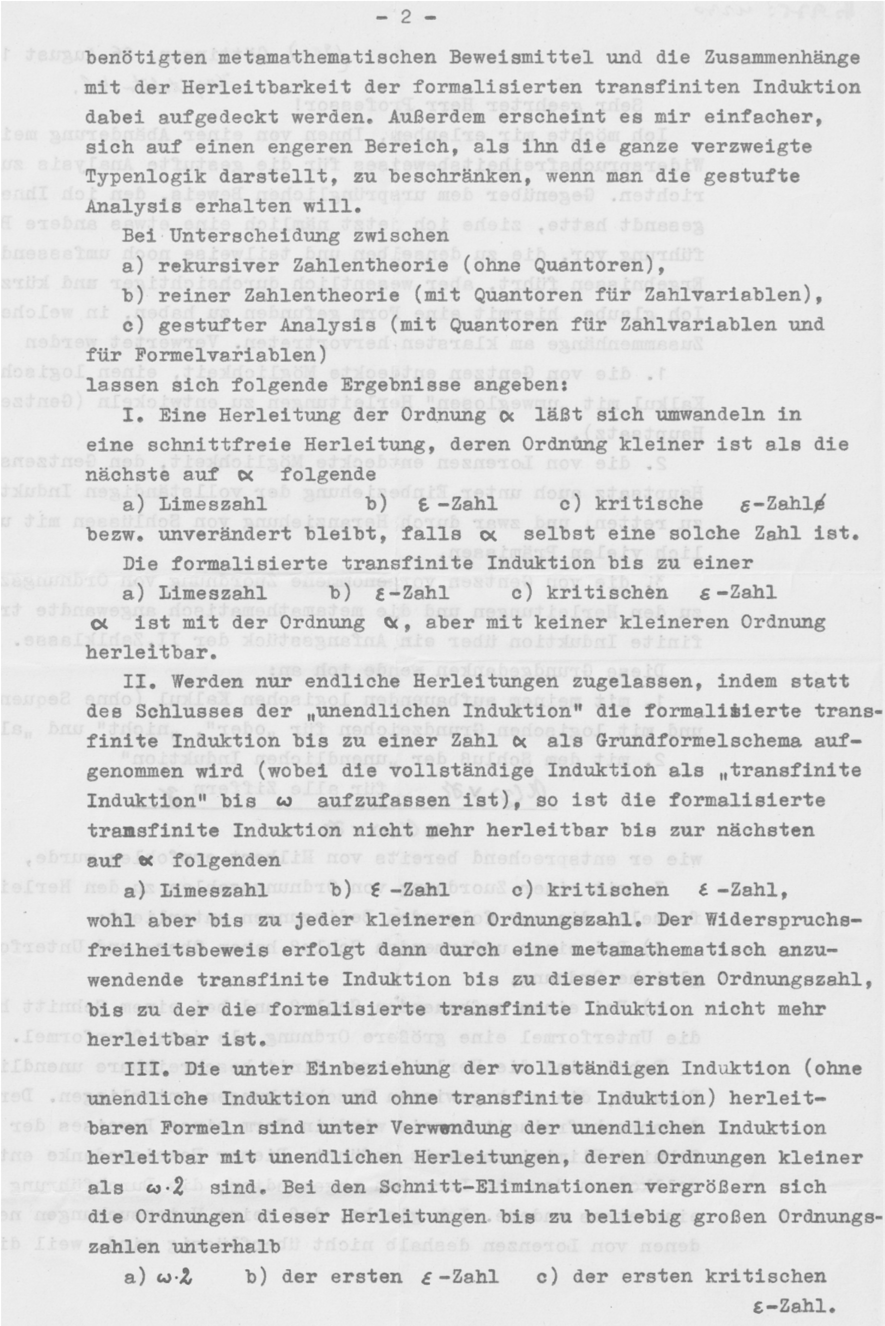
1. with my buildup logical calculus (without sequences, using the logical symbols for "or", "not", and "all"),
2. with the inference of "infinite induction"

$$\frac{\mathfrak{A}(\mathfrak{z}) \vee \mathfrak{N} \qquad \text{for all numerals } \mathfrak{z}}{(x)\mathfrak{A}(x) \vee \mathfrak{N}}$$

   as it was accordingly already recommended by Hilbert,
3. with an assignment of ordinals to the formulas of the derivation which is only subject to the following conditions:
   a) in the case of a structural inference, premise and conclusion have the same order,
   b) in case of a buildup inference as well as a cut the conclusion has a higher order than each of the premises.

In this process, the derivations are finitistically describable infinite figures which are subject to certain constraints. This proof idea is entirely in keeping with the one applied by Lorenzen, but the implementation is a slightly different one. I think, that my investigations are not redundant when put next to those of Lorenzen because the

[10] "Geschichtete Analysis", as Schütte calls it, is here translated as "stratified analysis". In his 1952 paper, however, he calls it "verzweigte Analysis". The word "verzweigt" also occurs in this letter and we translate it as "ramified".



– 2 –

benötigten metamathematischen Beweismittel und die Zusammenhänge mit der Herleitbarkeit der formalisierten transfiniten Induktion dabei aufgedeckt werden. Außerdem erscheint es mir einfacher, sich auf einen engeren Bereich, als ihn die ganze verzweigte Typenlogik darstellt, zu beschränken, wenn man die gestufte Analysis erhalten will.

Bei Unterscheidung zwischen

a) rekursiver Zahlentheorie (ohne Quantoren),

b) reiner Zahlentheorie (mit Quantoren für Zahlvariablen),

c) gestufter Analysis (mit Quantoren für Zahlvariablen und für Formelvariablen)

lassen sich folgende Ergebnisse angeben:

I. Eine Herleitung der Ordnung α läßt sich umwandeln in eine schnittfreie Herleitung, deren Ordnung kleiner ist als die nächste auf α folgende

a) Limeszahl b) ε-Zahl c) kritische ε-Zahl~~e~~

bezw. unverändert bleibt, falls α selbst eine solche Zahl ist.

Die formalisierte transfinite Induktion bis zu einer

a) Limeszahl b) ε-Zahl c) kritischen ε-Zahl

α ist mit der Ordnung α, aber mit keiner kleineren Ordnung herleitbar.

II. Werden nur endliche Herleitungen zugelassen, indem statt des Schlusses der „unendlichen Induktion" die formalisierte transfinite Induktion bis zu einer Zahl α als Grundformelschema aufgenommen wird (wobei die vollständige Induktion als „transfinite Induktion" bis ω aufzufassen ist), so ist die formalisierte transfinite Induktion nicht mehr herleitbar bis zur nächsten auf α folgenden

a) Limeszahl b) ε-Zahl c) kritischen ε-Zahl,

wohl aber bis zu jeder kleineren Ordnungszahl. Der Widerspruchsfreiheitsbeweis erfolgt dann durch eine metamathematisch anzuwendende transfinite Induktion bis zu dieser ersten Ordnungszahl, bis zu der die formalisierte transfinite Induktion nicht mehr herleitbar ist.

III. Die unter Einbeziehung der vollständigen Induktion (ohne unendliche Induktion und ohne transfinite Induktion) herleitbaren Formeln sind unter Verwendung der unendlichen Induktion herleitbar mit unendlichen Herleitungen, deren Ordnungen kleiner als $\omega\cdot 2$ sind. Bei den Schnitt-Eliminationen vergrößern sich die Ordnungen dieser Herleitungen bis zu beliebig großen Ordnungszahlen unterhalb

a) $\omega\cdot 2$ b) der ersten ε-Zahl c) der ersten kritischen ε-Zahl.

Fig. 1.3 Page 2 of Schütte's letter to Bernays.

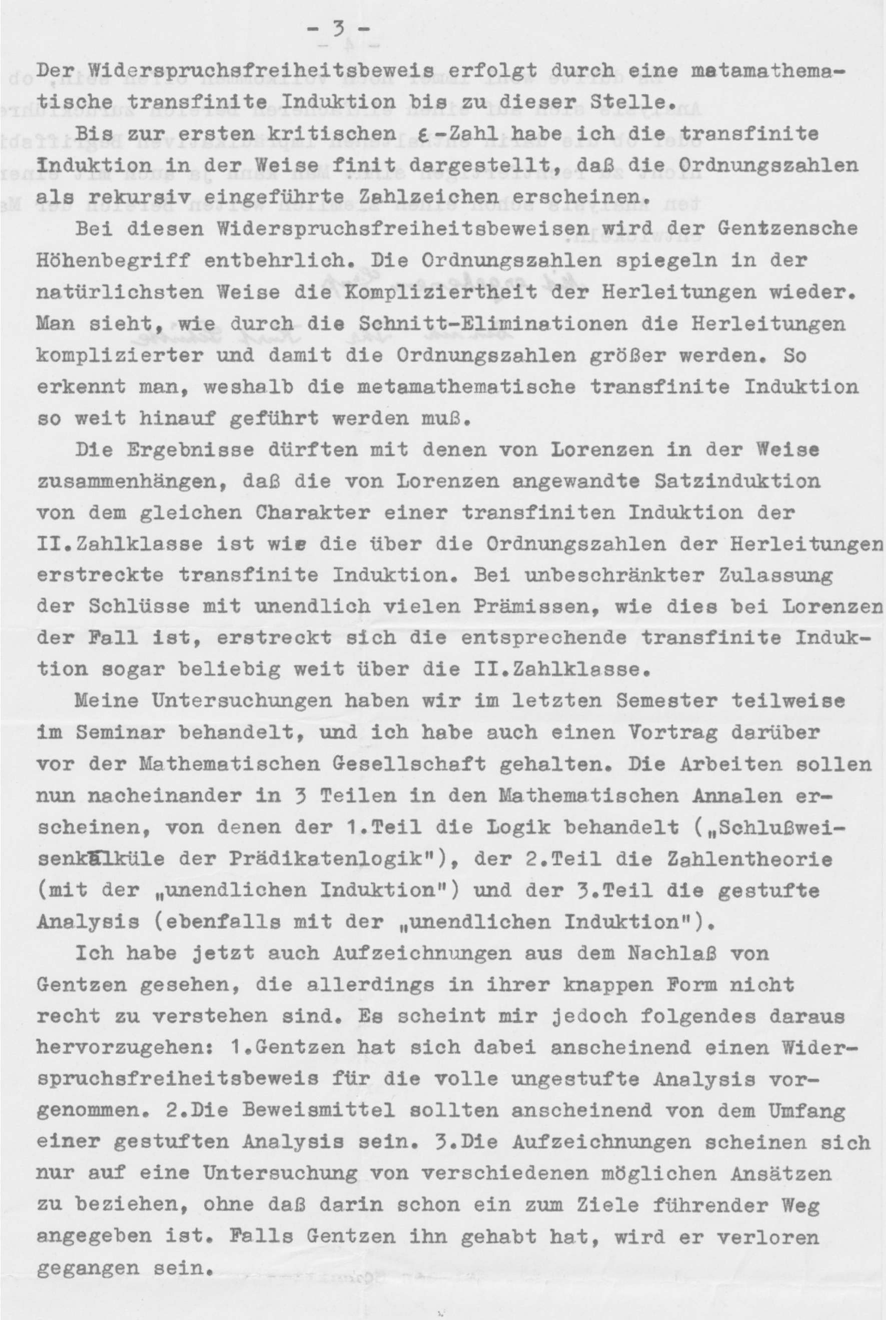
necessary metamathematical means of proof and the connections with the deducibility of the formalized transfinite induction are revealed in this way. Moreover, it seems to me easier to restrict oneself to a narrower domain than presented by the system of full ramified type if one wants to preserve stratified analysis (gestufte Analysis).

By distinguishing between

a) recursive number theory (without quantifiers),
b) pure number theory (with quantifiers for number variables)
c) stratified analysis (with quantifiers for number variables and for formula variables),

one can state the following results:

I. A derivation of order α can be transformed into a cut-free derivation whose order is smaller than, respectively, the next
   a) limit number b) ε number c) criticial ε number
   after α, or remains unaltered, if α happens to such a number.
   The formalized transfinite induction up to a, respectively,
   a) limit number b) ε number c) criticial ε number
   α ist derivable with order α, but not with a smaller order.
II. If one permits only finite derivations in that instead of the inference of "infinite induction" one adopts only formalized transfinite induction up to the number α (where ordinary formal induction has to be conceived as "transfinite induction" up to ω), then the formalized transfinite induction is no longer derivable for the next
   a) limit number b) ε number c) criticial ε number, respectively,
   that comes after α, yet is derivable for every smaller ordinal. Thus the consistency proof follows then via a metamathematically applied transfinite induction up to this first ordinal for which the formalized transfinite induction is no longer derivable.
III. The formulae derivable under inclusion of ordinary formal induction (but without the inference of infinite induction and also without transfinite induction) become derivable by means of infinite induction, yielding infinite derivations whose orders are smaller than $\omega \cdot 2$. Subjected to cut elimination, the orders of these derivations increase up to arbitrarily large ordinals below, respectively,
   a) $\omega \cdot 2$ b) the first ε number c) the first criticial ε number.



\- 3 -

Der Widerspruchsfreiheitsbeweis erfolgt durch eine metamathematische transfinite Induktion bis zu dieser Stelle.

Bis zur ersten kritischen ε-Zahl habe ich die transfinite Induktion in der Weise finit dargestellt, daß die Ordnungszahlen als rekursiv eingeführte Zahlzeichen erscheinen.

Bei diesen Widerspruchsfreiheitsbeweisen wird der Gentzensche Höhenbegriff entbehrlich. Die Ordnungszahlen spiegeln in der natürlichsten Weise die Kompliziertheit der Herleitungen wieder. Man sieht, wie durch die Schnitt-Eliminationen die Herleitungen komplizierter und damit die Ordnungszahlen größer werden. So erkennt man, weshalb die metamathematische transfinite Induktion so weit hinauf geführt werden muß.

Die Ergebnisse dürften mit denen von Lorenzen in der Weise zusammenhängen, daß die von Lorenzen angewandte Satzinduktion von dem gleichen Charakter einer transfiniten Induktion der II.Zahlklasse ist wie die über die Ordnungszahlen der Herleitungen erstreckte transfinite Induktion. Bei unbeschränkter Zulassung der Schlüsse mit unendlich vielen Prämissen, wie dies bei Lorenzen der Fall ist, erstreckt sich die entsprechende transfinite Induktion sogar beliebig weit über die II.Zahlklasse.

Meine Untersuchungen haben wir im letzten Semester teilweise im Seminar behandelt, und ich habe auch einen Vortrag darüber vor der Mathematischen Gesellschaft gehalten. Die Arbeiten sollen nun nacheinander in 3 Teilen in den Mathematischen Annalen erscheinen, von denen der 1.Teil die Logik behandelt („Schlußweisenkalküle der Prädikatenlogik"), der 2.Teil die Zahlentheorie (mit der „unendlichen Induktion") und der 3.Teil die gestufte Analysis (ebenfalls mit der „unendlichen Induktion").

Ich habe jetzt auch Aufzeichnungen aus dem Nachlaß von Gentzen gesehen, die allerdings in ihrer knappen Form nicht recht zu verstehen sind. Es scheint mir jedoch folgendes daraus hervorzugehen: 1.Gentzen hat sich dabei anscheinend einen Widerspruchsfreiheitsbeweis für die volle ungestufte Analysis vorgenommen. 2.Die Beweismittel sollten anscheinend von dem Umfang einer gestuften Analysis sein. 3.Die Aufzeichnungen scheinen sich nur auf eine Untersuchung von verschiedenen möglichen Ansätzen zu beziehen, ohne daß darin schon ein zum Ziele führender Weg angegeben ist. Falls Gentzen ihn gehabt hat, wird er verloren gegangen sein.

Fig. 1.4 Page 3 of Schütte's letter to Bernays.

> The consistency proof follows by a metamathematical transfinite induction up to this place.

Up to the first critical ε number, I have provided a finitistic representation of the corresponding transfinite induction in which the ordinals make their appearance as certain recursively introduced numerals.

In these consistency proofs, Gentzen's notion of height can be dispensed with. The ordinals mirror in the most natural way the complexity of the derivations. One sees how cut elimination renders derivations more complex and how thereby the assigned ordinals grow larger. In this way one comprehends why the metamathematical transfinite induction must be performed thus far upwards.

The results should relate to those of Lorenzen in that the sentence induction [Satzinduktion] employed by Lorenzen is of the same character of a transfinite induction over the second number class as the transfinite induction ranging over the orders of derivations. If one allows arbitrary inferences with infinitely many premises, as is the case with Lorenzen, the corresponding transfinite induction will reach arbitrarily high levels of the second number class.

We have studied part of my investigations in a seminar in the last semester, and I also gave a talk about them at the *Mathematische Gesellschaft*. The work should now successively be published in three parts in the *Mathematische Annalen*, of which the first part covers the logic ("Inference calculi of predicate logic"), the second part number theory (with "infinite induction"), and the third part stratified Analysis (also using "infinite induction").

I have now also seen notes from the Nachlass of Gentzen, which due to their condensed form are hardly graspable. But it seems to me that the following can be extracted from them: 1. Gentzen apparently set out to provide a consistency proof for full unstratified Analysis. 2. The methods of proof should apparently lie within the scope of stratified analysis. 3. The notes seem to address only possible approaches to attacking the problem without already indicating a path that leads to its resolution. If Gentzen saw such a path, we will have lost it.