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Iitaka Conjecture

An Introduction

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To my parents, Yuzo and Michiko

Preface

The ambitious program for the birational classification of higher-dimensional algebraic varieties discussed by Shigeru Iitaka in [12] is usually called the Iitaka program. Now it is known that the heart of the Iitaka program is the Iitaka conjecture. The main purpose of this book is to make the Iitaka conjecture more accessible. This book is not an introductory textbook for the Iitaka program but is an introduction to the Iitaka conjecture.

Let X be a smooth projective variety defined over $\mathbb{C}$. We put

$$\kappa(X) = \limsup_{m\to\infty} \frac{\log \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mK_X))}{\log m}$$

if $H^0(X, \mathcal{O}_X(lK_X)) \neq 0$ for some positive integer l, where K_X is the canonical divisor of X. If $H^0(X, \mathcal{O}_X(lK_X)) = 0$ for every positive integer l, then we put $\kappa(X) = -\infty$. We call $\kappa(X)$ the Kodaira dimension of X and know that it is a very important birational invariant of X. We can check that

$$\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim X\}.$$

In [11], Shigeru Iitaka defined $\kappa(X)$ and introduced the Iitaka fibration, which is a generalization of the notion of elliptic surfaces. Then he reached the following conjecture.

Conjecture (Iitaka Conjecture C) Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Then the inequality

$$\kappa(X) \geq \kappa(X_y) + \kappa(Y)$$

holds, where X_y is a sufficiently general fiber of $f : X \to Y$.

This conjecture first appeared in [12] and is still an important open problem. More generally, we have:

Conjecture (Iitaka Conjecture $\overline{C}$*)* Let $f : X \to Y$ be a dominant morphism between algebraic varieties whose general fibers are irreducible. Then we have the following inequality:

$$\overline{\kappa}(X) \geq \overline{\kappa}(Y) + \overline{\kappa}(X_y),$$

where X_y is a sufficiently general fiber of $f : X \to Y$.

Note that $\overline{\kappa}(X)$ denotes the logarithmic Kodaira dimension of X, which was also defined by Shigeru Iitaka in [13] in order to extend his framework, that is, the Iitaka program, for noncomplete algebraic varieties. We note that $\kappa(X) = \overline{\kappa}(X)$ holds when X is a smooth projective variety. Therefore, Conjecture C is a special case of Conjecture $\overline{C}$.

Around 1980, Shigefumi Mori initiated his theory of extremal rays (see [Mo1]). Then the minimal model program (MMP, for short) soon became the standard theory for the birational classification of higher-dimensional algebraic varieties. In some sense, it superseded the Iitaka program. Roughly speaking, we have already known that the above conjectures follow from the good minimal model conjecture (see [Kaw6], [N4], [F13], and [Has2]). We note that the good minimal model conjecture is the main goal of the minimal model program. For the details of the minimal model program, see [F12].

Theorem *Assume that the good minimal model conjecture holds true. Then Conjectures* C *and* $\overline{C}$ *also hold true.*

Or, more generally, we have:

Theorem *Conjectures* C *and* $\overline{C}$ *follow from the generalized abundance conjecture.*

The above two theorems strongly support Conjectures C and $\overline{C}$ because the experts believe that the good minimal model conjecture undoubtedly holds true. We note that the good minimal model conjecture is still widely open, although the minimal model program has developed drastically during the last decade.

Conjecture C was generalized as follows.

Conjecture (Generalized Iitaka Conjecture C^+*)* Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that $\kappa(Y) \geq 0$. Then the inequality

$$\kappa(X) \geq \kappa(X_y) + \max\{\mathrm{Var}(f), \kappa(Y)\}$$

holds, where X_y is a sufficiently general fiber of $f : X \to Y$ and $\mathrm{Var}(f)$ denotes Viehweg's variation of $f : X \to Y$.

Eckart Viehweg posed the following conjecture and proved that Conjecture C^+ follows from Conjecture Q.

Conjecture (Viehweg Conjecture Q) Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers such that $\kappa(X_{\overline{\eta}}) \geq 0$, where $X_{\overline{\eta}}$ is the geometric generic fiber of $f : X \to Y$. Assume that $\mathrm{Var}(f) = \dim Y$, where $\mathrm{Var}(f)$ denotes Viehweg's variation of $f : X \to Y$. Then $f_*\omega_{X/Y}^{\otimes k}$ is big in the sense of Viehweg for some positive integer k.

In this book, we describe Viehweg's theory of weakly positive sheaves and big sheaves and show that Conjecture C^+ follows from Conjecture Q. Then we prove the Iitaka Conjecture C in the following cases:

- the base space Y is of general type (see [Vi3]),
- the geometric generic fiber is of general type (see [Kol3]), and
- general fibers are elliptic curves (see [Vil]).

Finally, we prove the Iitaka Conjecture $\overline{C}$ under the assumption:

- $\dim X - \dim Y = 1$ (see [Kaw2]).

All the above cases are well known to the experts. However, the proof is not so easy to access for the younger generation. The author hopes that this book will make the Iitaka Conjectures C and $\overline{C}$ more accessible. Our choice of topics is biased and reflects the author's taste. In this book, we do not treat the Hodge theoretic part or the complex analytic part of the Iitaka Conjectures C and $\overline{C}$. We will simplify and generalize known arguments and some results with the aid of the weak semistable reduction theorem by Abramovich–Karu (see [Abk]) and the existence of relative canonical models for fiber spaces whose geometric generic fiber is of general type by Birkar–Cascini–Hacon–M^cKernan (see [BCHM]).

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Contents

Chapter 1
Overview

1.1 What is the Iitaka Program?

Let us start with a quick review of the *Iitaka program* in the 1970s for the reader's convenience. The basic references for this section are [I2], [U1], and [Ft2] (see also [Es], [I6], and [U2]).

Let X be a smooth projective variety defined over $\mathbb{C}$. We put

$$\kappa(X) = \begin{cases} \max_{m \in \mathbb{Z}_{>0}} \{\dim \Phi_{|mK_X|}(X)\} & \text{if } |mK_X| \neq \emptyset \text{ for some } m \in \mathbb{Z}_{>0}, \\ -\infty & \text{otherwise,} \end{cases}$$

where $\Phi_{|mK_X|}(X)$ denotes the closure of the image of the rational map

$$\Phi_{|mK_X|} : X \dashrightarrow \mathbb{P}^{\dim |mK_X|},$$

and call $\kappa(X)$ the *Kodaira dimension* of X. We note that $\kappa(X)$ is a birational invariant of X. By definition, we have

$$\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim X\}.$$

The following definition is due to Iitaka.

Definition 1.1.1 (*Varieties of hyperbolic type, parabolic type, and elliptic type*) A smooth projective variety X is called a variety of *hyperbolic type*, *parabolic type*, and *elliptic type* if $\kappa(X) = \dim X$, $\kappa(X) = 0$, and $\kappa(X) = -\infty$, respectively.

When $\kappa(X) > 0$, Iitaka proved that the rational maps

$$\Phi_{|mK_X|} : X \dashrightarrow Y_m = \Phi_{|mK_X|}(X)$$

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for all sufficiently large integers m with $|mK_X| \neq \emptyset$ are birationally equivalent to a fixed surjective morphism

$$\Phi_\infty : X_\infty \to Y_\infty$$

between smooth projective varieties with connected fibers such that $\kappa(F) = 0$ holds, where F is a sufficiently general fiber of $\Phi_\infty : X_\infty \to Y_\infty$. We call $\Phi_\infty : X_\infty \to Y_\infty$ the *Iitaka fibration* of X. It is a generalization of the notion of elliptic surfaces. By taking Iitaka fibrations, we can reduce the birational classification of higher-dimensional smooth projective varieties to:

- the study of smooth projective varieties of hyperbolic, parabolic, and elliptic type, and
- the study of fiber spaces whose sufficiently general fibers are of parabolic type.

In [I2, Sect. 4], Iitaka explained his ideas and 12 interesting open problems. Although Iitaka discussed the bimeromorphic classification of compact complex manifolds in [I2, Sect. 5], we will treat only algebraic varieties in this book.

Problem 1.1.2 (*cf.* [I2, Problem 6]) Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers and let F be a sufficiently general fiber of $f : X \to Y$. Assume that $\kappa(F) = \dim F$ and $\kappa(Y) = \dim Y$ hold. Then does the equality $\kappa(X) = \dim X$ hold true?

Problem 1.1.3 (*cf.* [I2, Problem 8]) Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers and let F be a sufficiently general fiber of $f : X \to Y$. Assume that $\kappa(F) = 0$ and $\kappa(Y) = \dim Y$ hold. Then does the equality $\kappa(X) = \dim Y$ hold true?

In [I2, Footnote 45)], Iitaka posed his famous Conjecture C and said that Problems 1.1.2 and 1.1.3 are special cases of Conjecture C.

Conjecture 1.1.4 (Iitaka Conjecture C) *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Then the inequality*

$$\kappa(X) \geq \kappa(X_y) + \kappa(Y)$$

holds, where X_y is a sufficiently general fiber of $f : X \to Y$.

Note that the questions in Problems 1.1.2 and 1.1.3 are now known to be true (see, for example, Theorems 1.2.9 and 1.2.12 below), although Conjecture C is still open.

Remark 1.1.5 (cf. [U2]) The original Iitaka Conjecture C was formulated for compact complex manifolds in [I2]. In Ueno's supplementary footnote 7 in [I2], it was pointed out that Conjecture C does not always hold true for compact complex manifolds. The reader can find an explicit example in [U1, Remark 15.3]. Kenji Ueno said that the importance of Conjecture C had not been fully understood when Iitaka proposed many conjectures in [I2]. The Iitaka Conjecture C has become one of the central issues in the various attempts to solve the several problems in [I2].