VLADIMIR I. ARNOLD

Collected Works

Springer

Vladimir I. Arnold, 1961
Photograph by Jürgen Moser

VLADIMIR I. ARNOLD

Collected Works

VLADIMIR I. ARNOLD

Collected Works

VOLUME I

Representations of Functions, Celestial Mechanics and KAM Theory, 1957–1965

VLADIMIR I. ARNOLD

Collected Works

VOLUME I

Representations of Functions, Celestial Mechanics
and KAM Theory, 1957–1965

Edited by

Alexander B. Givental
Boris A. Khesin
Jerrold E. Marsden
Alexander N. Varchenko
Victor A. Vassilev
Oleg Ya. Viro
Vladimir M. Zakalyukin

Springer

Vladimir I. Arnold

Russian Academy of Sciences
Steklov Mathematical Institute
ul. Gubkina 8
Moscow 117966
Russia

Editors

A.B. Givental, B.A. Khesin, J.E. Marsden, A.N. Varchenko,
V.A. Vassilev, O.Ya. Viro, V.M. Zakalyukin

ISBN 978-3-642-01741-4 e-ISBN 978-3-642-01742-1
DOI 10.1007/978-3-642-01742-1
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2009931933

Cover design: WMXDesign GmbH, Heidelberg

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Vladimir Igorevich Arnold is one of the most influential mathematicians of our time. V.I. Arnold launched several mathematical domains (such as modern geometric mechanics, symplectic topology, and topological fluid dynamics) and contributed, in a fundamental way, to the foundations and methods in many subjects, from ordinary differential equations and celestial mechanics to singularity theory and real algebraic geometry. Even a quick look at a partial list of notions named after Arnold already gives an overview of the variety of such theories and domains:

KAM (Kolmogorov–Arnold–Moser) theory,
The Arnold conjectures in symplectic topology,
The Hilbert–Arnold problem for the number of zeros of abelian integrals,
Arnold's inequality, comparison, and complexification method in real algebraic geometry,
Arnold–Kolmogorov solution of Hilbert's 13th problem,
Arnold's spectral sequence in singularity theory,
Arnold diffusion,
The Euler–Poincaré–Arnold equations for geodesics on Lie groups,
Arnold's stability criterion in hydrodynamics,
ABC (Arnold–Beltrami–Childress) flows in fluid dynamics,
The Arnold–Korkina dynamo,
Arnold's cat map,
The Arnold–Liouville theorem in integrable systems,
Arnold's continued fractions,
Arnold's interpretation of the Maslov index,
Arnold's relation in cohomology of braid groups,
Arnold tongues in bifurcation theory,
The Jordan–Arnold normal forms for families of matrices,
The Arnold invariants of plane curves.

Arnold wrote some 700 papers, and many books, including 10 university textbooks. He is known for his lucid writing style, which combines mathematical rigour with physical and geometric intuition. Arnold's books on *Ordinary differential equations* and *Mathematical methods of classical mechanics* became mathematical bestsellers and integral parts of the mathematical education of students throughout the world.

Some Comments on V.I. Arnold's Biography and Distinctions

V.I. Arnold was born on June 12, 1937 in Odessa, USSR. In 1954–1959 he was a student at the Department of Mechanics and Mathematics, Moscow State University. His M.Sc. Diploma work was entitled "On mappings of a circle to itself." The degree of a "candidate of physical-mathematical sciences" was conferred to him in 1961 by the Keldysh Applied Mathematics Institute, Moscow, and his thesis advisor was A.N. Kolmogorov. The thesis described the representation of continuous functions of three variables as superpositions of continuous functions of two variables, thus completing the solution of Hilbert's 13th problem. Arnold obtained this result back in 1957, being a third year undergraduate student. By then A.N. Kolmogorov showed that continuous functions of more variables can be represented as superpositions of continuous functions of three variables. The degree of a "doctor of physical-mathematical sciences" was awarded to him in 1963 by the same Institute for Arnold's thesis on the stability of Hamiltonian systems, which became a part of what is now known as KAM theory.

After graduating from Moscow State University in 1959, Arnold worked there until 1986 and then at the Steklov Mathematical Institute and the University of Paris IX.

Arnold became a member of the USSR Academy of Sciences in 1986. He is an Honorary member of the London Mathematical Society (1976), a member of the French Academy of Science (1983), the National Academy of Sciences, USA (1984), the American Academy of Arts and Sciences, USA (1987), the Royal Society of London (1988), Academia Lincei Roma (1988), the American Philosophical Society (1989), the Russian Academy of Natural Sciences (1991). Arnold served as a vice-president of the International Union of Mathematicians in 1999–2003.

Arnold has been a recipient of many awards among which are the Lenin Prize (1965, with Andrey Kolmogorov), the Crafoord Prize (1982, with Louis Nirenberg), the Lobachevsky Prize of Russian Academy of Sciences (1992), the Harvey prize (1994), the Dannie Heineman Prize for Mathematical Physics (2001), the Wolf Prize in Mathematics (2001), the State Prize of the Russian Federation (2007), and the Shaw Prize in mathematical sciences (2008).

One of the most unusual distinctions is that there is a small planet Vladarnolda, discovered in 1981 and registered under #10031, named after Vladimir Arnold. As of 2006 Arnold was reported to have the highest citation index among Russian scientists.

In one of his interviews V.I. Arnold said: "The evolution of mathematics resembles the fast revolution of a wheel, so that drops of water fly off in all directions. Current fashion resembles the streams that leave the main trajectory in tangential directions. These streams of works of imitation are the most noticeable since they constitute the main part of the total volume, but they die out soon after departing the wheel. To stay on the wheel, one must apply effort in the direction perpendicular to the main flow."

With this volume Springer starts an ongoing project of putting together Arnold's work since his very first papers (not including Arnold's books.) Arnold continues to do research and write mathematics at an enviable pace. From an originally planned 8 volume edition of his Collected Works, we already have to increase this estimate to 10 volumes, and there may be more. The papers are organized chronologically. One might regard this as an attempt to trace to some extent the evolution of the interests of V.I. Arnold and cross-fertilization of his ideas. They are presented using the original English translations, whenever such were available. Although Arnold's works are very diverse in terms of subjects, we group each volume around particular topics, mainly occupying Arnold's attention during the corresponding period.

Volume I covers the years 1957 to 1965 and is devoted mostly to the representations of functions, celestial mechanics, and to what is today known as the KAM theory.

Acknowledgements. The Editors thank the Göttingen State and University Library and the Caltech library for providing the article originals for this edition. They also thank the Springer office in Heidelberg for its multilateral help and making this huge project of the Collected Works a reality.

March 2009

Alexander Givental
Boris Khesin
Jerrold Marsden
Alexander Varchenko
Victor Vassiliev
Oleg Viro
Vladimir Zakalyukin

Contents

ON THE REPRESENTATION OF FUNCTIONS OF TWO VARIABLES IN THE FORM $\chi[\phi(x) + \psi(y)]^\star$

V.I. Arnol'd
translated by Gerald Gould

1. Kolmogorov proved [1] that the set of functions of two variables representable as a certain combination of continuous functions of one variable and addition is everywhere dense in the space $C(E^2)$ of continuous functions defined on the square E^2. It follows immediately from our result proved below that this is not true for the simplest combinations: the set of functions of the form $\chi[\phi(x) + \psi(y)]$ even turns out to be nowhere dense in $C(E^2)$.

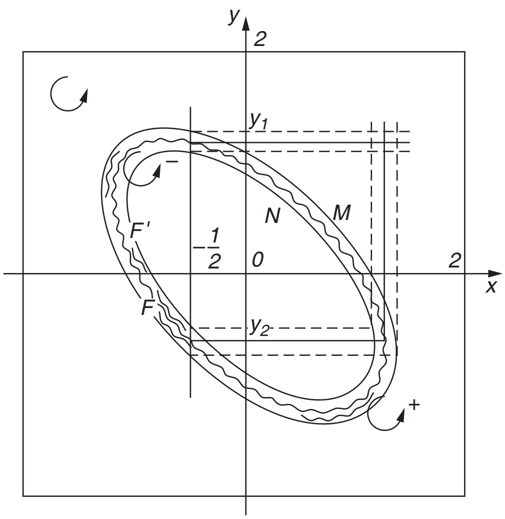

Fig. 1.

We shall indicate *a closed subset* N *of the square* $|x| \leqslant 2$, $|y| \leqslant 2$ (*Fig. 1*) *such that for any continuous function* $f(x, y)$ *vanishing on* (*and only on*) N *there exists* $\delta(f) > 0$ *such that* $|f(x, y) - \chi[\phi(x) + \psi(y)]| \geqslant \delta$ *at some point of this square for any continuous functions* χ, ϕ *and* ψ; *every function having*

* Uspekhi Math. Nauk **12**, No. 2, 119–121 (1957)

N as its level set is 'with a neighbourhood' non-representable in the form $\chi[\phi(x)+\psi(y)]$. An example of such a set N is the ellipse $(x+y)^2+\frac{(x-y)^2}{4}=1$.

We shall prove this. Since $f(x,y)$ is of constant sign outside the ellipse we can assume that $f(x,y)>0$ there. Then clearly there exists $\delta>0$ such that $f(x,y)>2\delta$ at all points in the region $G \overset{\text{def}}{=} (x+y)^2+\frac{(x-y)^2}{4}>\frac{5}{4}$, that is, outside the ellipse $M \overset{\text{def}}{=} (x+y)^2+\frac{(x-y)^2}{4}=\frac{5}{4}$. Suppose that there exist continuous functions $\phi(x)$, $\psi(y)$, $\chi(z)$ such that $|f(x,y)-\chi[\phi(x)-\psi(y)]|<\delta$ † for all (x,y), $2\leqslant x,\ y\leqslant 2$. Then the inequality $\chi[\phi(x)+\psi(y)]<\delta$ holds on N and the inequality $\chi[\phi(x)+\psi(y)]>\delta$ holds on M.

The largest open connected sets $G^-\supset N$ and $G^+\supset G$,⋆ where $\chi[\phi(x)+\psi(y)]<\delta$ and $\chi[\phi(x)+\psi(y)]>\delta$, respectively, are separated by the closed set F where $\chi[\phi(x)+\psi(y)]=\delta$ (that is, each continuum intersecting G^- and G^+ also intersects F), because the continuous function $\chi[\phi(x)+\psi(y)]$ on a continuum takes all values between any two given values. By a well-known theorem (Theorem E in [2]) the boundary of G^+ has a component $F'\subseteq F$ already separating G^- and G^+, and hence M and N. We claim that the continuous function $\phi(x)+\psi(y)$ is constant on F'. Indeed, suppose that, on the contrary, $z_1=\phi(x)+\psi(y)|_a<\phi(x)+\psi(y)|_b=z_2$, where $a,b\in F'$. Then in a sufficiently small neighbourhood of a there is a point $a'\in G^+$ where $\phi(x)+\psi(y)<z_1+\frac{z_2-z_1}{3}$, and in a sufficiently neighbourhood of b there is a point $b'\in G^+$ where $\phi(x)+\psi(y)>z_2-\frac{z_2-z_1}{3}$. Therefore on a polygonal line joining a' and b' in G^+ there is a point c where $\phi(x)+\psi(y)=\frac{z_1+z_2}{2}$; also there is a point c on the continuum F' where $\phi(x)+\psi(y)=\frac{z_1+z_2}{2}$. Consequently, $\chi[\phi(x)+\psi(y)]|_{c'}=\chi[\phi(x)+\psi(y)]|_c$, which contradicts the conditions $c'\in G^+$, $c\in F'$.

We denote by z the unique value of $\phi(x)+\psi(y)$ at points of F'. Then on the intervals $x=-\frac{1}{2}$, $y\in[1.1,1.22]$ and $x=-\frac{1}{2}$, $y\in[-0.62,-0.5]$ intersecting M and N there are points $(-\frac{1}{2},y_1)$ and $(-\frac{1}{2},y_2)$ at which $\phi(x)+\psi(y)=z$. There is such a point (x_1,y_2) on the interval on which the line $y=y_2$ intersects the strip between M and N for $x>0$.

It follows from the equalities⋆⋆

$$\phi(-\tfrac{1}{2})+\psi(y_1)=z\,,$$
$$\phi(-\tfrac{1}{2})+\phi(y_2)=z\,,$$
$$\phi(x_1)+\psi(y_2)=z$$

that $\phi(x_1)+\psi(y_1)=z$ and $\chi[\phi(x_1)+\psi(y_2)]=\delta$. However, it is easy to see that the point (x_1,y_1) lies in G, therefore $\chi[\phi(x_1)+\psi(y_2)]>\delta$. This contradiction proves the 'stable' non-representability of $f(x,y)$ in the form $\chi[\phi(x)+\psi(y)]$;

† *Translator's note*: This should be $|f(x,y)-\chi[\phi(x)+\psi(y)]|<\delta$.

⋆ *Translator's note*: This should be $G^+\supset M$.

⋆⋆ *Translator's note*: The second of these inequalities contains a misprint. It should read $\phi(-\frac{1}{2})+\psi(y_2)=z$.

in particular, for the function $f(x,y) = (x+y)^2 + \frac{1}{4}(x-y)^2 - 1$ we can choose $\delta > \frac{1}{4}$.

2. I.A. Weinstein proved that the class of continuous functions of the form $\chi[\phi(x)+\psi(y)]$ that are strictly monotone in each variable is a closed subset of $C(E^2)$. Here the strict monotonicity is essential: we claim that *the function xy is not representable in the form $\chi[\phi(x)+\psi(y)]$ even though it is the uniform limit of the sequence of functions* $\exp\big(\ln(x+\frac{1}{n})+\ln(y+\frac{1}{n})\big)$, *which do have the form* $\chi[\phi(x)+\psi(y)]$ (where $\phi_n(x) = \psi_n(x) = \ln(x+\frac{1}{n})$ and $\chi(z) = \exp(z)$).

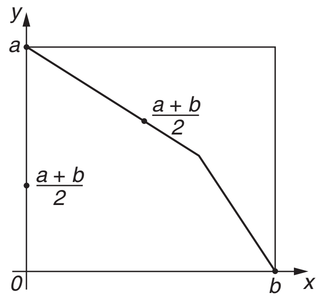

Fig. 2.

In fact, if $\chi[\phi(x)+\psi(y)] = xy$ everywhere in the square $x, y \in [0,1]$, then the function $\phi(x)+\psi(y)$ would take the same value at the points $(0,0)$, $(0,1)$, and $(1,0)$. Indeed, any two of these three points can be joined by a polygonal line having no common points with the set $xy = 0$ apart from the end points, and also by a polygonal line lying entirely in this set. If $\phi(x)+\psi(y)$ took different values a and b at these end points (see Fig. 2), then the intermediate value $\frac{a+b}{2}$ would be taken both on the set $xy = 0$ and outside this set, which would mean that $\chi(\frac{a+b}{2}) = 0$ and $\chi(\frac{a+b}{2}) > 0$ simultaneously. This contradiction proves that $\phi(0)+\psi(0) = \phi(0)+\psi(1) = \phi(1)+\psi(0)$; hence $\phi(0)+\psi(0) = \phi(1)+\psi(1)$ and therefore

$$0 = \chi[\phi(0)+\psi(0)] = \chi[\phi(1)+\psi(1)] = 1\,.$$

In other words, there do not exist any functions $\phi(x)$, $\psi(y)$, $\chi(z)$ such that $\chi[\phi(x)+\psi(y)] = xy$.

We also point out that the first example of a continuous function not representable in the form $\chi[\phi(x)+\psi(y)]$ (obtained simultaneously by A.A. Kirillov and the author), namely, the function $f(x,y) = \min(x,y)$ (where $x, y \in [0,1]$) can also be approximated to arbitrary precision by functions of the form $\chi[\phi(x)+\psi(y)]$.

Received 26 December 1956

References

[1] Kolmogorov, A.N.: On the representation of continuous functions of several variables as a superposition of continuous functions of a smaller number of variables. Dokl. Akad. Nauk SSSR **108**, No, 2 (1956).

[2] Krondov, A.S.: On functions of two variables. Usp. Mat. Nauk **5**, No, 1 (1950).

ON FUNCTIONS OF THREE VARIABLES*

V.I. ARNOL'D

In the present paper there is indicated a method of proof of a theorem which yields a complete solution of the 13th problem of Hilbert (in the sense of a denial of the hypothesis expressed by Hilbert).

Theorem 1. *Every real, continuous function $f(x_1,x_2,x_3)$ of three variables which is defined on the unit cube E^3 can be represented in the form*

$$f(x_1,x_2,x_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} h_{ij}[\varphi_{ij}(x_1,x_2), x_3], \tag{1}$$

where the functions h_{ij} and φ_{ij} of two variables are real and continuous.

A.N. Kolmogorov [1] obtained recently the representation

$$f(x_1,x_2,x_3) = \sum_{i=1}^{3} h_i[\varphi_i(x_1,x_2), x_3], \tag{2}$$

where the functions h_i and φ_i are continuous, the function h_i is real, and the function φ_i takes on values which belong to some tree Ξ. In the construction of A.N. Kolmogorov (for the case of functions of three variables), the tree Ξ can be taken not as a universal tree, but such that all of its points have a branching index not greater than 3. For this, the functions u^r_{km} of the fundamental lemma [1] (for $n = 2$) must be chosen so that in addition to the indicated five properties they must have the following properties.

(6) *The boundary of each level set of each function u^r_{km} divides the plane into not more than 3 parts.*

(7) *For every r, $G^r_{11} \supseteq E^2$.*

On the basis of this remark, Theorem 1 is a consequence of the existence of the representation (2) and of the next theorem.

Theorem 2. *Let F be any family of real equicontinuous functions $f(\xi)$ defined on a tree Ξ all of whose points have a branching index $\leqslant 3$. One can realize the tree as a subset X of the three-dimensional cube E^3 in such a way that any function of the family F can be represented in the form*

$$f(\xi) = \sum_{k=1}^{3} f_k(x_k),$$

where $x = (x_1,x_2,x_3)$ is the image of $\xi \in \Xi$ in the tree X; the $f_k(x_k)$ are continuous real functions of one variable, while the f_k depend continuously

---

* Editor's note: translation into English published in Amer. Math. Soc. Transl. (2) 28 (1963), 51–54. Translation of V.I. Arnol'd: On functions of three variables Dokl. Akad. Nauk SSSR 114:4 (1957), 679–681

on f *(in the sense of uniform convergence).*

We will introduce certain auxiliary concepts. Let K be a finite complex of segments contained in E^3 and consisting of segments which are not parallel to any coordinate plane.

Definition 1. A system of points

$$a_0 \neq a_1 \neq \ldots \neq a_{n-1} \neq a_n$$

belonging to K will be called a *zigzag* (lightning) if the segments $\overline{a_{i-1}a_i}$ are perpendicular to the axes X_{α_i}, respectively, and

$$\alpha_1 \neq \alpha_2 \neq \ldots \neq \alpha_{n-1} \neq \alpha_n .$$

The finite system of the pairwise distinct points $a_{i_1 i_2 \ldots i_n}$ tagged by the corteges of indices $i_1 i_2 \ldots i_n$, will be called a *branching scheme* if (1) there exists only one point a_0 tagged with one index; (2) the presence of $a_{i_1 i_2 \ldots i_{n-1} i_n}$ in the system implies the presence of $a_{i_1 \ldots i_{n-1}}$ in the system.

Definition 2. A branching system of points $a_{i_1 \ldots i_n}$ contained in K will be called a *generating scheme* if for a given cortege $i_1 \ldots i_n$ the set of points of the form $a_{i_1 \ldots i_n i_{n+1}}$ lies on the plane passing through $a_{i_1 \ldots i_n}$ and perpendicular to some coordinate axis $x_{\alpha_{i_1 \ldots i_n}}$, and contains all points of intersection of this plane with K, that are distinct from $a_{i_1 \ldots i_n}$.

The tree Ξ can be represented in the form

$$\Xi = \overline{\bigcup_{n=1}^{\infty} D_n}, \quad D_n \subset D_{n+1},$$

where D_n is a finite tree, D_1 is a simple arc, and D_{n+1} is obtained from D_n by attaching segments S_n at certain points p_n that are not branch points or endpoints of d_n [2].

We will denote by ω_n the upper boundary of the oscillations of the functions $f \in F$ on the components of the difference $\Xi \setminus D_n$. It is easy to see that

$$\omega_n \to 0 \quad \text{when} \quad n \to \infty .$$

Therefore, one can select a sequence

$$n_1 < n_2 < \ldots < n_r < \ldots ,$$

so that

$$\omega_n \leqslant \frac{1}{r^2} \quad \text{when} \quad n \geqslant n_r .$$

The realization X of the tree Ξ in E^3 is constructed in the form

$$X = \overline{\bigcup_{n=1}^{\infty} D'_n},$$

where D'_n is a complex of segments which realize D_n in such a way that the images S'_n of the arcs S_n are segments that are not perpendicular to the coordinate axes.

The inductive construction of D'_n is performed so that $\overline{\bigcup_{n=1}^{\infty} D'_n}$ is a tree [2], and that the following conditions are satisfied.

(1) Every function $f \in F$ can be represented on D_n in the form

$$f(\xi) = \sum_{k=1}^{3} f_k^n(x_k), \tag{3}$$

where the $f_k^n(x_k)$ depend continuously on f.

(2) The tree D'_n has for every point a_0 a generating system issuing from a_1, and whose initial direction α_0 can be chosen arbitrarily.

(3) Let A_n be the set of points D'_n which is the image of the branch points of Ξ. There exists a denumerable set $B_n \subseteq D'_n$, $B_n \cap A_n = 0$ such that the zigzag $a_0 \ldots a_m$, which begins at $a_0 \in D'_n \setminus B_n$, has no points in common with A_n and no coincident points $a_i = a_j$, $i \neq j$.

(4) If $n_r < n \leqslant n_{r+1}$, then

$$|f_k^n(x_k) - f_k^{n_r}(x_k)| \leqslant \left(3 + \frac{n - n_r}{n_{r+1} - n_r}\right) \frac{1}{r^2}. \tag{4}$$

This proof of the possibility of the inductive construction of the trees D'_n, and of the functions f_k^n with properties (1) to (4), is too complicated to be given here. Roughly speaking, at each step the attached segment S'_{n+1} is chosen of very short length; its direction, and the way of mapping of S_{n+1} on S'_{n+1} are selected so as to guarantee the fulfillment of properties (2) and (3) by D'_{n+1}. The preservation of equality (3), in the transition from n to $n+1$, on the newly attached segment S_{n+1}, requires the introduction of a correction $f_k^{n+1} - f_k^n$, for at least one of the functions f_k^n, on the projection S'_{n+1} on the axis x_k. For the preservation of equality (3) on the earlier constructed tree D'_n, it is necessary to compensate for this correction by means of new corrections for the functions f_k^n on a number of other segments. The exact method of the introduction of these corrections, we will not present here. We only note the following: these corrections must be such that inequality (4) will be preserved for $n' = n + 1$; if S'_{n+1} is chosen small enough, and if its direction is chosen appropriately, it must be possible to produce it for every function f_k^n on a finite system of non-intersecting segments of the axis x_k. In the proof of this possibility one makes use of the fact that the tree D_n' has properties (2) and (3).

The proof of the existence of the continuous function

$$f_k(x_k) = \lim_{n \to \infty} f_k^n(x_k)$$

and of the validity of the equation

$$f(\xi) = \sum_{k=1}^{3} f_k(x_k)$$

on the entire X, is not complicated.

I express my sincere thanks to A.N. Kolmogorov for the aid and advice I have received from him in the preparation of this work.

Bibliography

[1] A.N. Kolmogorov, *On the representation of continuous functions of several variables by superpositions of continuous functions of a smaller number of variables*, Dokl. Akad. Nauk SSSR **108** (1956), 179-182. (Russian)

[2] K. Menger, *Kurventheorie*, Teubner, Leipzig, 1932.

Translated by:
H.P. Thielman

THE MATHEMATICS WORKSHOP FOR SCHOOLS AT MOSCOW STATE UNIVERSITY*

V.I. Arnol'd
Moscow
translated by Gerald Gould

The mathematics workshop for schools at Moscow State University in the name of Lomonosov came into existence in 1935. The organizers of the workshop were: the now-deceased Corresponding Member of the Academy of Sciences of the USSR L.G. Shnirel'man, Professor L.A. Lyusternik (now Corresponding Member of the Academy of Sciences of the USSR), and Doctor I.M. Gel'fand (now Corresponding Member of the Academy of Sciences of the USSR).

The activities of the workshop proceed in two streams: twice a month (on Sundays) *lectures on mathematics* are given by professors and instructors at Moscow State University and other institutes (separately for the pupils of the 7–8 class and for pupils of the 9–10 class) and *sections of the mathematics workshop* meet weekly under the guidance of students and (more rarely) postgraduate students of the university.[1] The annual *Mathematical Olympiad* is, in a certain sense, the culmination of the activities of the circle; here the directors of the mathematics workshop traditionally play a large role in bringing this about.

General information on the activities of the mathematics workshop in the 1955/56 academic year is given in the preceding issue of "Matematicheskiĭ Prosveshchenie"; there one can find the list of lectures given in that year.[2] The series "Popular lectures on mathematics" published by Gostekhizdat will give an idea of the character of these lectures.[3] The main part of this series of books by Moscow authors consists in expositions of the lectures given in the mathematical circle for schools at Moscow State University. Here we wish to shed light on the activities of the sections of the circle (the early part of these

* Mat. Prosveshchenie **2**, 241–245 (1957)

[1] It was only at the very beginning of the activities of the workshop that professors of Moscow State University were also involved in the work of the sections.

[2] Dynkin, E.B., Girsanov, I.V.: The nineteenth School Mathematical Olympiad in Moscow. Mat. Prosveshch. **2**, No. 1, 187 (1957).

[3] *Editor's note*: See the paper by N.B. Beskin on pp. 275–290 of this issue.

activities is well reflected in the series of books "Library of the mathematical circle", also published by Gostekhizdat).

The 'lessons' of a section take place in the form of a discussion: the supervisor of the section introduces the topic of study to the participants; 5–10 minutes is set aside for each problem; then the solution is explained and the supervisor continues his talk on the topic being studied. Each individual problem is not difficult (most of the pupils manage it in 5–10 minutes). At the end of the lesson the pupils are given (usually more difficult and sometimes very difficult) homework problems, which are collected at the beginning of the next lesson.

Below we give a summary account of two lessons of the workshop (a section for 10 pupils) on the themes "Variation of a curve" and "Harmonic functions".

Variation of a curve

We are given a line segment AB of length 1. If this line segment is illuminated by parallel rays, then the length of the shadow thrown onto various lines will vary from 0 to 1. More precisely, the length of the projection of the segment onto lines lying in the same plane will, in general be different for different lines; however in all cases it will be between 0 and 1. The length of the projection of AB onto a line l is called the *variation of the segment AB in the direction l* (Fig. 1); we shall denote it by $V_l(AB)$ or simply by V_l if it is clear which segment we are referring to.

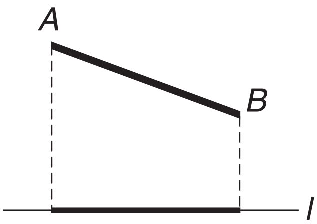

Fig. 1.

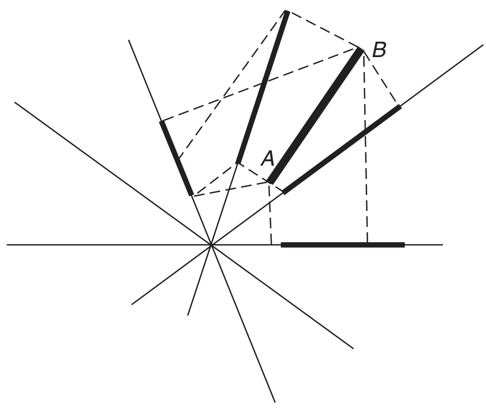

Fig. 2.

It is intuitively obvious that the mean value of the 'shadow' over all directions exists and that it is between 0 and 1. More precisely, this means that if we divide the 360° angle into n equal parts, and take the arithmetic mean

$$V_n = \frac{V_{l_1} + V_{l_2} + \cdots V_{l_n}}{n}$$

of the variations of the segment AB in the directions $l_1, l_2, \ldots, l_n$ (Fig. 2), then the limit

$$\lim_{n\to\infty} V_n = K$$

exists and K lies between 0 and 1.

This number K is called the *mean variation* or simply the *variation* of the unit segment AB.

This number is not very difficult to find;[4] it is equal to $\frac{2}{\pi} \approx 0.637$. However, we shall not find it now, but calculate it later via an indirect route (Problem 7) Nevertheless, we shall use the fact that this limit exists from the very outset.

Problem 1. What is the variation of a segment of length a?

Solution. Since, clearly, the variation of such a segment in any given direction is a times as large as that of a unit segment parallel to it, and the limit of this quantity, that is, the mean variation of the segment of length a, is equal to Ka.

We define the *variation of a polygonal line in some direction* to be the sum of the lengths of the projections of its component line-segments ('links') in this direction (Fig. 3).

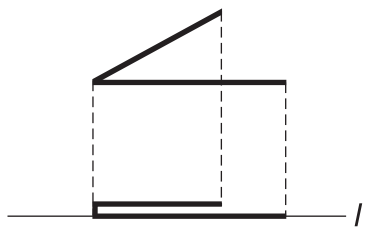

Fig. 3.

Problem 2. Determine the variation of a square of side 1 in the directions of its sides and its diagonals.

Solution. Clearly, the variation of the square in the direction of each side is equal to 2, and in the direction of a diagonal is equal to $2\sqrt{2}$.

The mean variation of a polygonal line over all directions, or simply the *variation of the polygonal line* over all directions is defined, as above, via the passage to the limit: $V = \lim_{n\to\infty} V_n$, where V_n is the arithmetic mean of the variations of the polygonal line along the n directions of the sides of a regular n-gon.

[4] See, for example, the book Yaglom, A.M., Yaglom, I.M.: Elementary problems in a non-elementary setting. Gostekhizdat, Moscow (1954), Problem 147b.

Problem 3. Determine the variation of a polygonal line of length a.

Solution. Clearly the variation of a polygonal line in each direction is the sum of the variations of the projections of its links in this direction, and since the mean value of a sum is equal to the sum of its mean values,[5] the variation of the polygonal line is the sum of the variations of its links. Since, by Problem 1 the variation of each link is equal to the product of the length of this link by K, the variation of the polygonal line is Ka.

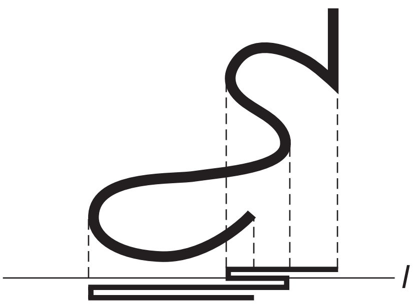

Fig. 4.

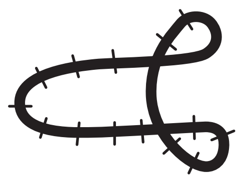

Fig. 5.

In order to transfer the definition of variation to *curves* we need to make precise the notion of a curve. This is difficult to do in the general case. However, we shall assume that the curve is either convex or can be divided into finitely many convex pieces. Then when one projects the curve in any given direction one can divide it into finitely many pieces each of which is intersected just once by each of the projecting lines.[6] Then *the variation of the curve in the chosen direction is, by definition, the sum of the lengths of the projections of its pieces in this direction* (Fig. 4). It can be shown that there exists a mean value of this quantity over all directions. We call this the mean variation or simply *variation* of the curved line.

It is clear that if the curve is a polygonal line, then we arrive at the previous definition.

[5] The precise meaning of this phrase is as follows: the arithmetic mean of the variations of a polygonal line over n directions is equal to the sum of the arithmetic means of the variations of its links over these directions. Therefore the limit as $n \to \infty$ of arithmetic means of the variations of the polygonal line over the different directions is equal to the sum of the limits of the arithmetic means of the variations of the individual links.

[6] Here we do not rule out the case when such a piece is a straight-line segment, so that when projecting in one of the directions the straight-line segment lies entirely in the projecting line.

Problem 4. Find the variation of a circle of diameter D.

Solution. First we choose some direction. The diameter having this direction divides the circle into two pieces, namely, into two arcs each of which is intersected by any line perpendicular to the chosen direction in at most one point. Hence the variation of the circle in the chosen direction is equal to $2D$. Clearly the variation in any other direction is the same, therefore the mean variation of the circle is equal to $2D$.

We now select several points on the curve and join them successively by straight lines (Fig. 5). Then we obtain a polygonal line. It can be shown that for sufficiently good curves (for example, for all convex curves) the limit of the lengths of these polygonal lines exists, provided that as these polygonal lines vary the length of the largest link of the lines tends to zero. This limit is called the *length of the curve*.

It can also be proved that for curves that can be divided into finitely many convex pieces the limit of the variations of these polygonal lines exists as the length of the largest link tends to zero.

Problem 5. Find the limit which the variation of a polygonal line inscribed in a "sufficiently good" curve of length a tends to when the polygonal line varies so that the length of its largest link tends to zero.

Solution. Since for each polygonal line of length a_n the variation is equal to Ka_n and $a_n \to a$ for "sufficiently good" curves, the limit of the variations of the polygonal lines is equal to Ka.

Problem 6. Prove that the variation of a ('sufficiently good') curve of length a is equal to Ka.

Solution. It suffices to observe that one can inscribe in such a curve a polygonal line with arbitrarily small links whose variation along each of the n given directions coincides with the variation of the curve. Therefore, once the limit in Problem 5 exists it is equal to the variation of the curve.

Problem 7. Find the numerical value of K, that is, the variation of a segment of length 1.

Solution. Since, on the one hand, a circle of diameter D has length D and hence variation $K\pi D$ (Problems 5 and 6) while, on the other hand (Problem 4), the variation of this circle is equal to $2D$, it follows that $K = \frac{2}{\pi}$.

By the *width of a curve with respect to a given direction* we mean the smallest distance between two lines of this direction that enclose the curve.

A curve has *constant width* if its width with respect to all directions is the same. Examples of a curve of constant width are the circle and the so-called *Rello triangle* consisting of three equal arcs of a circle (Fig. 6).[7] With the help

[7] There is a lot of information about curves of constant width in the book: Yaglom, I.M., Boltyanskii, V.G.: Convex figures. Gostekhizdat, Moscow (1951).

of variation one can obtain a very elegant proof of the following *Barbier's Theorem*:

Problem 8. Prove that all curves of constant width h have the same length πh.

Solution. This follows from the fact that the variation of each such curve in any direction is equal to $2h$; see the reults of Problems 6 and 7.

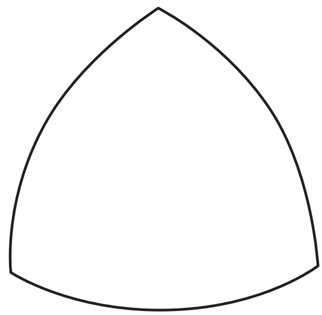

Fig. 6.

Here is another problem which at first glance appears to be rather complicated:

Problem 9. A curve L of length 22 is contained in a circle C of radius 1. Prove that there is a line intersecting L in at least 8 points.

Solution. The variation of L is equal to $\frac{2}{\pi} \cdot 22 > 14$ (Problems 6 and 7). On the other hand, the length of the projection of L in any direction does not exceed 2 (L is contained in C!). Hence for some directions certain parts of the projection of L will be covered by the projections of separate arcs of L more than 7 times (that is, at least 8 times). This completes the proof.

We now turn to an account of the lesson devoted to the topic "Harmonic functions".

The conclusion of this article will appear in the next issue

В ШКОЛЬНОМ МАТЕМАТИЧЕСКОМ КРУЖКЕ ПРИ МГУ*

В. И. Арнольд

(Москва)

(Окончание)

Гармонические функции

Две первые задачи не имели отношения к основной теме. Для полноты освещения занятия кружка мы приводим их; близкая к ним по методу решения третья задача являлась подготовительной к четвертой, с которой, по существу, и начиналась тема.

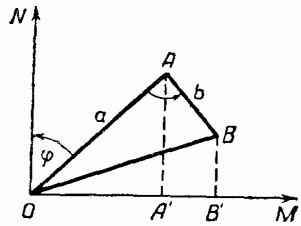

Рис. 1.

Задача 1. Найти наибольшее и наименьшее значения выражения

$a \sin\varphi + b\cos\varphi$ (a и b положительны).

Решение. Проведем два взаимно-перпендикулярных луча OM и ON и построим прямоугольный треугольник OAB с катетами $OA = a$ и $AB = b$, расположив их так, как на рис. 1 (прямые углы MON и OAB ориентированы против часовой стрелки). Обозначим угол AON через φ, тогда, проектируя ломаную OAB на ось OM (проекции направленные!), получаем [1]:

$$(OB') = \text{пр.}\, OB = \text{пр.}\, OA + \text{пр.}\, AB = a\sin\varphi + b\cos\varphi.$$

Если вращать треугольник OAB вокруг вершины O, то угол φ изменяется; наибольшее и наименьшее значения проекции (OB') достигаются, когда отрезок OB коллинеарен OM, т. е. когда $\operatorname{tg}\varphi = \frac{a}{b}$; они равны $\sqrt{a^2+b^2}$ и $-\sqrt{a^2+b^2}$.

Задача 2. Доказать, что если

$$a_1\cos\varphi_1 + a_2\cos\varphi_2 + \dots + a_n\cos\varphi_n = 0 \tag{1}$$

и

$$a_1\cos(\varphi_1+1) + a_2\cos(\varphi_2+1) + \dots + a_m\cos(\varphi_m+1) = 0 \tag{2}$$

(все коэффициенты a_i положительны), то и при любом α

$$a_1\cos(\varphi+\alpha) + a_2\cos(\varphi_2+\alpha) + \dots + a_m\cos(\varphi_m+\alpha) = 0. \tag{3}$$

[1]) (OB') — величина направленной проекции.

* Editor's note: V.I. Arnol'd: The school mathematical circle at Moscow State University: harmonic functions. Published in Mat. Prosveshchenie 3 (1958), 241–250

Решение. Выберем в плоскости луч OM и построим ломаную линию $OA_1A_2\ldots A_m$ (на рис. 2 $m=3$), где $OA_1=a_1$, $A_1A_2=a_2$, ..., $A_{m-1}A_m=a_m$, причем векторы $\overline{OA_1}$, $\overline{A_1A_2}$, ..., $\overline{A_{m-1}A_m}$ образуют с лучом OM соответственно углы φ_1, φ_2, ..., φ_m. Легко видеть, что условие (1) означает, что $OA_m \perp OM$, а условие (2) — что $O\tilde{A}_m \perp OM$, где $O\tilde{A}_m$ получается из OA_m вращением

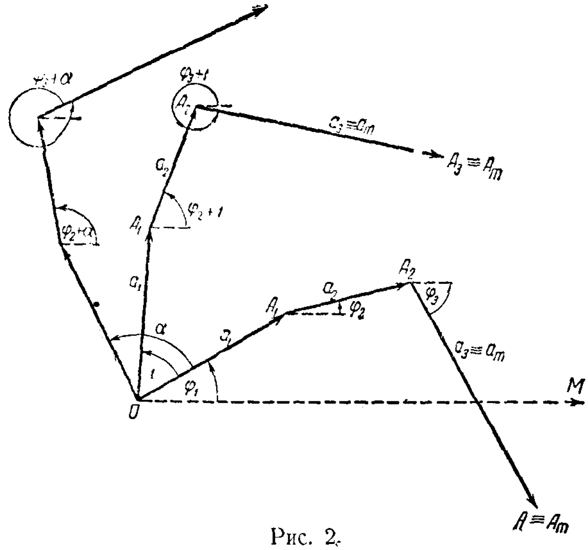

Рис. 2.

против часовой стрелки (при обычном направлении отсчета углов) на угол 1 (радиан). Оба условия вместе означают поэтому, что $\overline{OA_m}=0$, т. е. A_m совпадает с O. Но в таком случае проекция вектора $\overline{OA_m}$, повернутого на угол α [т. е. выражение $\sum_{i=1}^{m} a_i \cos(\varphi_i+\alpha)$], тоже равна нулю, что и доказывает (3).

Задача 3. Вычислить сумму m векторов с общим началом в центре правильного m-угольника и с концами в его вершинах (рис. 3, *а*).

Было предложено три решения.

Решение 1. Пусть сумма этих векторов — вектор $\overline{OA}$. Повернем многоугольник вокруг точки O на угол $\frac{2\pi}{m}$. Каждый вектор-слагаемое повернется на $\frac{2\pi}{m}$; тогда и сумма $\overline{OA}$ повернется на тот же угол, приняв положение $\overline{OA'}$. Вместе с тем каждый вектор перейдет при таком повороте в следующий, так что сумма не изменится, следовательно, $\overline{OA'}=\overline{OA}$. Но эти векторы образуют угол $\frac{2\pi}{m}$. Это может быть лишь при условии $\overline{OA}=0$.

Решение 2. Складывая векторы по «правилу треугольника» в порядке следования вершин, получим, очевидно, m-звенную ломаную, все звенья которой равны (они равны радиусу окружности, описанной около многоугольника) и все внешние углы равны (они равны $\frac{2\pi}{m}$, рис. 3, *б*). Отсюда следует что ломаная образует правильный m-угольник; так как он замкнут, то искомая сумма равна нулю.

Решение 3. Достаточно доказать это для правильного m-угольника, расположенного в комплексной плоскости так, что его вершины изображают все корни m-й степени из 1: 1, ε, ε^2, ..., ε^{m-1} (рис. 3, *в*). Такой правильный m-угольник мы в дальнейшем будем называть основным m-угольником. Центр основного m-угольника изображает число 0, а одна из вершин — число 1.

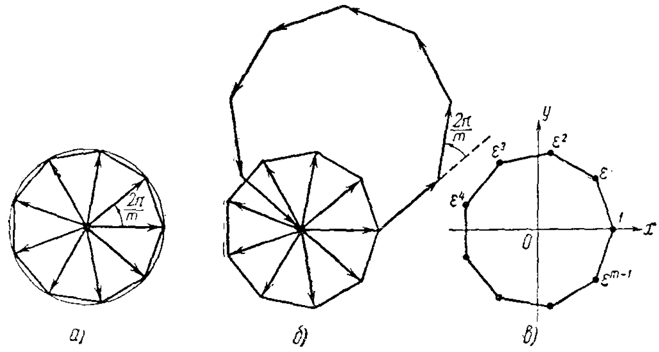

Рис. 3.

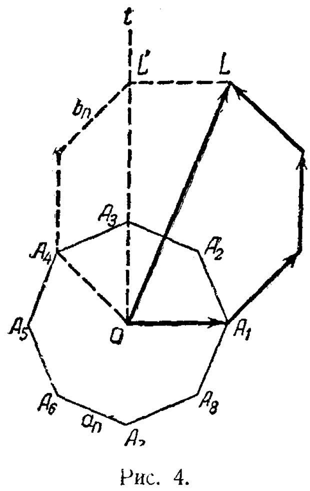

Рис. 4.

Как известно, вершины основного m-угольника изображают все решения уравнения $z^m - 1 = 0$. По теореме Виета, сумма этих решений равна нулю, ибо коэффициент при z^{m-1} в этом уравнении равен нулю. Но комплексные числа складываются по правилу сложения изображающих их векторов. Следовательно, сумма векторов, о которых говорится в условии задачи, равна нулю.

Задача 4. Вычислить предел K

$$K = \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right)}{n}$$

— *среднее значение* функции $y = \sin x$ на отрезке $0 \leqslant x \leqslant \pi$.

Решение. Рассмотрим снова правильный m-угольник, о котором говорилось в предыдущей задаче; на этот раз будем считать радиус описанной окружности равным 1, а число его сторон четным: $m = 2n$ (на рис. 4 $m = 8$). Сложим теперь только «правую половину» векторов: $\overline{OA_1} + \overline{AO_2} + \ldots + \overline{OA_n} = \overline{OL}$. Замыкающая OL рассматриваемой суммы будет совпадать с диаметром D_n окружности, описанной около нового m-угольника. Легко видеть, что если вектор $\overline{OA_1}$ направить горизонтально, то эта замыкающая при большом m близка к ее проекции OL' на вертикальную прямую Ot. А так как проекции единичных векторов $\overline{OA_1}$, $\overline{OA_2}$, ..., $\overline{OA_n}$ на эту вертикаль равны как раз

$$\sin 0 = 0, \quad \sin\frac{2\pi}{m} = \sin\frac{\pi}{n}, \quad \sin\frac{4\pi}{m} = \sin\frac{2\pi}{n}, \quad \ldots, \quad \sin\frac{(n-1)\pi}{n},$$

то среднее значение K равно пределу, к которому стремится частное $\frac{D_n}{n}$. Но из подобия m-угольников, изображенных на рис. 4, ясно, что $\frac{b_n}{a_n} = \frac{D_n}{2}$ (радиус $OA_1 = 1$), где $a_n = 2\sin\frac{\pi}{n}$, а $b_n = 1$. Следовательно,

$$\lim_{n\to\infty}\frac{\sum\limits_{k=0}^{n-1}\sin\left(\frac{k\pi}{n}\right)}{n} = \lim_{n\to\infty}\frac{|OL|}{n} = \lim_{n\to\infty}\frac{D_n}{n} = \lim_{n\to\infty}\frac{2}{n\cdot 2\sin\frac{\pi}{2n}} =$$

$$= \lim_{n\to\infty}\left(\frac{2}{\pi} : \frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}\right) = \frac{2}{\pi} : \lim_{n\to\infty}\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}} = \frac{2}{\pi} : 1 = \frac{2}{\pi}\ ^{1)}.$$

З а м е ч а н и е. Полученный результат имеет следующий геометрический смысл: предел, к которому стремится площадь ступенчатой фигуры, изображенной на рис. 5, между полуволной синусоиды и осью абсцисс, равен 2.

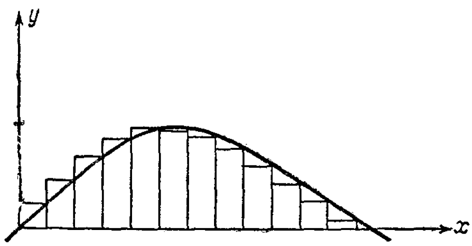

Рис. 5.

З а д а ч а 5. Доказать, что среднее значение произвольного многочлена с комплексными коэффициентами

$$P_k(z) = z^k + a_1 z^{k-1} + \ldots + a_k \qquad (1)$$

в n вершинах правильного n-угольника на комплексной плоскости, при $n > k$, равно значению многочлена в центре этого многоугольника.

Решение производится в три этапа.

---

[1]) Таким образом, этот предел оказался равным тому значению K, который мы раньше (см. «Математическое просвещение», вып. 2, стр. 242) назвали *средней вариацией* единичного отрезка. Это не случайно; решение всего цикла задач о вариациях кривых может рассматриваться как косвенное вычисление указанного в этой задаче предела.