RONALD J. ANDERSON

INTRODUCTION TO MECHANICAL VIBRATIONS

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Ronald J. Anderson
Queen's University
Kingston
Canada

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To June

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Preface

When I first studied vibrations, as an undergraduate student, its importance was clear to our class because it was a required course for mechanical engineers. A few years later, when I started teaching vibrations and new topics were entering the field of mechanical engineering, a course on vibrations was no longer seen as being important enough to be a required so it became an elective. Now, although "mechanical engineering" is still used as an umbrella term, the students who graduate are mechanical engineers with a specialization. Students in the specialized streams do not have time to cover all of the topics that used to be expected of mechanical engineers so some graduate without thermodynamics, others without vibrations, and so on. Specialization like this is inevitable given the expanding scope of knowledge in engineering and the limited time available to undergraduate students but it means even fewer students are learning about vibrations and other important topics. While preparing this introduction to vibrations, I kept in mind the need for undergraduate students to have a better understanding of two topics that are ubiquitous in today's engineering workplace – finite element analysis (FEA) and fast Fourier transforms (FFT). FEA and FFT software tools are readily available to both students and practicing engineers and they need to be used with understanding and a degree of caution.

I was never able to find a textbook that covered just enough, and the right, material for a semester length introductory course in vibrations. I used many textbooks over the years but there was never a fit with what I thought should be in an introduction to vibrations. I was looking for something student-friendly in that it should be readable, almost conversational, but still be mathematically rigorous. What I found on the market were mainly "reference books" as opposed to "teaching books". Many of the textbooks I tried are very good at covering, in depth, a broad range of topics in vibrations, but students have difficulty using them as a first text in the subject, mainly because of the overwhelming amount of material presented.

This book grew from my attempt to accomplish two things in a single course in vibrations. The primary goal is, of course, to present the basics of vibrations in a manner that promotes understanding and interest while building a foundation of knowledge in the field. To do this, I have had to give only brief coverage of many important topics with the hope that some students will go on to expand their knowledge in these areas if their interest is piqued. As mentioned earlier, a secondary goal is to give students a good understanding of finite element analysis and Fourier transforms. While these two subjects fit nicely into vibrations, this book presents them in a way that emphasizes understanding of the underlying principles so that students are aware of both the power and the limitations of the methods.

Chapter 1 addresses the way in which a student has to think about previous undergraduate dynamics knowledge in order to make the transition to analysis of vibrating systems. It introduces the idea of small motions about a stable equilibrium state and addresses the details of linearization. Lagrange's Equations are introduced here and students take to them very quickly as an alternative to Newton's Laws.

Chapter 2 considers the details of analyzing single degree of freedom systems. While much of this material is obvious to those skilled in vibrations, it is vital material for developing the students' abilities. It covers topics such as preloads in springs and why gravitational forces don't need to be included because they are canceled out by the constant preloads. It looks at the constitutive relationship for a spring and shows how to draw free body diagrams consistently and accurately.

Chapter 3 is about free vibrations of single degree of freedom systems. It covers systems with and without damping and tries to make sense of what it means to solve a second-order, linear differential equation without being too prescriptive about it.

Chapter 4 looks at time response when applying a harmonic forcing function to an undamped single degree of freedom system, thereby introducing the phenomena of beating and resonance. This is a short chapter although the subject of time response, if presented in detail, could make for a very long one. Time response is an area that I see as being of secondary importance in an introduction to vibrations.

Chapter 5 considers steady state forced vibrations, covering harmonic forcing functions, harmonic base motion, systems with a rotating unbalance, and accelerometer design. This is really the essence of vibrational analysis and is covered in detail.

Chapter 6 is devoted to the very important subject of damping. Linear viscous damping is discussed and the concept of modeling other energy removal devices as "equivalent linear viscous dampers" is introduced. Coulomb damping is covered. The concept of logarithmic decrement is introduced.

Chapter 7 recognizes that systems often have more than one degree of freedom. Deriving the equations of motion for systems with many degrees of freedom is discussed. The concept of multiple natural frequencies, each associated with a different mode shape, is covered in detail. Description of mode shapes is given a lot of time because of its importance in the field. Forced vibrations, vibration absorbers, and the method of normal modes are covered.

Chapter 8 moves on into the study of continuous systems and uses vibrations of a taut string and a cantilever beam as examples of two continuous systems where solutions can be found. The concept of infinitely many degrees of freedom is introduced.

Chapter 9 recognizes that solutions cannot always be found for continuous systems so the finite element method is introduced as an alternate way to get solutions. Shape functions, element mass and stiffness matrices, and assembly of global mass and stiffness matrices are covered, as well as application of boundary conditions and applied forces. Derivations here are handled using Lagrange's Equation because the students are familiar with that approach by the time we get to finite elements. This is certainly not the approach taken by experts in finite elements but it is a useful and appropriate way to get the students to understand the assumptions made in using FEA.

Chapter 10 is devoted to a relatively new device called the “inerter”. This device provides a force that is proportional to the relative acceleration across it. A concept for how to construct such a device and analyses showing its effects on vibrations are presented. This is presented to give the students a look at developing technology with which they can have some fun while realizing that pat answers coming from what they have learned about vibrations to this point may not apply to this device.

Chapter 11 presents a detailed description of how to analyze experimental data in studying vibrations. Topics covered include: Discrete Fourier transforms; sampling and aliasing; leakage and windowing. The approach I have taken here is very old-fashioned in that it treats the Discrete Fourier Transform as something that is derived from a least squares curve fit. This harks back to methods used more than fifty years ago but it enhances students' understanding of what lies behind the transformation to the frequency domain. This is a long chapter that presents methods that the students can program themselves so they don't have to be tied down to packaged FFT programs. Once they have their software working, they can experiment with data sets that clearly demonstrate aliasing, leakage, and so on.

Chapter 12 discusses a variety of topics in vibrations such as how to handle the mass of a spring, flow-induced vibrations, self-excited vibrations in rail vehicles, rigid body modes, and things you can determine from the static deflection of a system. I find that I am usually able to get somewhere into this chapter before the semester is over. The material in this chapter is interesting but certainly doesn't need to be covered to have a complete introduction to vibrations.

It is my hope that this book strikes the right balance for professors teaching introductory vibrations and for their students. I wish them all well.

September, 2019 — *Ronald J. Anderson*
Kingston, Canada

About the Companion Website

The companion website for this book is at

www.wiley.com/go/anderson/introduction-to-vibrations

The website includes:

- Animated GIFs
- PDFs and
- Modes software files

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1

The Transition from Dynamics to Vibrations

Introductory undergraduate courses on dynamics typically consider large scale motions of systems of particles and/or rigid bodies and instantaneous solutions to their nonlinear, governing equations. You may recall working on dynamics problems where a system of bodies starts from rest at a prescribed position and your task was to determine, for example, the angular acceleration of a body or the forces acting on some part of the system. Solutions like this, while having some utility, provide only part of the understanding of the system that is required for a successful design. In most cases, the derived governing equations are complete enough but the “snapshot” solutions don’t help much with the design process.

There are, in fact, many things that can be done with the equations governing the dynamic motion of the system. Briefly, they can be used to

1. Find where the bodies in the system would be if the system were at rest. These are the *Equilibrium States*.
2. Determine whether the equilibrium states are stable or unstable.
3. Determine how the system behaves for small motions away from a stable equilibrium state.
4. Determine the response of the system in the time domain through the use of numerical simulations. This is the most complex type of analysis and, perhaps surprisingly, gives the least information to the designer until the design has reached the fine tuning phase. The simulations are the analog of “cut and try” experiments where an unsuccessful result gives little information on what to change in order to improve the design.

While going through the material presented in this book, you will be concentrating on very small motions of systems about stable equilibrium states. In doing so, you will see connections to topics you may have covered in courses on statics, on dynamics, and on control systems. You will become very familiar with the linearized, differential, equations of motion for dynamic systems moving around stable equilibrium states and methods for deriving and solving them. This is the essence of *Vibrations*.

To get started and as a review of sorts we begin with the dynamic analysis to a relatively simple system – a bead sliding on a rotating semicircular wire.

Introduction to Mechanical Vibrations, First Edition. Ronald J. Anderson.

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1.1 Bead on a Wire: The Nonlinear Equations of Motion

First courses on the subject of Dynamics, whether for particles or rigid bodies, are primarily concerned with teaching the basics of kinematics, free body diagrams, and applications of Newton's Laws of Motion. Applying these three concepts sequentially will lead to a set of simultaneous force and moment balance equations that take account of kinematic constraints.

There are different ways of approaching these problems. One can use a formal vector-based approach and we will start with that here because it gives a complete set of governing equations including solutions for all constraint forces that are required to enforce kinematic constraints on the motion. A shorthand version of this approach which may be called an "informal vector approach" is often used in practice and that will be the second method addressed here. It typically works with two-dimensional views and leads to the governing equations of motion without necessarily solving for all constraint forces. The third approach will see the equations of motion derived using Lagrange's Equations. This is a work/energy approach that leads to the nonlinear differential equation of motion with minimal effort on the part of the analyst. The kinematic constraint forces are automatically eliminated as the governing equations are derived, leaving a designer with no information about forces acting on elements of the system unless extra work is done to find them. Lagrange's Equations are not typically introduced to undergraduate engineers as often as Newton's Laws are, so extra effort is made in this chapter to introduce the procedures for applying Lagrange's Equations to mechanical systems.

As an example, consider Figure 1.1. The figure shows a small bead with mass, m, sliding on a frictionless semicircular wire that rotates about a vertical axis with a constant angular velocity, ω. The wire has radius R. Gravity acts to pull the mass to the bottom of the semicircle while centripetal effects try to move it to the top. The single angular degree of freedom, θ, is sufficient to describe the motion of the bead on the wire.

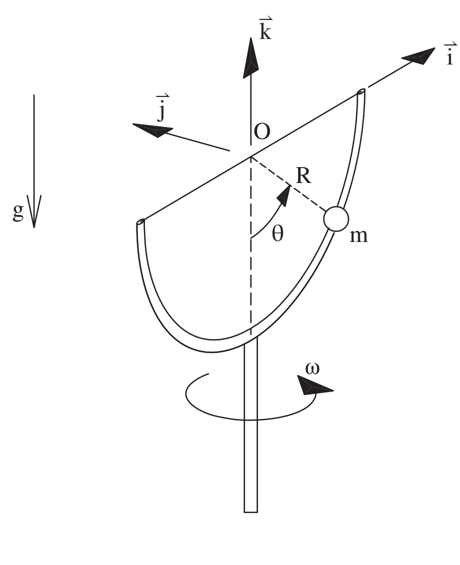

Figure 1.1 A bead on a wire.

1.1.1 Formal Vector Approach using Newton's Laws

Using the formal vector approach, the first step in the kinematic analysis is to choose a coordinate system (i.e. a set of unit vectors) that is convenient for expressing the vectors that will be used. The coordinate system may be fixed or rotating with some known angular velocity. In this case, we will use the $(\vec{\imath}, \vec{\jmath}, \vec{k})$ system shown in Figure 1.1. This is a rotating system fixed in the wire so that $\vec{\imath}$ and $\vec{k}$ stay in the plane of the wire and $\vec{\jmath}$ is perpendicular to the plane. Furthermore, $\vec{\imath}$ and $\vec{\jmath}$ remain horizontal and $\vec{k}$ is always vertical. The angular velocity of the coordinate system is $\vec{\omega} = \omega\vec{k}$.

We use the general approach to differentiating vectors, as follows, where $\vec{r}$ can be a position vector, a velocity vector, an angular momentum vector, or any other vector.

$$\frac{d\vec{r}}{dt} = \underbrace{\dot{\vec{r}}}_{\substack{\text{rate of change}\\ \text{of magnitude}\\ \text{of the vector}}} + \underbrace{\vec{\omega}\times\vec{r}}_{\substack{\text{rate of change}\\ \text{of direction}\\ \text{of the vector}}} \tag{1.1}$$

It is important to understand that the angular velocity vector, $\vec{\omega}$, is the absolute angular velocity of the coordinate system in which the vector, $\vec{r}$, is expressed. There is a danger that the rate of change of direction terms will be included twice if the angular velocity of the vector relative to the coordinate system in which it is measured is used instead.

We start the kinematic analysis by locating a fixed point, in this case point O, and writing an expression for the position vector that locates m with respect to O.

$$\vec{p}_{m/O} = R\sin\theta\vec{\imath} - R\cos\theta\vec{k} \tag{1.2}$$

The absolute velocity of m is

$$\vec{v}_m = \vec{v}_O + \frac{d}{dt}\vec{p}_{m/O} \tag{1.3}$$

Then, using Equation 1.1 and recognizing that $\vec{v}_O = 0$ since O is a fixed point and that $\dot{R} = 0$ since the radius of a semicircle is constant,

$$\vec{v}_m = (R\dot{\theta}\cos\theta\vec{\imath} + R\dot{\theta}\sin\theta\vec{k}) + \omega\vec{k}\times(R\sin\theta\vec{\imath} - R\cos\theta\vec{k}) \tag{1.4}$$

which can be simplified to

$$\vec{v}_m = R\dot{\theta}\cos\theta\vec{\imath} + \omega R\sin\theta\vec{\jmath} + R\dot{\theta}\sin\theta\vec{k} \tag{1.5}$$

The absolute acceleration of m is then

$$\vec{a}_m = \frac{d}{dt}\vec{v}_m = \dot{\vec{v}}_m + \omega\vec{k}\times\vec{v}_m \tag{1.6}$$

which simplifies to

$$\begin{aligned}\vec{a}_m = {} & (R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta - R\omega^2\sin\theta)\vec{\imath} \\ & + (2R\omega\dot{\theta}\cos\theta)\vec{\jmath} + (R\ddot{\theta}\sin\theta + R\dot{\theta}^2\cos\theta)\vec{k}\end{aligned} \tag{1.7}$$

Once an expression for the absolute acceleration has been found, the kinematic analysis is complete and we move on to drawing a Free Body Diagram (FBD). For this example, the FBD is shown in Figure 1.2.

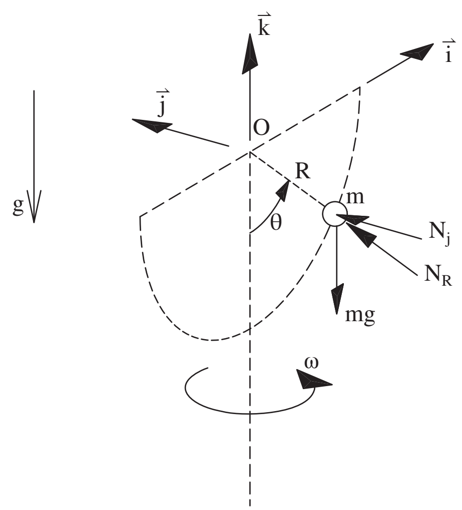

Figure 1.2 Free Body Diagram of a bead on a wire.

Constraints are taken into account when showing the forces acting on the bead. The forces shown and the rationale behind them are:

- $-mg\vec{k}$ = the weight of the body acting vertically downward. This is the effect of gravity.
- $N_j\vec{j}$ = one component of the normal force that the wire transmits to the mass. Since $\vec{j}$ is perpendicular to the plane of the wire, there can be a normal force in that direction.
- $\vec{N}_R = -N_R \sin\theta\vec{i} + N_R \cos\theta\vec{k}$ = the other component of the normal force. We let it have an unknown magnitude N_R and align it with the radial direction since that direction is normal to the wire.
- Note that there is no friction force because the system is frictionless. If there were, we would need to show a friction force acting in the direction that is tangential to the wire.

Once the FBD is complete, we can proceed to write Newton's Equations of Motion by simply summing forces in the positive coordinate directions and letting them equal the mass multiplied by the absolute acceleration in that direction. The result is three scalar equations as follows

$$\sum F_{\vec{i}} : -N_R \sin\theta = m(R\ddot{\theta}\cos\theta - R\dot{\theta}^2 \sin\theta - R\omega^2 \sin\theta) \tag{1.8}$$

$$\sum F_{\vec{j}} : N_j = m(2R\omega\dot{\theta}\cos\theta) \tag{1.9}$$

$$\sum F_{\vec{k}} : -mg + N_R \cos\theta = m(R\ddot{\theta}\sin\theta + R\dot{\theta}^2 \cos\theta) \tag{1.10}$$

At this point in the majority of undergraduate Dynamics courses we would count the number of unknowns that we have in the three equations to see if there is sufficient information to solve the problem. We would find five unknowns

$$N_R, N_j, \theta, \dot{\theta}, \ddot{\theta}$$

and say that we are unable to solve this without further information since we have only three equations. A typical textbook problem would say, for example, that the mass is released from rest (i.e. $\dot{\theta} = 0$) at a specified angle, θ_0, thereby removing two of the unknowns and letting you solve for N_R, N_j and $\ddot{\theta}$.

This solution gives an instantaneous look at the system that really doesn't point out the value of the equations derived. Equations 1.8 through 1.10 do not have five unknowns. They have two unknown constraint forces, N_R and N_j, and a group of variables $(\theta, \dot{\theta}, \ddot{\theta})$ that are related by differentiation. Rather than counting five unknowns as we did earlier, we should say that there are three unknowns

$$N_R, N_j, (\theta, \dot{\theta}, \ddot{\theta})$$

and three equations.

We can combine the three equations to eliminate N_R and N_j and we will be left with a single differential equation containing θ, $\dot{\theta}$, and $\ddot{\theta}$. This nonlinear, ordinary differential equation is the *equation of motion* for the system. Given initial conditions for θ and $\dot{\theta}$, we can solve the equation of motion as a function of time and predict the angle, its derivatives, and the two normal forces at any time. The solution of nonlinear differential equations is not a trivial exercise but can be handled fairly easily using numerical techniques.

The equation of motion for this system can be found by multiplying Equation 1.8 by $\cos\theta$ and adding the result to Equation 1.10 multiplied by $\sin\theta$, giving

$$mR\ddot{\theta} - mR\omega^2 \sin\theta\cos\theta + mg\sin\theta = 0 \tag{1.11}$$

Equation 1.9 is useful only for determining N_j during the motion. An expression for N_R can be found by multiplying Equation 1.8 by $\sin\theta$ and subtracting it from Equation 1.10 multiplied by $\cos\theta$. As a result, we could solve the differential equation of motion (Equation 1.11) numerically and always have the ability to predict the two constraint forces. These forces provide useful design information that is difficult to get from the methods considered next.

1.1.2 Informal Vector Approach using Newton's Laws

Here we consider a two-dimensional view of the system as shown in Figure 1.3 and work out the kinematic expressions for the accelerations from our knowledge of kinematics. There

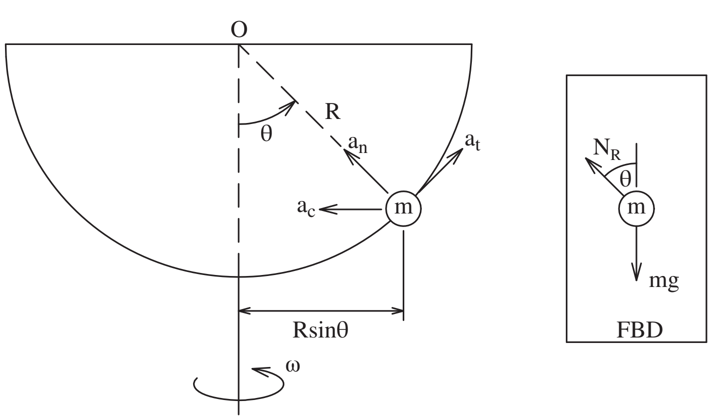

Figure 1.3 A 2D representation of the bead on a wire.

are three acceleration terms shown. They are a tangential acceleration, a_t, a normal acceleration, a_n, and a centripetal acceleration, a_c. Both a_t and a_n are due to the rates of change of the angle θ. Since the wire has constant radius, R, we can immediately write $a_t = R\ddot{\theta}$ and $a_n = R\dot{\theta}^2$ in the directions shown. Centripetal accelerations are of the form $a = \omega^2 r$ and a_c arises from the rotation of the wire with constant angular velocity ω. The relevant radius here is $R \sin\theta$ as shown. Therefore $a_c = \omega^2 R \sin\theta$ in the direction shown.

The inset in Figure 1.3 shows a FBD of the bead with the gravitational force and radial normal force being visible in this plane. There is another normal force perpendicular to the plane that can't be seen in this view. It is N_j in Figure 1.2 and was shown to be equal to $2mR\omega\dot{\theta}\cos\theta$ in Equation 1.9. The acceleration in this expression is a Coriolis acceleration. One needs quite a lot of experience with kinematic analysis to get the correct form of this term using an informal approach. Thankfully, it is perpendicular to the plane in which the bead moves relative to the wire, so it never appears in the equation of motion[1].

Summing forces in the vertical and horizontal directions gives

$$+\uparrow \sum F : N_R \cos\theta - mg = ma_n \cos\theta + ma_t \sin\theta \tag{1.12}$$

$$+\leftarrow \sum F : N_R \sin\theta = ma_c + ma_n \sin\theta - ma_t \cos\theta \tag{1.13}$$

To eliminate the constraining normal force from these two equations, we multiply Equation 1.12 by $\sin\theta$ and Equation 1.13 by $\cos\theta$ and subtract the resulting expressions. The result is

$$\begin{aligned} &N_R(\cos\theta\sin\theta - \sin\theta\cos\theta) - mg\sin\theta = \\ &ma_n(\cos\theta\sin\theta - \sin\theta\cos\theta) + ma_t(\sin^2\theta + \cos^2\theta) - ma_c\cos\theta \end{aligned} \tag{1.14}$$

where it is clear that both N_R and ma_n are multiplied by zero and disappear from further consideration whereas ma_t is multiplied by a trigonometric identity equal to 1. Simplifying and substituting the derived kinematic expressions for a_t and a_c gives

$$mR\ddot{\theta} - mR\omega^2 \sin\theta\cos\theta + mg\sin\theta = 0 \tag{1.15}$$

which is the same nonlinear equation of motion (Equation 1.11) found in Subsection 1.1.1.

1.1.3 Lagrange's Equations of Motion

In this section, we consider the use of Lagrange's[2] Equations of Motion.

Lagrange's Equations, since they are based on work/energy principles, give the analyst two distinct advantages when deriving the equations of motion. First, the vector kinematic analysis is shorter than it is with a direct application of Newton's Laws since acceleration vectors need not be found. This is because the kinetic and potential energy expressions can

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1 The normal force N_j is what determines the torque that some external mechanism must apply to the wire in order to enforce the constraint that it rotate with constant angular velocity. The analysis here, like many dynamic analyses, assumes that torque is available and simply works with the constraint on the angular velocity.

2 Joseph-Louis Lagrange (1736–1813), an Italian/French mathematician, is well known for his work on calculus of variations, dynamics, and fluid mechanics. In 1788 Lagrange published the *Mécanique Analytique* summarizing all the work done in the field of mechanics since the time of Newton, thereby transforming mechanics into a branch of mathematical analysis.

be derived from velocity vectors and position vectors respectively. Secondly, there is no need to draw free body diagrams for each of the rigid bodies in the system because the forces of constraint between the bodies do no work and are therefore not required for the analysis.

Of course, there are also disadvantages. The method requires a great deal of differentiation, sometimes of relatively complicated functions. Some analysts prefer the kinematics of Newton's method over the differentiation required when using Lagrange's Equations. Some point to a lack of physical feeling for problems without free body diagrams as being a disadvantage of the method. Finally, if the intent of analyzing the dynamics of a system is to predict loads, which could be carried forward into a structural analysis for instance, the forces of interaction between bodies are not available from a straightforward application of Lagrange's Equations.

1.1.3.1 The Bead on a Wire via Lagrange's Equations

We consider again the bead on the semicircular wire (Figure 1.1) and derive the equation of motion using Lagrange's Equations.

Lagrange's Equation is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = Q_q \tag{1.16}$$

where:

- T = the total kinetic energy of the system
- U = the total potential energy of the system
- q = a generalized coordinate
- $\dot{q}$ = the time derivative of q
- Q_q = the generalized force corresponding to a variation of q

We first determine the kinetic energy of the system. This requires that we have an expression for the absolute velocity of the mass. This was done previously and the result, from Equation 1.5, is

$$\vec{v}_m = R\dot{\theta}\cos\theta\vec{\imath} + \omega R\sin\theta\vec{\jmath} + R\dot{\theta}\sin\theta\vec{k} \tag{1.17}$$

The kinetic energy of the system is then

$$T = \frac{1}{2}m(\vec{v}_m \cdot \vec{v}_m) \tag{1.18}$$

which becomes, after substitution of Equation 1.17 and some simplification,

$$T = \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) \tag{1.19}$$

Alternatively, using the informal approach and referring to Figure 1.3, we can see that there will be a component of velocity equal to $R\dot{\theta}$ tangent to the wire and another component equal to $\omega R\sin\theta$ perpendicular to the wire and into the page. These two components are mutually perpendicular so we can write, by applying Pythagoras' theorem,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(R^2\dot{\theta}^2 + \omega^2R^2\sin^2\theta) \tag{1.20}$$

After factoring R^2 out of the brackets, this becomes exactly the same expression we had in Equation 1.19.

The potential energy of the system is due to gravity only. If the datum for potential energy is taken to be at point O, the potential energy, U, of the system is determined simply by the vertical distance from O to the bead. This distance is $R\cos\theta$ and, as the mass is below the datum, the potential energy is negative, leading to

$$U = -mgR\cos\theta \tag{1.21}$$

Having expressions for T and U and a single degree of freedom, θ, we can apply Lagrange's Equation (Equation 1.16) and find

$$\begin{aligned}
q &= \theta \\
\dot{q} &= \dot{\theta} \\
\frac{\partial T}{\partial \dot{\theta}} &= mR^2\dot{\theta} \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) &= mR^2\ddot{\theta} \\
\frac{\partial T}{\partial \theta} &= mR^2\omega^2\sin\theta\cos\theta \\
\frac{\partial U}{\partial \theta} &= mgR\sin\theta \\
Q_\theta &= 0
\end{aligned} \tag{1.22}$$

Substituting the expressions from Equation 1.22 into Lagrange's Equation (Equation 1.16) gives the desired equation of motion

$$mR^2\ddot{\theta} - mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta = 0 \tag{1.23}$$

where we note that this equation when divided throughout by R yields the same result as Equations 1.11 and 1.15, the equation of motion derived using Newton's Laws.

Clearly, Equation 1.23 could be further simplified by factoring out the group mR but this would take away the ability to look at the individual terms and give a physical explanation for them. Whenever an equation is derived, the first test for correctness is to see if all of the terms have the same dimensions. In this case, the first term has dimensions of ML^2/T^2 where M is *mass*, L is *length*, and T is *time*. Note that angles such as θ are in radians, which are dimensionless since they are defined by an arc length divided by a radius. It follows that trigonometric functions such as $\sin\theta$ and $\cos\theta$ are also dimensionless. Angular velocities therefore have dimensions derived from angles divided by time, $1/T$, and angular accelerations are expressed as $1/T^2$. Using these conventions, it is easy to see that all three terms in Equation 1.23 have the same dimensions[3].

The dimensions of force are ML/T^2 or mass times acceleration. Taking this into account, we can see that the three terms in Equation 1.23 all have dimensions of FL or force times length. The terms are, in fact, all moments. The third term is the most obvious because it contains the gravity force mg multiplied by a moment arm of $R\sin\theta$. The moment arm

3 Note the difference between dimensions and units. *Dimensions* refer to physical characteristics such as mass, length, or time. *Units* refer to the system of measurement we use to substitute numbers into an equation. Examples are kilograms for mass, feet for length, and minutes for time.

is simply the horizontal distance between the mass and point O. Lagrange's Equation has produced an equation of motion based on a dynamic moment balance about the stationary point O and it did so without requiring the derivation of acceleration expressions, the drawing of free body diagrams, or the production of force and moment balance relationships. This is the power of using Lagrange's Equation for deriving equations of motion.

Following are explanations of terms that arise when using Lagrange's Equations.

1.1.3.2 Generalized Coordinates

The generalized coordinates are simply the degrees of freedom of the system with the condition that they be independently variable. That is, given a system with N generalized coordinates ($q_i, i = 1, N$), any generalized coordinate, q_j, must be able to undergo an arbitrary small variation, δq_j, with all of the other generalized coordinates being held constant.

In the example just considered, we might try to specify the position of the bead on the wire by using two coordinates – the vertical distance from O and the horizontal distance from O. We would soon find that these coordinates are not independent because the bead is constrained to stay on the circular wire so changing the horizontal position requires a change in the vertical position. These two coordinates are therefore not generalized coordinates.

1.1.3.3 Generalized Forces

The generalized force, Q_{q_r}, associated with the generalized coordinate, q_r, accounts for the effect of externally applied forces that are not included in the potential energy. We normally include elastic (i.e. spring) forces and gravitational forces in the potential energy and all others enter through the use of generalized forces.

Given a three-dimensional applied force

$$\vec{F} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} \tag{1.24}$$

with a position vector

$$\vec{p}_F = x\vec{\imath} + y\vec{\jmath} + z\vec{k} \tag{1.25}$$

relative to a fixed point, we define the right-hand side of Lagrange's Equation for generalized coordinate q_r to be the *generalized force*, Q_{q_r}, where

$$Q_{q_r} = F_x\frac{\partial x}{\partial q_r} + F_y\frac{\partial y}{\partial q_r} + F_z\frac{\partial z}{\partial q_r} \tag{1.26}$$

The two most common methods for finding the generalized forces are as follows.

1. **The formal method**
   The most formal approach, and one that always works, starts with the vector expression of the absolute position of the point of application of the force

   $$\vec{p}_F = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

   and then writes the generalized force as

   $$Q_{q_r} = \vec{F} \cdot \frac{\partial \vec{p}_F}{\partial q_r} \tag{1.27}$$

Equation 1.27 can be written for each of N applied forces and the resulting scalar generalized forces can be added together to give the total generalized force for generalized coordinate q_r as

$$Q_{q_r} = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{p}_{F_i}}{\partial q_r} \tag{1.28}$$

2. **The intuitive approach**

Let there be n generalized coordinates specifying the position of a force acting on a dynamic system in Cartesian Coordinates. The force will be acting at the point (x, y, z) where the coordinates x, y and z are functions of the generalized coordinates q_1 through q_n and of time, t, as follows

$$\begin{aligned} x &= x(q_1, q_2, \dots, q_r, \dots, q_n, t) \\ y &= y(q_1, q_2, \dots, q_r, \dots, q_n, t) \\ z &= z(q_1, q_2, \dots, q_r, \dots, q_n, t) \end{aligned} \tag{1.29}$$

Variations in the position of the force as the generalized coordinates are varied while time is held constant can be written as

$$\begin{aligned} \delta x &= \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2 + \dots \frac{\partial x}{\partial q_r}\delta q_r + \dots \frac{\partial x}{\partial q_n}\delta q_n \\ \delta y &= \frac{\partial y}{\partial q_1}\delta q_1 + \frac{\partial y}{\partial q_2}\delta q_2 + \dots \frac{\partial y}{\partial q_r}\delta q_r + \dots \frac{\partial y}{\partial q_n}\delta q_n \\ \delta z &= \frac{\partial z}{\partial q_1}\delta q_1 + \frac{\partial z}{\partial q_2}\delta q_2 + \dots \frac{\partial z}{\partial q_r}\delta q_r + \dots \frac{\partial z}{\partial q_n}\delta q_n \end{aligned} \tag{1.30}$$

If we are trying to find the generalized force corresponding to only one of the generalized coordinates, say q_r, we rewrite Equation 1.30 with $\delta q_i = 0; i = 1, n; i \neq r$ and $\delta q_r \neq 0$, giving

$$\begin{aligned} \delta x &= \frac{\partial x}{\partial q_r}\delta q_r \\ \delta y &= \frac{\partial y}{\partial q_r}\delta q_r \\ \delta z &= \frac{\partial z}{\partial q_r}\delta q_r \end{aligned} \tag{1.31}$$

Now consider Equation 1.26 with each side multiplied by δq_r

$$Q_{q_r}\delta q_r = F_x \frac{\partial x}{\partial q_r}\delta q_r + F_y \frac{\partial y}{\partial q_r}\delta q_r + F_z \frac{\partial z}{\partial q_r}\delta q_r \tag{1.32}$$

The terms from Equation 1.31 can be substituted into the right-hand side of Equation 1.32 to yield

$$Q_{q_r}\delta q_r = F_x \delta x + F_y \delta y + F_z \delta z \tag{1.33}$$

The right-hand side of Equation 1.33 can be seen to be the work done by the applied force as its position varies due to changes in the generalized coordinate q_r while all other generalized coordinates and time are held constant.

Using the intuitive approach to finding generalized forces, the analyst will consider, in sequence, the variation of individual generalized coordinates and will write expressions for the total work done during each variation. The generalized force associated with each generalized coordinate will be the work done during the variation of that coordinate, δW_{q_r}, divided by the variation in the coordinate. That is,

$$Q_{q_r} = \delta W_{q_r} / \delta q_r \tag{1.34}$$

1.1.3.4 Dampers – Rayleigh's Dissipation Function

Devices called "dampers" are common in mechanical systems. These are elements that dissipate energy and they are modeled as producing forces that are proportional to their rate of change of length. The rate of change of length is the relative velocity across the damper. "Proportional" implies linearity and a force proportional to speed implies laminar, viscous flow. As a result, these elements are often referred to as "linear viscous dampers".

Figure 1.4 shows a system where a body is attached to ground by a damper. The body is moving to the right with speed v and the damping coefficient (constant of proportionality) is c. The physical connection of the damper to both the ground and the body dictates that the rate of change of length of the damper is equal to the speed v. The force in the damper will therefore be $F_d = cv$. The direction of the force will be such that it causes the damper to increase in length as shown in the lower part of Figure 1.4. By Newton's 3rd Law, the force on the body must be equal and opposite to the force acting on the damper. The force F_d therefore acts to the left on the body. In other words, the damping force opposes the velocity of the body.

Consider now the more general case of a particle where the velocity of the body is given by

$$\vec{v}_m = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} \tag{1.35}$$

Given this velocity, the force that the damper applies to the particle will be

$$\vec{F}_d = -c\vec{v}_m = -c\dot{x}\vec{i} - c\dot{y}\vec{j} - c\dot{z}\vec{k} \tag{1.36}$$

The components of $\vec{F}_d$ can be substituted into Equation 1.26 to get the following expression for the generalized force arising from the damper

$$Q^d_{q_r} = -c\dot{x}\frac{\partial x}{\partial q_r} - c\dot{y}\frac{\partial y}{\partial q_r} - c\dot{z}\frac{\partial z}{\partial q_r} \tag{1.37}$$

Figure 1.4 A linear viscous damper.