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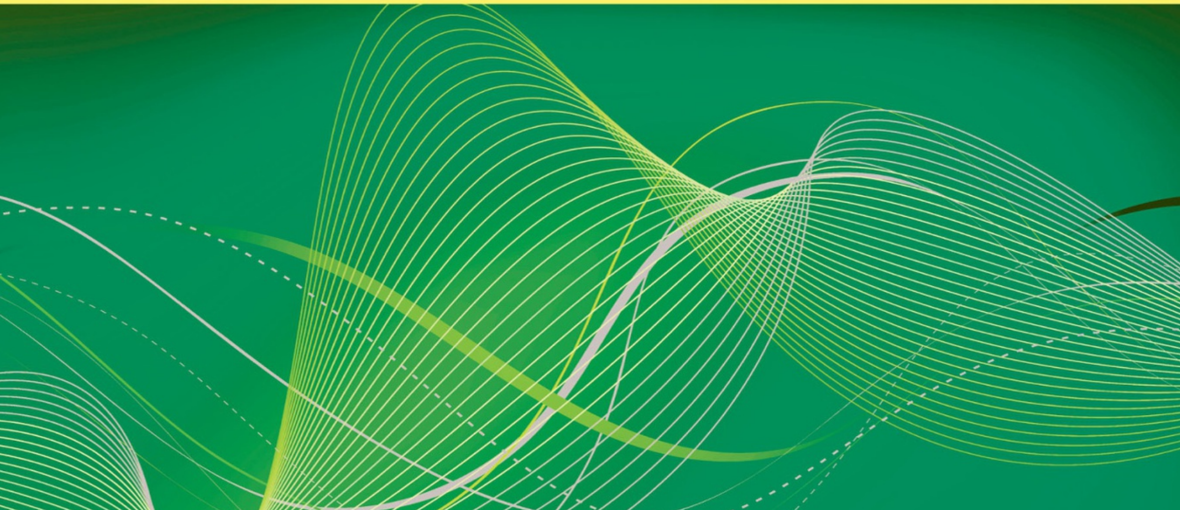
Mathematical Modeling of Random and Deterministic Phenomena

Edited by
Solym Mawaki Manou-Abi
Sophie Dabo-Niang
Jean-Jacques Salone



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Nikolaos Limnios

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Preface

In order to identify mathematical modeling and interdisciplinary research issues in evolutionary biology, epidemiology, epistemology, environmental and social sciences encountered by researchers in Mayotte, the first international conference on mathematical modeling (CIMOM'18) was held in Dembéni, Mayotte, from November 15 to 17, 2018, at the *Centre Universitaire de Formation et de Recherche*. The objective was to focus on mathematical research with interdisciplinarity.

This book aims to highlight some of the mathematical research interests that appear in real life, for example the study of random and deterministic phenomena. It also aims to contribute to the future emergence of mathematical modeling tools that can provide answers to some of the specific research questions encountered in Mayotte. In Mayotte and its region, including the coastal zone of Africa, climate change has impacted ecological, biological, epidemiological, environmental, social and natural systems. There is an urgent need to use mathematical tools to understand what is happening and what may happen and to help decision-makers. The modeling of such complex systems has therefore become a necessity, in particular, to preserve the ecological, environmental, economic, social and natural environments of Mayotte. Mayotte is, in fact, a research laboratory, where the scientific fields converge. The CIMOM'18 conference was an effective opportunity to present not only recent advances in mathematical modeling, with an emphasis on epidemiology, ecology, the environment, evolution biology and socio-economic issues, but also new interdisciplinary research questions.

Most of the documents presented in this book have been collected from a variety of sources, including communication documents at the CIMOM'18. It contains not only chapters related to the research questions above-mentioned, but also potential mathematical modeling tools for some important research questions.

After the CIMOM'18, we invited the original authors (or speakers) to write journal articles to provide contributions on these questions, with a common structure for each chapter, in terms of pointing out mathematical models, illustrative examples and applications on advanced topics, with a view to publishing this Wiley Mathematics and Statistics series book. Each chapter has been reviewed by one or two independent reviewers and the book publishers. Some chapters have undergone major revisions based on the reviews before being definitively accepted.

We hope that this book will promote mathematical modeling tools in real applications and inspire more researchers in Mayotte and other regions to further explore emerging research issues and impacts.

Solym Mawaki MANOU-ABI
Sophie DABO-NIANG
Jean-Jacques SALONE
November 2019

Acknowledgments

This book was made possible through the collaboration of many people and institutions whom we would like to thank. The idea for its drafting was born from the organization of the international conference on mathematical modeling in Mayotte (CIMOM'18). Very quickly it became clear to us that it was necessary to write articles in the form of a collective book that could serve as a basis for the development of mathematical tools for the modeling of complex systems. Mathematics is the foundation of science, and it is essential for the economic development of a region or a country. Mayotte can, and must, participate more in access to mathematic and scientific research.

We would like to thank the *Centre Universitaire de Mayotte* and its Scientific Commission, the University of Montpellier and the Vice-Rectorate of Mayotte for their scientific, financial and logistical support. We would like to thank all the authors, speakers, guest speakers and people who have contributed to this beautiful project, namely: Etienne Pardoux, Benoîte De Saporta, Jean Dhombres, Abdennebi Omrane, Loïc Louison, William Dimbour, Gwladys Toulemonde, Dominique Hervé, Angelo Raherinirina, Sylvain Dotti, Éloïse Comte, André Mas, Christian Delhommé, Jean Diatta, Bertrand Cloez, Jean-Michel Marin, Aurélien Siri, Elliott Sucré, Abal-Kassim Cheik Ahamed, Laurent Souchard and all the students involved.

We also thank Nikolaos Limnios, who was the capable editor for this book. In addition to being very familiar with the subject of mathematical modeling, he was able to help us during the various stages of the book's production. Many renowned anonymous researchers helped to review the chapters of this book and we would also like to thank them a lot.

A special thanks to Cédric Villani and Charles Torossian for their exceptional lectures at the CIMOM'18 and for supporting this project.

Introduction

This book, entitled "Mathematical Modeling of Random and Deterministic Phenomena", was written to provide details on current research in applied mathematics that can help to answer many of the modeling questions encountered in Mayotte. It is aimed at expert readers, young researchers, beginning graduate and advanced undergraduate students, who are interested in statistics, probability, mathematical analysis and modeling. The basic background for the understanding of the material presented is timely provided throughout the chapters.

This book was written after the international conference on mathematical modeling in Mayotte, where a call for chapters of the book was made. They were written in the form of journal articles, with new results extending the talks given during the conference and were reviewed by independent reviewers and book publishers.

This book discusses key aspects of recent developments in applied mathematical analysis and modeling. It also highlights a wide range of applications in the fields of biological and environmental sciences, epidemiology and social perspectives. Each chapter examines selected research problems and presents a balanced mix of theory and applications on some selected topics. Particular emphasis is placed on presenting the fundamental developments in mathematical analysis and modeling and highlighting the latest developments in different fields of probability and statistics. The chapters are presented independently and contain enough references to allow the reader to explore the various topics presented. The book is primarily intended for graduate students, researchers and educators; and is useful to readers interested in some recent developments on mathematical analysis, modeling and applications.

The book is organized into two main parts. The first part is devoted to the analysis of some advanced mathematical modeling problems with a particular focus on

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Introduction written by Solym Mawaki MANOU-ABI, Sophie DABO-NIANG and Jean-Jacques SALONE.

epidemiology, environmental ecology, biology and epistemology. The second part is devoted to a mathematical modelization with interdisciplinarity in ecological, socio-economic, epistemological, natural and social problems.

In Chapter 1, we present large population approximations for several deterministic and stochastic epidemic models. The hypothesis of constant population of susceptibles is explained through some realistic situations. After recalling the definition of SIS, SIRS and SIR models, a law of large numbers (LLN) is presented as well as a central limit theorem (CLT) to estimate the time of extinction of an epidemic and a principle of great deviation to estimate the error. This chapter then describes the principle of moderate deviations. These results are then used to deduce the critical population sizes for launching an epidemic. It explains how it can be used to predict the time taken for an epidemic to cease.

Chapter 2 is devoted to the study of non-parametric prediction of biomass of demersal fish in a coastal area, with a case study in Senegal. The inputs of the regression model are spatio-functional, i.e. the temperature and salinity of the water are depth curves recorded at different fishing locations. The prediction is done through a dual kernel estimator accounting the proximity between the temperature or salinity observations and locations. The originality of the approach lies in the functional nature of the exogeneous variables. Some theoretical asymptotic results on the predictor are provided.

Chapter 3 is concerned with the study of urban flood risk in urban areas caused by heavy rainfall, that may trigger considerable damage. The simulated water depths are very sensitive to the temporal and spatial distribution of rainfall. Besides, rainfall, owing in particular to its intermittency, is one of the most complex meteorological processes. Its simulation requires an accurate characterization of the spatio-temporal variability and intensity from available data. Classical stochastic approaches are not designed explicitly to deal with extreme events. To this end, spatial and spatio-temporal processes are proposed in the sound asymptotic framework provided by extreme value theory. Realistic simulation of extreme events raises a number of issues such as the ability to reproduce flexible dependence structure and the simulation of such processes.

In Chapter 4, we consider a problem of change-point detection for a continuous-time stochastic process in the family of piecewise deterministic Markov processes. The process is observed in discrete-time and through noise, and the aim is to propose a numerical method to accurately detect both the date of the change of dynamics and the new regime after the change. To do so, we state the problem as an optimal stopping problem for a partially observed discrete-time Markov decision process, taking values in a continuous state space, and provide a discretization of the state space based on quantization to approximate the value function and build a tractable stopping policy. We provide error bounds for the approximation of the value function and numerical simulations to assess the performance of our candidate policy. An application concerns treatment optimization for cancer patients. The change point then corresponds to a sudden deterioration of the health of the patient. It must be detected early, so that the treatment can be adapted.

The context of Chapter 5 is the nutrient transfer mechanism in croplands. The authors study the case of an additional nutrient which comes from a "service plant" (meaning a natural input), as a control function. The Nye-Tinker-Barber model is introduced with a perturbation as an unknown source of nutrient. An optimal control formulation of this problem is studied and adapted for the incomplete data case. A characterization of the low-regret optimal control is provided

In Chapter 6, basic stochastic evolution equations in long-time periodic environment are developed. Periodicity often appears in implicit ways in various phenomena. For instance, this is the case when we study the effects of fluctuating environments on population dynamics. Some classical books gave a nice presentation of various extensions of the concepts of periodicity, such as almost periodicity, asymptotically periodicity, almost automorphy, as well as pertinent results in this area. Recently, there has been an increasing interest in extending certain results to stochastic differential equations in separable Hilbert space. This is due to the fact that almost all problems in a real life situation, to which mathematical models are applicable, are basically stochastic rather than deterministic. In this chapter, we deal with a stochastic fractional integro-differential equation, for which a result of existence and uniqueness of an asymptotically periodic solution is given.

In Chapter 7, we study the existence of solutions in semilinear evolution equations with impulse, where the differential operator generates a strongly compact semi-group. The chapter generalizes a recent published work by one of the co-authors to the non-local initial condition case. In the previous work, the existence, stability and smoothness of bounded solutions for impulsive semilinear parabolic equations with Dirichlet boundary conditions, are obtained using the Banach fixed point theorem, under the classical Lipschitz assumptions.

In Chapter 8, we discuss the history and criticisms of a mathematical model, namely the diffusion of heat. The starting point is a "thought experiment" on the diffusion of heat through an infinite rectangular flat lamina. This is the path along which Fourier invented the representation of functions that bears his name; and we mainly treat the typical example of the periodic step function. Fourier thus invented the notion of proper modes, also known today as eigen modes, and found the orthogonality relations. Following Fourier, we then consider an example, the diffusion of heat in a sphere like the Earth, and come up with the required adaptation that, for the first time, allowed us to investigate the greenhouse effect. We then examine some of the criticisms related to Fourier's representation until functional analysis was created in the 20th Century, answering various questions. Still, an interesting creation came with a critique from quantum mechanics in the 1930s, perhaps not understood as such, but which led to wavelets as developed in the 21st Century, and a remarkable new tool that can be adapted to various situations. The text, in a story form, aims to combine mathematics, physics and also epistemology in a history that is rigorous with respect for original texts; it also tries to understand the meaning of a scientific posterity for the construction of science, as well as how a thought experiment has been transformed into a realistic modeling.

The second part is dedicated to the development of interdisciplinary modeling with mathematical approaches.

In Chapter 9, we present a methodology for interdisciplinary modeling of complex systems using hypergraphs. This project begins by setting out the research stakes related to the sustainable management of mangrove forests in Mayotte: Mangroves are coastal ecosystems that have undergone global upheavals while facing a number of issues regarding biodiversity, pressures for natural hazards and attractiveness for the socio-economic development of territories. The mangroves of Mayotte thus present high stakes of preservation and management. This sustainable management is conceived in a participatory framework where, “it seems necessary for the users of the mangrove and those involved in the management of these wetlands, to exchange their experiences and knowledge further”. The author proposes an interdisciplinary system approach in ecology, geography, literature and modeling that aims at “the identification of variables” and “interactions in order to co-construct conceptual models combining societal and ecological dimensions” and “the identification of key variables to guide reflection on the sustainable management of these mangroves. The author aims to contribute to the implementing of integrated management of Mayotte’s mangroves in order to preserve them and ensure the maintenance of their ecosystem services”.

In Chapter 10, we discuss modeling of post-forestry transitions in Madagascar and the Indian Ocean by setting up a dialogue between mathematics, computer science and environmental sciences. We discuss mathematical tools, implemented to model and analyze the dynamics of complex socio-ecological systems, made up of cultivated and inhabited areas after deforestation in Madagascar.

In Chapter 11, the authors propose a descriptive analysis and a modelization of the evolution of the birth rate in Mayotte.

Finally, in Chapter 12, we develop the idea that excessive mathematical modeling of the Mahoran economy would be ineffective to really take into account the weight of informal economy sectors, even though a systemic modeling seems to be an interesting perspective. The argument is based, in a historical and epistemological approach, on the critical discussion of two classic arguments for mathematization economy: the ontological argument that the economy is based on numbers (and laws) and is therefore arithmetic-algebraic in nature, and the linguistic argument that considers mathematical language as a bearer minima of universality, logic and rigor. Examples of economic situations encountered in Mayotte support this argument, showing the complex links that exist between the formal and informal economy, between modern society and traditional practices. The statistician drift is denounced. The diversity and multiplicity of stakeholders and economic factors also appear as obstacles to mathematical modeling.

Last but not least, we are grateful to our families for their continued support, encouragement and especially for supporting us during all the long hours we spent away from them while working on this book.

PART 1

Advances in Mathematical Modeling

Mathematical Modeling of Random and Deterministic Phenomena,
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and Jean-Jacques Salone.

1

Deviations From the Law of Large Numbers and Extinction of an Endemic Disease

1.1. Introduction

We consider epidemic models with a constant flux of susceptibles, either because an infected individual becomes susceptible immediately after healing, or after some time when the individual becomes immune to the illness, or because there is a constant flux of newborn or immigrant susceptibles.

In the above-mentioned three cases, for certain values of the parameters, there is an endemic equilibrium, which is a stable equilibrium of the associated deterministic epidemic model. The deterministic model can be considered as the law of large numbers limit (as the size of the population tends to ∞) of a stochastic model, where infections, healings, births and deaths happen according to Poisson processes, whose rates depend upon the numbers of individuals in each compartment.

Since the disease-free states are absorbing, it follows from an irreducibility property, which is clearly valid in our models, that the epidemic will stop sooner or later in the more realistic stochastic model. However, the time which the stochastic perturbances will need to stop the epidemic may be enormous when the size N of the population is large. The aim of this chapter is to describe, based on the central limit theorem (CLT), large and moderate deviations (LD, MD), the time it takes for the epidemic to stop in the stochastic model.

The chapter is organized as follows. In section 1.2, we describe the three deterministic and stochastic models which we have in mind, namely, the SIS, SIRS

Chapter written by Étienne PARDOUX.

and SIR model with demography. In section 1.3, we give the general formulation of the stochastic models, and recall the law of large numbers, the central limit theorem and the theory of large deviations, and their application to the time of extinction of an epidemic. Finally, in section 1.4, we present the moderate deviations result for the SIS model (which is the simplest of our three models), and explain how it can be used to predict the time taken for an epidemic to cease. Those results will be proved in more generality, with full details of the proofs in Pardoux (2019).

The results concerning the law of large numbers and the large deviations can be found in Kratz and Pardoux (2018), Pardoux and Samegni-Kepgnou (2017), and Britton and Pardoux (2019b), where the central limit theorem is also established. Note that the three above-mentioned references present different approaches to the large deviations results. The moderate deviations results will appear in Pardoux (2019).

We conclude this introduction with a short history and a few references to books and lecture notes which describe models of infectious diseases and epidemics. Mathematical modeling of infectious diseases has a long history of being useful. The first such mathematical model was probably the one proposed by Bernoulli in Bernoulli (1760), with a model of smallpox. A little more than one hundred years ago, Sir Ronald Ross, a British medical doctor and Nobel laureate, who contributed to the understanding of malaria wrote:

> As a matter of fact all epidemiology, concerned as it is with variation of disease from time to time and from place to place, must be considered mathematically (...) and the mathematical method of treatment is really nothing but the application of careful reasoning to the problems at hand.

As a matter of fact, Ross deduced, from mathematical arguments, conclusions concerning malaria, which his physician colleagues found hard to accept. One of the first books devoted to mathematical modeling of infectious diseases is Bailey (1975). A book which has had huge impact is Anderson and May (1991), which deals exclusively with deterministic models. Since then, there has been steady production of new research monographs, for example Andersson and Britton (2000) also looking at inference methodology, Daley and Gani (1999) focusing mainly on stochastic models, Keeling and Rohani (2008) dealing also with animal populations, and Diekmann, Heesterbeek, and Britton (2013) covering both deterministic and stochastic modeling. Finally, Britton and Pardoux (2019a) will soon present the broadest treatment of stochastic epidemic models ever published in one volume, covering both classical and new results and methods, from mathematical models to statistical procedures.

1.2. The three models

1.2.1. *The SIS model*

The deterministic SIS model is the following. Let $s(t)$ (respectively $i(t)$) denote the proportion of susceptible (respectively infectious) individuals in the population. Given an infection parameter λ and a recovery parameter γ, the deterministic SIS model can be written as

$$\begin{cases} s'(t) = -\lambda s(t)i(t) + \gamma i(t), \\ i'(t) = \lambda s(t)i(t) - \gamma i(t). \end{cases}$$

Since clearly $s(t) + i(t) \equiv 1$, the system can be reduced to a one-dimensional ordinary differential equation. If we let $z(t) = i(t)$, we have $s(t) = 1 - z(t)$, and we obtain the ordinary differential equation

$$z'(t) = \lambda z(t)(1 - z(t)) - \gamma z(t)\,.$$

It is easy to verify that this ordinary differential equation has a so-called "disease-free equilibrium", which is $z(t) = 0$. If $\lambda > \gamma$, this equilibrium is unstable, and there is a stable endemic equilibrium $z(t) = 1 - \gamma/\lambda$.

The corresponding stochastic model is as follows. Let S_t^N (respectively I_t^N) denote the proportion of susceptible (respectively infectious) individuals in a population of total size N.

$$\begin{cases} S_t^N = S_0^N - \dfrac{1}{N} P_{inf}\left(\lambda N \displaystyle\int_0^t S_r^N I_r^N dr\right) + \dfrac{1}{N} P_{rec}\left(\gamma N \displaystyle\int_0^t I_r^N dr\right), \\ I_t^N = I_0^N + \dfrac{1}{N} P_{inf}\left(\lambda N \displaystyle\int_0^t S_r^N I_r^N dr\right) - \dfrac{1}{N} P_{rec}\left(\gamma N \displaystyle\int_0^t I_r^N dr\right). \end{cases}$$

Here $P_{inf}(t)$ and $P_{rec}(t)$ are two mutually independent standard (i.e. rate 1) Poisson processes. Let us give some explanations, first concerning the modeling, then concerning the mathematical formulation.

Let $\mathcal{S}_t^N$ (respectively $\mathcal{I}_t^N$) denote the number of susceptible (respectively infectious) individuals in the population. The equations for those quantities are the above equations, multiplied by N. The argument of $P_{inf}(t)$ can be written as

$$\lambda \int_0^t \frac{\mathcal{S}_r^N}{N} \mathcal{I}_r^N dr\,.$$

The justification for such a rate of infections in the total population is as follows. Each infectious individual meets other individuals in the population at some rate β.

The encounter results in a new infection with probability p if the partner of the encounter is susceptible, which happens with probability S_t^N/N, since we assume that each individual in the population has the same probability of being that partner, and with probability 0 if the partner is an infectious individual. Letting $\lambda = \beta p$ and summing over the infectious at time t gives the above rate. Concerning recovery, it is assumed that each infectious recovers at rate γ, independently of the others.

REMARK 1.1.– *Let us comment about the fact that we write our stochastic models in terms of Poisson processes. The fact that the infection events happen according to a Poisson process is a rather natural assumption. However, concerning the recovery from infection, our model assumes that the duration of the infectious period follows an exponential distribution. This is not realistic. We are forced to make such an assumption if we want to have a Markov model. We must confess that this assumption is done for mathematical convenience. However, we expect to extend our results to non-Markovian models in forthcoming publications.*

Note that there is an equivalent, but slightly more complicated way of writing the Poisson terms, which we now present. Let $\mathcal{M}_{inf}$ and $\mathcal{M}_{rec}$ denote two mutually independent Poisson random measures on $(0, +\infty)^2$, with mean measure the Lebesgue measure.

$$P_{inf}\left(\lambda N \int_0^t S_r^N I_r^N dr\right) \text{ can be rewritten as } \int_0^t \int_0^\infty \mathbf{1}_{u \le \lambda N S_r^N I_r^N dr} \mathcal{M}_{inf}(dr, du)$$

and

$$P_{rec}\left(\gamma N \int_0^t I_r^N dr\right) \text{ can be rewritten as } \int_0^t \int_0^\infty \mathbf{1}_{u \le \gamma N I_r^N dr} \mathcal{M}_{rec}(dr, du)\,.$$

Again we have $S_t^N + I_t^N = 1$, and $Z_t^N = I_t^N$ satisfies

$$Z_t^N = Z_0^N + \frac{1}{N} P_{inf}\left(\lambda N \int_0^t (1 - Z_r^N) Z_r^N dr\right) - \frac{1}{N} P_{rec}\left(\gamma N \int_0^t Z_r^N dr\right).$$

1.2.2. *The SIRS model*

In the SIRS model, contrary to the SIS model, an infectious who heals is first immune to the illness, he is “recovered”, and only after some time does he lose his immunity and turn susceptible. The deterministic SIRS model can be written as

$$\begin{cases} s'(t) = -\lambda s(t) i(t) + \rho r(t), \\ i'(t) = \lambda s(t) i(t) - \gamma i(t), \\ r'(t) = \gamma i(t) - \rho r(t), \end{cases}$$

while the stochastic SIRS model can be written as

$$\begin{cases} S_t^N = S_0^N - \frac{1}{N}P_{inf}\left(\lambda N\int_0^t S_r^N I_r^N dr\right) + \frac{1}{N}P_{loim}\left(\rho N\int_0^t R_r^N dr\right), \\ I_t^N = I_0^N + \frac{1}{N}P_{inf}\left(\lambda N\int_0^t S_r^N I_r^N dr\right) - \frac{1}{N}P_{rec}\left(\gamma N\int_0^t I_r^N dr\right) \\ R_t^N = R_0^N + \frac{1}{N}P_{rec}\left(\gamma N\int_0^t I_r^N dr\right) - \frac{1}{N}P_{loim}\left(\rho N\int_0^t R_r^N dr\right). \end{cases}$$

These two models could be reduced to two-dimensional models for $z(t) = (i(t), s(t))$ (respectively $Z_t^N = (I_t^N, S_t^N)$).

1.2.3. *The SIR model with demography*

In this model, recovered individuals remain immune forever, but there is a flux of susceptibles by births at rate μN, while individuals from each of the three compartments die at rate μ. Thus, the deterministic model

$$\begin{cases} s'(t) = \mu - \lambda s(t)i(t) - \mu s(t) \\ i'(t) = \lambda s(t)i(t) - \gamma i(t) - \mu i(t) \\ r'(t) = \gamma i(t) - \mu r(t), \end{cases}$$

whose stochastic variant can be written as

$$\begin{cases} S_t^N = S_0^N - \frac{1}{N}P_{inf}\left(\lambda N\int_0^t S_r^N I_r^N dr\right) + \frac{1}{N}P_{birth}(\rho N t) - \frac{1}{N}P_{ds}\left(\mu N\int_0^t S_r^N dr\right), \\ I_t^N = I_0^N + \frac{1}{N}P_{inf}\left(\lambda N\int_0^t S_r^N I_r^N dr\right) - \frac{1}{N}P_{rec}\left(\gamma N\int_0^t I_r^N dr\right) \\ \qquad - \frac{1}{N}P_{di}\left(\mu N\int_0^t I_r^N dr\right), \\ R_t^N = R_0^N + \frac{1}{N}P_{rec}\left(\gamma N\int_0^t I_r^N dr\right) - \frac{1}{N}P_{dr}\left(\mu N\int_0^t R_r^N dr\right). \end{cases}$$

REMARK 1.2.– *We may think that it would be more natural to decide that births happen at rate μ times the total population. Then the total population process would be a critical branching process, which would go extinct in finite time a.s., which we do not want. Next it might seem more natural to replace, in the infection rate, the ratio $\mathcal{S}_t^N/N$ by $\mathcal{S}_t^N/(\mathcal{S}_t^N + \mathcal{I}_t^N + \mathcal{R}_t^N)$, which is the actual ratio of susceptibles in the population at time t. It is easy to show that $\mathcal{S}_t^N + \mathcal{I}_t^N + \mathcal{R}_t^N$ is close to N, so we choose the simplest formulation.*

Again, we can reduce these models to two-dimensional models for $z(t) = (i(t), s(t))$ (respectively $Z_t^N = (I_t^N, S_t^N)$), by deleting the r (respectively R^N) component.

1.3. The stochastic model, LLN, CLT and LD

1.3.1. *The stochastic model*

The three above-mentioned stochastic models are of the following form.

$$\begin{aligned} Z_t^N &= z_N + \frac{1}{N}\sum_{j=1}^{k} h_j P_j\left(N\int_0^t \beta_j(Z_s^N)ds\right) \\ &= z_N + \int_0^t b(Z_s^N)ds + \frac{1}{N}\sum_{j=1}^{k} h_j M_j\left(N\int_0^t \beta_j(Z_s^N)ds\right), \end{aligned} \qquad [1.1]$$

where $\{P_j(t),\, t \geq 0\}_{0\leq j\leq k}$ are mutually independent standard Poisson processes, $M_j(t) = P_j(t) - t$, and $b(z) = \sum_{j=1}^{k} \beta_j(z)h_j$. Z_t^N takes its values in $\mathbb{R}^d$.

In the case of the SIS model, $d = 1$, $k = 2$, $h_1 = 1$, $\beta_1(z) = \lambda z(1-z)$, $h_2 = -1$ and $\beta_2(z) = \gamma z$.

In the case of the SIRS model, $d = 2$, $k = 3$, $h_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\beta_1(z) = \lambda z_1 z_2$, $h_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\beta_2(z) = \gamma z_1$ and $h_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\beta_3(z) = \rho(1 - z_1 - z_2)$.

In the case of the SIR model with demography, we can restrict ourselves to $d = 2$, while $k = 4$, $h_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\beta_1(z) = \lambda z_1 z_2$, $h_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\beta_2(z) = (\gamma + \mu)z_1$, $h_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\beta_3(z) = \mu$, $h_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\beta_4(z) = \mu z_2$.

While the above formulation has the advantage of being concise, for certain purposes it is more convenient to rewrite [1.1] using the equivalent formulation already described in the case of the SIS model. Let $\{\mathcal{M}_j,\, 1 \leq j \leq k\}$ be mutually independent Poisson random measures on $\mathbb{R}_+^2$ with mean measure the Lebesgue