
MULTIPLICATIVE IDEAL THEORY IN COMMUTATIVE ALGEBRA

A Tribute to the Work of Robert Gilmer

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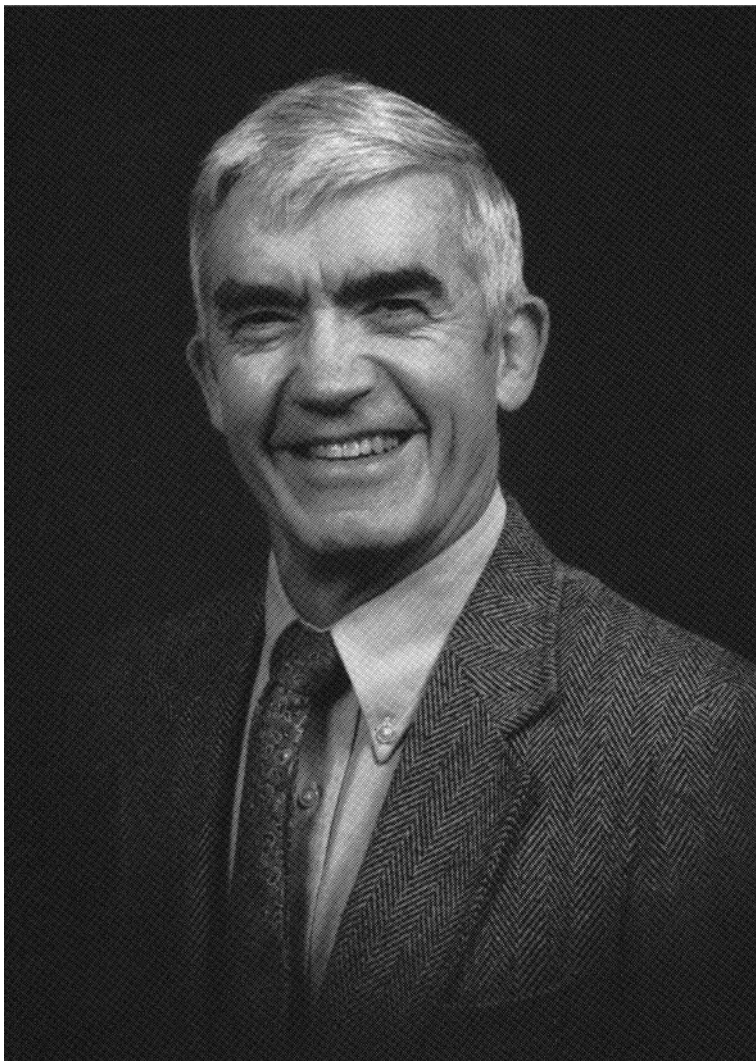
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Robert Gilmer

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Preface

This volume consists of a collection of articles centered around topics in commutative ring theory influenced or inspired by Robert Gilmer. The articles were solicited by the editors from experts in the field and each represents the state of the topic at the time of publication. Some of the articles are original research papers and some are expository. Several of the expository articles also contain original research. It is the editors' hope and intention that the volume will be useful to current researchers in the field and an inspiration to others to study this beautiful area of mathematics of which we are so fond.

The volume is also intended to be a tribute to our friend and colleague, Robert Gilmer, whose work and presence has inspired us as well as others.

We would like to thank the authors for their contributions, the referees for their work and Dr. John Martindale, Senior Editor, and Mr. Robert Saley, Assistant Editor, of Springer for their encouragement and help with this project. We thank also Jim Coykendall, Mitch Keller and The Mathematics Genealogy Project for providing us with Robert Gilmer's mathematics genealogy.

Finally, we would like to express our appreciation to Robert himself for his contributions to the volume, both inspirational and actual.

Boca Raton, Florida
Storrs, Connecticut
West Lafayette, Indiana
Las Cruces, New Mexico

Jim Brewer
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Commutative rngs

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1 Introduction

The purpose of this article is to discuss commutative rngs (that is, commutative rings that do not have an identity) and especially Robert Gilmer's work in this area. To avoid confusion we adopt the following terminology. The word "ring" will be used in the neutral sense that there may or may not be an identity. The term "rng" or "ring without identity" will mean a ring that does not have an identity and we will always explicitly say "ring with identity" when that is the case. The term "rng" appears in Jacobson [37, Section 2.17] where he suggests the pronunciation *rü*ng and says that the term was suggested to him by Louis Rowen. Bourbaki [6, Chapter 1] uses the term "pseudo-ring" for rings without identity. While we will be mostly concerned with commutative rings, in several places we consider noncommutative rings. This should be clear from context.

Today the word "ring" usually means ring with identity. This was not the case so long ago. I remember reading Lambek's book *Lectures on Rings and Modules* [38] in 1968 and being taken back by the fact that he assumed the existence of an identity element. When I taught material on the Jacobson radical thirty years ago I did it via quasi-regular elements. Today I usually assume the existence of an identity. I suspect the insistence on an identity today is in part due to the larger role played by homological methods in ring theory. Also, the existence of an identity entails the quasi-compactness of the spectrum of a ring, a useful property for algebraic geometry.

Now there are plenty of good rings that don't have an identity such as the even integers $2\mathbb{Z}$ and an infinite direct sum $\bigoplus R_\alpha$ of rings. On the other hand, since any ring can naturally be embedded in a ring with identity (see Section 2), many mathematicians no doubt take the point of view that there is no loss in generality in assuming the existence of an identity. The extent to which this is true depends on the context.

When asked to write an article for this volume, I gave considerable thought to the choice of topics. Certainly a number of suitable topics came to mind:

dimension theory, Prüfer domains and valuation domains, polynomial rings and power series rings, and semigroup rings. But I finally chose commutative rings without identity. About thirty of Robert's almost two hundred papers involve rngs to some extent or another – twenty-six which involve rngs in a significant way are listed in the references. One can of course debate which papers to include since many results on rings do not depend on the existence of an identity. Perhaps I picked commutative rngs for a topic because I remembered the material on rngs in Chapter 1 of *Multiplicative Ideal Theory* [25] and consulting his paper “Eleven nonequivalent conditions on a commutative ring” [15] (maybe even chuckling at the title).

I e-mailed Robert informing him of my article and asked for any thoughts. He sent the following. “The first thought that comes to mind is that there was always a bit of tension, to me, in whether to look at a given condition/problem/whatever under consideration in rngs, or only in rings. In most cases this wasn't an issue. Sometimes it was. I remember discussing with Jack Ohm on numerous occasions whether or not rngs were worthy of study. If you know Jack very well, you may know that he will take either side of a question, just for the sake of a good ‘discussion.’ A summary of the two positions would generally come down to something like (1) The fact that rngs were good enough for Emmy Noether (and other illustrious ones, though Noether was the usual citation) to consider validates their worth or (2) Life is too short for me to consider such ignoble algebraic structures; besides most questions about rngs can be reduced to questions about rings.”

Finally, let me quote from the Preface of *Multiplicative Ideal Theory* [25]. “Some readers may find the failure to restrict to rings with identity noisome, so perhaps a word of explanation on this subject is in order. Examples of rings without identity abound, and on many occasions the author has found them to be useful, especially in the construction of examples (even examples of commutative rings with identity). Moreover, in many results, perhaps even a majority, the assumption that the ring contains an identity is not pertinent; it plays no role in the proof. Like most commutative algebraists, the author prefers to work in rings with identity, with all modules and homomorphisms unitary as well, but certainly is not averse to dropping these conditions, either.”

In Section 2 we discuss adjoining an identity to a ring. Let R be a ring that is a subring of the ring S with identity 1. We say that S is a *unital extension of R* if $S = R[1] = \{r + n1 \mid r \in R, n \in \mathbb{Z}\}$. Besides giving Dorroh's [9] construction of a unital ring extension we give an algebra version. We also survey the work of Brown and McCoy [7] and Arnold and Gilmer [5] on unital extensions of commutative rings. In Section 3 we consider conditions on a commutative ring that are weaker than the existence of an identity. This section closely follows Gilmer's paper “Eleven nonequivalent conditions on a commutative ring” [15]. A new result is that a ring R has the property that for each $a \in R$, there exists an $r_a \in R$ with $r_a a = a$ if and only if for each

ring S , every left ideal of $R \times S$ has the form $I \times J$ for left ideals I and J of R and S , respectively.

In Section 4 we consider some classical rings of commutative ideal theory. We give a complete characterization of rings (general Z.P.I.-rings) in which every ideal is a product of prime ideals, rings (π -rings) in which every principal ideal is a product of prime ideals, and rings in which every ideal has a unique normal (primary) decomposition. We also discuss Gilmer's work on multiplication rings and related rings and on commutative rings in which every prime ideal is principal. In the final Section 5 a number of miscellaneous results (mostly due to Gilmer) on commutative rings are given.

2 Adjunction of an Identity

In this section we discuss different ways of adjoining an identity to a ring A . This amounts to embedding A into a ring B with identity 1. For then $A^* = A[1] = \{a + m1 \mid a \in A, m \in \mathbb{Z}\}$ is a ring with identity. We call B a *unital ring extension* of A if $B = A^*$. Most of the results of this section do not require the rings to be commutative. We first handle the case where A is a commutative ring with a regular element and then give Dorroh's unital extension. We also give an "algebra version" of Dorroh's construction. We end this section with a survey of the work of Brown and McCoy [7] and Arnold and Gilmer [5] that determines all the unital extensions of a commutative ring.

Let A be a commutative ring containing a regular element r_0 . Let N be the set of regular elements (that is, nonzero divisors) of A . Then $A_N = T(A)$ the total quotient ring of A has an identity element, namely $1 = r_0/r_0$. Now $A^* = A[1] = \{a + m1 \mid a \in A, m \in \mathbb{Z}\}$ is a commutative ring with identity. So $A \subseteq A^*$ is a unital extension. Note that $\text{char } A = \text{char } A^*$. Moreover, if A had an identity to start with, then $A^* = A$. Note that $T(A) = T(A^*)$ and $F(A) = F(A^*)$ ($F^*(A) = F^*(A^*)$) where $F(A)$ ($F^*(A)$) is the set of (finitely generated) fractional ideals of A or A^* . Thus A and A^* are simultaneously Noetherian. Here A is an ideal of A^* and $A^*/A \approx \mathbb{Z}/I$ where I is the ideal $\{m \in \mathbb{Z} \mid m1 \in A\}$.

But what do we do if A doesn't have a regular element or isn't even commutative? Dorroh [9] gave a simple way to embed any ring into a ring with identity. Let A be a ring and let $A^* = A \times \mathbb{Z}$. On A^* define $(a, n) + (b, m) = (a + b, n + m)$ and $(a, n)(b, m) = (ab + nb + ma, nm)$. It is easily checked the A^* is a ring with identity $(0, 1)$ with a natural embedding $A \rightarrow A^*$ given by $a \rightarrow (a, 0)$ with image $\bar{A} = \{(a, 0) \mid a \in A\}$ which is an ideal of A^* . Note that $\bar{A}[(0, 1)] = \bar{A} + \mathbb{Z}(0, 1) = A^*$ and $A^*/\bar{A} \approx \mathbb{Z}$. So A^* is a unital extension of A . Note that unlike the construction in the previous paragraph, if A has an identity, the Dorroh construction does not preserve the identity of A and $A \neq A^*$. Here A and A^* are \mathbb{Z} -algebras. Dorroh remarks that if A is a \mathbb{Q} -algebra, then we can replace \mathbb{Z} by \mathbb{Q} and $A \times \mathbb{Q}$ is then a

\mathbb{Q} -algebra with identity. In [10] Dorroh mentions that if $\text{char } A = n$, then A^*/nA^* is a ring with identity where $\text{char } A = \text{char}(A^*/nA^*)$ and that A can be naturally embedded into A^*/nA^* . Observe that A^*/nA^* is naturally isomorphic to $A \times (\mathbb{Z}/n\mathbb{Z})$ with the product similar to that of the Dorroh extension. Of course $A \subseteq A^*/nA^*$ is a unital extension. Put another way, if A is a $\mathbb{Z}/n\mathbb{Z}$ -algebra, so is A^*/nA^* . Stone [47] showed that a Boolean ring can be embedded in a Boolean ring with identity. Brown and McCoy [7] gave a detailed study of unital ring extensions. They observed that if $\text{char } A = n > 0$ and $k \geq 1$, then we can replace \mathbb{Z} by $\mathbb{Z}/kn\mathbb{Z}$ so that $A^* = A \times \mathbb{Z}/kn\mathbb{Z}$ is a ring with identity where $\text{char } A^* = kn$. Observe that if $\text{char } A = n$, then A is a $\mathbb{Z}/kn\mathbb{Z}$ -algebra as is A^* .

Of course in hindsight it is clear where the addition and multiplication in the Dorroh extension are coming from. Formally adjoin 1 to A to get $A^* = A[1] = \{a + n1 \mid n \in \mathbb{Z}\}$. Then addition and multiplication are forced on us by $(a + n1) + (b + m1) = a + b + (m + n)1$ and $(a + n1)(b + m1) = ab + (n1)b + (m1)a + (n1)(m1) = ab + nb + ma + (nm)1$. So 1 becomes $(0, 1)$ and a becomes $(a, 0)$ as reminiscent in the construction of the complexes from the reals.

In each of the cases in the previous paragraph we are starting with a ring A that is an R -algebra ($R = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{Z}/n\mathbb{Z}$) and are embedding A into an R -algebra A^* with identity. This suggests the following theorem. The details of the proof are left to the reader.

Theorem 2.1. *Let R be a ring with identity and let A be a ring that is a unitary R - R -bimodule. On $A^* = A \times R$ define $(a_1, r_1) + (a_2, r_2) = (a_1 + a_2, r_1 + r_2)$ and $(a_1, r_1)(a_2, r_2) = (a_1a_2 + r_1a_2 + a_1r_2, r_1r_2)$. Then A^* is a ring with identity $(0, 1)$. Here A^* is a unitary R - R -bimodule with $r(a, s) = (ra, rs)$ and $(a, s)r = (ar, sr)$. Hence if R is a commutative ring with 1 and A is a unitary R -algebra, A^* is a unitary R -algebra with identity. The map $A \rightarrow A^*$ given by $a \rightarrow (a, 0)$ is an R -algebra embedding, $A \times 0$ is a two-sided ideal of A^* , $R(0, 1) + (A \times 0) = A^*$, and $A^*/(A \times 0) \approx R$.*

Let R be a ring with identity and let A be an ideal of R . Then A is a ring. Observe that A has an identity if and only if $A = Re$ is generated by a central idempotent e of R . Suppose we apply the R -algebra version of the Dorroh extension to A . Here $A^* = A \times R$. Considering $R \times R$ with the usual sum and product, we get an R -algebra monomorphism $A^* \rightarrow R \times R$ given by $(a, r) \rightarrow (a, 0) + r(1, 1) = (a + r, r)$ with image $\{(r, s) \in R \times R \mid r - s \in A\}$. Thus in the case of $R = \mathbb{Z}$ and $A = 2\mathbb{Z}$, the Dorroh extension of $2\mathbb{Z}$ is naturally isomorphic to $\{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid n \equiv m \pmod{2}\}$. This is quite different from the first method of adjoining an identity in the case where the ring contains a regular element; for here $(2\mathbb{Z})[1] = \mathbb{Z}$.

For the next example let $A = \bigoplus_{n=1}^{\infty} \mathbb{Z}$. Here A contains no regular elements, but $A \subseteq \prod_{n=1}^{\infty} \mathbb{Z}$, a ring with identity $(1, 1, \dots)$. We have $(\bigoplus_{n=1}^{\infty} \mathbb{Z})[(1, 1, \dots)] = \{(m_i) \in \prod_{n=1}^{\infty} \mathbb{Z} \mid m_i \text{ is eventually constant}\}$. We

leave it to the interested reader to check that this last ring is isomorphic to the ring obtained by the Dorroh extension.

We next observe that adjoining an identity does not commute with adjoining an indeterminate. Consider $2\mathbb{Z}$, so $(2\mathbb{Z})[1] = \mathbb{Z}$. Now $((2\mathbb{Z})[X])[1] = \mathbb{Z} + 2X\mathbb{Z}[X] \subsetneq \mathbb{Z}[X] = ((2\mathbb{Z})[1])[X]$. A similar result holds for power series adjunction. For the Dorroh extension we have $(2\mathbb{Z})^* \approx \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid n \equiv m \pmod{2}\}$ and $(2\mathbb{Z})^*[X] \approx \{(f(X), g(X)) \in \mathbb{Z}[X] \times \mathbb{Z}[X] \mid f(X) \equiv g(X) \pmod{2\mathbb{Z}[X]}\}$. It is easily checked that the map $((2\mathbb{Z})[X])^* = (2\mathbb{Z})[X] \times \mathbb{Z} \longrightarrow (2\mathbb{Z})^*[X]$ given by $(f(X), n) \longrightarrow (f(X), 0) + n(1, 1) = (f(X) + n, n)$ is a ring monomorphism with image $\{(f(X), g(X)) \in \mathbb{Z}[X] \times \mathbb{Z}[X] \mid f(X) \equiv g(X) \pmod{2\mathbb{Z}[X]}, \deg g(X) \leq 0\}$. For any unital extension $R \subseteq S = R[1], R[X][1] = R[X] + \mathbb{Z}1 = S + XR[X] \subseteq S[X]$ with equality if and only if $R = S$.

These above examples seem to indicate that all questions concerning rngs cannot be resolved by just adjoining an identity. Note that for any unital extension $R \subseteq S$, R is Noetherian if and only if S is Noetherian. Indeed, a subset E of R is an (left, right, two-sided) ideal of R if and only if E is an ideal of S . So if S is Noetherian, so is R . Next suppose that R is Noetherian. Let A be an ideal of S . Then $A \cap R$ is finitely generated. But $A/A \cap R \approx (A+R)/R \subseteq S/R$ and hence is a cyclic abelian group; so A is finitely generated. Hence S is Noetherian. But Gilmer [16] has shown that $R[X]$ is Noetherian if and only if R is Noetherian with an identity.

Recall that a ring extension $R \subseteq S$ is *unital* if S is a ring with identity 1 and $S = R + \mathbb{Z}1 = R[1]$. Note that if S is a unital extension of R , then R is an ideal of S . If R_1 is a subring of R_2 , we say that an ideal A_2 of R_2 *lies over* an ideal A_1 of R_1 if $A_2 \cap R_1 = A_1$. We next determine the unital extensions of a commutative ring R . This was done by Brown and McCoy [7], but we follow Arnold and Gilmer's treatment [5].

Proposition 2.2. *Let R be a unital extension of the commutative ring A and let B be an ideal of A . Then B is an ideal of R . If C is an ideal of R lying over B , then $C \subseteq (B:{}_R A)$ and C is principal modulo B . Now $(B:{}_R A)$ lies over B if and only if $(B:{}_A A) = B$. If $(B:{}_A A) = B$ and hence $(B:{}_R A) = B + R\alpha$ for some $\alpha \in R$, then $\{B + Rn\alpha\}_{n=0}^\infty$ is the set of ideals of R lying over B in A .*

Proof. This is essentially [5, Proposition 2.2]. Clearly B is an ideal of R , if C lies over B , then $C \subseteq (B:{}_R A)$, and $(B:{}_R A)$ lies over B if and only if $(B:{}_A A) = B$ since $(B:{}_A A) = (B:{}_R A) \cap A$. Observe that if C lies over B , then $C/B = C/C \cap A \approx (C+A)/A \subseteq R/A \approx \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 0$. Hence C/B is a cyclic R/A -module and hence a cyclic abelian group. So if $(B:{}_A A) = B$, $(B:{}_R A) = B + R\alpha$ for some $\alpha \in R$. Observe that the set of R/A -submodules of $(B:{}_R A)/B$ is $\{Rn\bar{\alpha}\}_{n=0}^\infty$.

Let $R^* = R \times \mathbb{Z}$ be the Dorroh extension of R where R is identified with the ideal $R \times 0$ of R^* . So $R \subset R^*$ is a unital extension. Let S be a unital

extension of R . Then the map $f: R^* \rightarrow S$ given by $(r, n) \rightarrow r + n1$ is a ring epimorphism. Note that $\ker f$ is an ideal of R^* lying over 0 in R and $S \approx R^*/\ker f$. Conversely, if C is an ideal of R^* lying over 0 in R , R^*/C is a unital extension of R . Hence if $\{C_\beta\}$ is the set of ideals of R^* lying over 0 in R , $\{R^*/C_\beta\}$ is the set of unital extensions of R . We can now apply Proposition 2.2 for the case $B = 0$. Now each C_β is a principal ideal of R^* and is a cyclic abelian group. So $C_\beta = R^*(r_\beta, k_\beta) = \{(nr_\beta, nk_\beta) \mid n \in \mathbb{Z}\}$ for some $r_\beta \in R$ and $k_\beta \geq 0$. If some $k_\beta > 0$, then the smallest such positive integer k_{β_0} is a divisor of each k_β (use the division algorithm) and is called the *mode of R* . If $k_\beta = 0$, then $C_\beta \subseteq R$, so that $C_\beta = C_\beta \cap R = 0$ and there is only one such C_β . If each $k_\beta = 0$, we say that R has *mode 0*. Observe that for any $(r_\beta, k_\beta) \in R^*$ with $k_\beta > 0$, $R^*(r_\beta, k_\beta) \cap R = 0$ if and only if $(-r_\beta)x = k_\beta x$ for each $x \in R$. Thus R has mode 0 if no nonzero element of R acts as a nonzero integer under multiplication and R has mode $k > 0$ if k is the smallest positive integer such that for some $r \in R$, $rx = kx$ for all $x \in R$. Note that the mode of R divides $\text{char } R$ and R has an identity if and only if R has mode 1. If R has mode 0, R^* is the unique unital extension of R .

Further suppose that $(0:{}_R R) = 0$. By Proposition 2.2 an ideal C of R^* lies over 0 in R if and only if $C \subseteq (0:_{R^*} R)$; and $(0:_{R^*} R) = (\alpha)$ where $\alpha = (r, k) \in R^*$ and k is the mode of R . Now $\{(n\alpha)\}_{n=0}^\infty$ is the set of ideals of R^* lying over 0 in R and $\{R^*/(n\alpha)\}_{n=0}^\infty$ is the set of unital extensions of R . Put $R_n = R^*/(n\alpha)$. If $k = 0$, then $\alpha = 0$; and so each $R_n = R^*$. If $k > 0$, then $R_n/R = R^*/(R + (n\alpha)) = R^*/R \times nk\mathbb{Z} \approx \mathbb{Z}/nk\mathbb{Z}$. So up to R -isomorphism, the R_n 's are distinct. We sum up these results in the next theorem.

Theorem 2.3. *Let R be a commutative ring and let R^* be the Dorroh extension of R .*

- (1) *If $\{C_\beta\}$ is the set of ideals of R^* lying over 0 in R , then $\{R^*/C_\beta\}$ is the set of unital extensions of R . Each C_β is principal as an ideal of R^* and is cyclic as an abelian group. If some C_β is nonzero, then there is a positive integer k , the mode of R , such that each $C_\beta = ((r_\beta, kn_\beta))$.*
- (2) *If $(0:{}_R R) = 0$, then there is an $\alpha = (r, k) \in R^*$ such that $((n\alpha))_{n=0}^\infty$ is the set of ideals of R^* lying over 0 in R . If $k = 0$, then R^* is the unique unital extension of R . If $k > 0$, then $\{R_n := R^*/(n\alpha)\}_{n=0}^\infty$ is the set of unital extensions of R and the rings R_n are distinct up to R -isomorphism.*

Suppose that R is a commutative ring with regular element r . So $(0:{}_R R) = 0$. Consider the unital extension $R[1] = R + \mathbb{Z}1 \subseteq T(R)$. We claim that $R[1] \approx R_1$. Define $\varphi: R^* \rightarrow R[1]$ by $\varphi((s, m)) = s + m1$. Then $\ker \varphi \subseteq (\alpha) = (0:_{R^*} R)$ where $\alpha = (x, k) \in R^*$. Now $(r, 0)(x, k) \in (R \times 0) \cap (\alpha) = 0$, so $0 = rx + kr$. Then $0 = (rx + kr)r^{-1} = x + k1 = \varphi(\alpha)$. Hence $\ker \varphi = (\alpha)$, so $R[1] \approx R^*/(\alpha) = R_1$.

We end this section with the following example from [5].

Example 2.4. [5, Example 2.6] A ring T of mode $k > 0$ with $(0:{}_T T) = 0$ so that $T_n \approx T_m$ for all $n, m \geq 0$ even though T_n and T_m are not T -isomorphic

for $n \neq m$. Let S be a ring with mode k (for example, $S = k\mathbb{Z}$). Set $S^{(n)} = \prod_{m=1}^{\infty} S_n$ and $R = \prod_{n=1}^{\infty} S^{(n)}$, so R is a ring with identity. Put $T = R \times S$, so $(0:_{T}T) = 0$ and T has mode k . Now for each $n \geq 0$, $T_n \approx R \times S_n$ [5, Proposition 2.5]. But certainly $R \times S_n \approx R$. So $T_n \approx T_m$ for each $n, m \geq 0$.

3 Conditions Weaker than Having an Identity

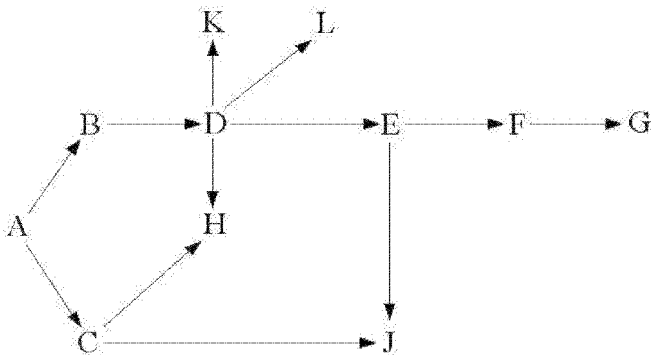
Commutative rings with identity have many familiar properties that we use everyday without much thought such as maximal ideals are prime, every proper ideal is contained in a maximal ideal, if \sqrt{A} is maximal then A is primary, and if A and B are comaximal ideals, then $A \cap B = AB$. An examination of the proof of each of these above properties will show that some consequence of the existence of an identity is used. For example, if A and B are comaximal ideals, then the fact that $R(A \cap B) = A \cap B$ is used: $A \cap B = R(A \cap B) = (A + B)(A \cap B) = A(A \cap B) + B(A \cap B) \subseteq AB$.

In [15], Gilmer considered eleven conditions on a commutative ring R , the first of which that R has an identity implies the other ten:

- A. R has an identity.
- B. R is generated by idempotents, that is, for $r \in R$, $r = r_1e_1 + \cdots + r_n e_n$ where $r_i \in R$ and e_i is idempotent. But since $Re_1 + \cdots + Re_n = Re$ for some idempotent e this is equivalent to $r = se$ where e is idempotent or to $r = re$ for some idempotent e .
- C. If A is a nonzero (principal) ideal of R such that $\sqrt{A} \neq R$, then R/A has an identity.
- D. If $x \in R$, there exists $y \in R$ with $x = xy$. Equivalently, given $x_1, \dots, x_n \in R$ there exists $y \in R$ with $x_i = x_i y$; $RA = A$ for each ideal A of R ; or $AB = A \cap B$ for comaximal ideals A and B of R .
- E. R is a *u-ring*, that is, if A is a proper ideal of R then $\sqrt{A} \neq R$, or equivalently, if A is a proper ideal of R , then A is contained in a proper prime ideal of R .
- F. $R = R^2$.
- G. Maximal ideals are prime.
- H. If P is a nonzero prime ideal of R , R/P has an identity. Equivalently, if Q is P -primary where $P \neq R$, then R/Q has an identity.
- J. If A is an ideal of R with \sqrt{A} maximal, then A is primary, or equivalently, R/A has an identity.
- K. Each ideal is contained in a maximal ideal.
- L. If A and B are comaximal proper ideals, then $AB = A \cap B$.

In the above conditions with equivalences such as D , it is the first property listed that Gilmer labeled with the letter. He then showed that the other properties were equivalent. Note that all eleven conditions are preserved by homomorphism. By choosing an appropriate ring with zero products, it is easy to construct commutative rings that fail to satisfy A , B , D , E , F , G , K , or

L . (Note that C , H and J hold vacuously in such a ring.) Gilmer gave the following diagram which completely describes the implications between these properties.



One of the useful properties of rings with identity is that ideals of $R \times S$ have the form $I \times J$ where I is an ideal of R and J is an ideal of S . This is of course not true in general since $\{(\bar{0}, \bar{0}), (\bar{1}, \bar{1})\}$ is an ideal of the ring $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with the zero product. The next result (which the author believes is new) characterizes the rings whose ideals have the desired form.

Proposition 3.1. *For a commutative ring R the following conditions are equivalent.*

- (1) R satisfies condition D .
- (2) For each commutative ring S , every ideal of $R \times S$ has the form $I \times J$ where I is an ideal of R and J is an ideal of S .
- (3) Every ideal of $R \times R$ has the form $I \times J$ where I and J are ideals of R .

Proof. (1) \Rightarrow (2) Let L be an ideal of $R \times S$. Let $(a, b) \in L$. Choose $r \in R$ with $ra = a$. Then $(a, 0) = (ra, 0) = (r, 0)(a, b) \in L$ and $(0, b) = (a, b) - (a, 0) \in L$. It easily follows that L has the desired form. (2) \Rightarrow (3) Clear. (3) \Rightarrow (1) Let $a \in R$. Since $\langle (a, a) \rangle = I \times J$ for ideals I and J of R , $(a, 0) \in \langle (a, a) \rangle$. So $(a, 0) = (r, s)(a, a) + n(a, a)$ for some $r, s \in R$ and $n \in \mathbb{Z}$. Hence $a = ra + na$ and $0 = sa + na$. So $na = -sa$ gives $a = ra + na = ra - sa = (r - s)a$.

Remark 3.2. There is a noncommutative version of Proposition 3.1 for left and right ideals which we leave to the interested reader.

While there are no simple implications other than those given in the diagram, combinations of properties give other implications. For example, Gilmer shows that $K + G \Rightarrow E$, $C + F \Leftrightarrow A$, and $E + H \Leftrightarrow D$. Also, we get

other implications by putting finiteness conditions on R . For example, Gilmer shows that $E + ACC$ on prime ideals $\Rightarrow D$ and if R is finitely generated ($R = Rr_1 + \cdots + Rr_n$), then $G \Rightarrow A$. Also, a Noetherian ring (or more generally, finitely generated ring) or a ring containing a regular element satisfying D has an identity. Thus a commutative Noetherian ring R has the property that each ideal of $R \times S$ has the form $I \times J$ if and only if R has an identity.

4 Classical Rings of Ideal Theory

In this section we consider some of the classical rings of ideal theory without the hypothesis of them containing an identity. We begin with a discussion of π -rings and general Z.P.I.-rings.

Recall that a commutative ring R is called a (π -ring) *general Z.P.I.-ring* if every (principal) ideal of R is a product of prime ideals. In a series of papers, S. Mori [39]-[43] studied π -rings and general Z.P.I.-rings. In [42] he showed that a commutative ring with identity is a π -ring if and only if it is a finite direct product of π -domains and SPIR's (principal ideal rings with a single prime ideal and that prime ideal is nilpotent). A very reasonable account can be found in [25, Section 46]. Examples of π -domains with identity include UFD's and Dedekind domains. Many characterizations of π -domains with identity are collected in [1, Theorem 3.1]. For example, for an integral domain D with identity the following are equivalent: (1) D is a π -domain, (2) every invertible ideal of D is a product of invertible prime ideals, (3) every nonzero prime ideal of D contains an invertible prime ideal, (4) D is a Krull domain with each height-one prime ideal invertible, (5) D is a locally factorial Krull domain, and (6) $D(X) = \{f/g \mid f, g \in D[X], \text{ the coefficients of } g \text{ generate } D\}$ is a UFD.

Less well known is Mori's characterization of π -rings without identity. In [43], Mori showed that a π -ring without identity is either (1) an integral domain, (2) a ring $R = (p)$ where every ideal of R including 0 is a power of R , (3) $K \times R$, where K is a field and R is a ring as in (2), or (4) $K \times D$, where K is a field and $D = (p)$ is a π -domain where every nonzero ideal of D is a power of D . He showed that in a π -domain without identity every principal ideal is actually a product of principal prime ideals (here the whole ring itself is considered to be a prime ideal) and that an integral domain D without identity is a π -domain if and only if (1) every nonzero element of D is a product of irreducible elements and (2) each irreducible element of D generates a principal prime ideal. Mori also remarked that $2\mathbb{Z}_{(2)}$ is an example of a π -domain without identity in which the factorization of an element into irreducible elements is not unique but for each irreducible element p of $2\mathbb{Z}_{(2)}$, $(p) = 2\mathbb{Z}_{(2)}$. In his review of Mori's paper O. F. G. Schilling [46] stated that in a π -domain D without identity, $D = (p)$ for each irreducible element p of D . Mori did not make this assertion which we shall later see is not correct.

Certainly a general Z.P.I.-ring is a π -ring. It is well-known that an integral domain with identity is a general Z.P.I.-ring if and only if it is a Dedekind domain. Thus a commutative ring with identity is a general Z.P.I.-ring if and only if it is a finite direct product of Dedekind domains and SPIR's. Note that the π -rings without identity given in (2), (3), and (4) of the previous paragraph are actually general Z.P.I.-rings. Thus a general Z.P.I.-ring without identity is either (1) an integral domain, (2) a ring $R = (p)$ where every ideal of R including 0 is a power of R , (3) $K \times R$, where K is a field and R is a ring as in (2), or (4) $K \times D$, where K is a field and $D = (p)$ is a general Z.P.I.-ring which is an integral domain where every nonzero ideal of D is a power of D .

Now there are some difficulties with several of Mori's results in [41]; see Wood [48] for details. Wood gave two treatments of general Z.P.I.-rings; one based on Mori's results and one independent of them. Some of the results from [48] appear in [49]. In [12], Gilmer characterized the integral domains D without identity in which every ideal is a product of prime ideals (here D is considered to be a prime ideal). He showed that an integral domain without identity has the property that each ideal is a product of prime ideals if and only if each nonzero ideal of D is a power of D ; moreover D is a PID. He showed that $D^* = D[1]$ is a rank-one discrete valuation domain with D as its maximal ideal. Conversely, if (S, M) is a rank-one discrete valuation domain with $S = M[1]$, then M is an integral domain without identity. Here a subset $A \subseteq M$ is an ideal of M if and only if it is an ideal of S ; so $\{M^n\}_{n=1}^\infty \cup \{0\}$ is the set of ideals of M . Hence M is a general Z.P.I.-ring. Gilmer also classified these rank-one discrete valuation domains (V, M) with $V = M[1]$. See [12, page 582].

It is easily seen that if (D, M) is a quasilocal UFD with $D = M[1]$, then M is a π -domain without an identity. For let $0 \neq d \in M$. Then $d = p_1 \cdots p_n$ where p_i is a principal prime of D . Since $M[1] = D$, for $a \in M$, $aD = (a)$, the principal ideal of M generated by a . Hence $(d) = (p_1) \cdots (p_n)$ where each (p_i) is a prime ideal of D and hence of M . The converse is also true [2, Theorem 8]: let R be π -domain without identity, then $R[1]$ is a quasilocal UFD with R as its maximal ideal. Note that Gilmer's result characterizing domains that are general Z.P.I.-rings is a special case. For each n let $D = \mathbb{Z}_p[[X_1, \dots, X_n]]$, $p > 0$ prime, and $M = (X_1, \dots, X_n)$. Then M is a π -domain without identity, but is a general Z.P.I.-ring if and only if $n = 1$.

Recently, the author and John Kintzinger [3] have given a similar characterization of π -rings R of type (2): $R = (p)$ where every ideal of R including 0 is a power of R . Here there is an SPIR S containing R with $R[1] = S$ and R is the maximal ideal of S . Moreover, there is a complete DVR (D, M) with $M[1] = D$ and an epimorphism $\pi: D \rightarrow S$ with $\pi(M) = R$.

Let D be an integral domain with identity. It is well known that D is a π -domain (UFD) if and only if every nonzero prime ideal of D contains an invertible (nonzero principal) prime ideal. Thus it seems reasonable to conjecture that if D is an integral domain without identity, then D is a π -domain if and only if each nonzero prime ideal contains a nonzero principal

prime ideal. However, this need not be the case as shown by an example of Gilmer [19, Example 5.3]. Let $D = X\mathbb{Z}[[X]]$. Then $D^* = \mathbb{Z} + X\mathbb{Z}[[X]] = \mathbb{Z}[[X]]$ is not quasilocal, so D is not a π -domain, but every prime ideal of D is principal. This paper [19] contains a detailed study of rings in which every prime ideal is principal. Gilmer calls such rings F -ring. Of course an F -ring with identity is a principal ideal ring. He shows that an F -ring R without identity which contains a regular element has $R^* = R[1]$ (1 the identity in $T(R)$) a PIR if and only if R is a PIR and that if every primary ideal of R is principal, then R is a PIR.

We previously mentioned that $2\mathbb{Z}_{(2)}$ is a π -domain without unique factorization into irreducibles. Let D be an integral domain without identity. Since D has no units, two elements are associates if and only if they are equal. Thus to say that D has unique factorization into irreducible elements would mean that each nonzero element of D is a product of irreducible elements and that this factorization is unique up to order of factors. Anderson [2, Proposition 10] showed that an integral domain without identity cannot have unique factorization.

While an integral domain without identity cannot have unique factorization into irreducible elements, we have the following result from Gilmer [21]. Suppose that R is a commutative ring without identity and that B is a finitely generated regular ideal of R that is representable as a finite product of prime ideals of R (here R is considered to be a prime ideal). Then this representation is unique.

Let D be an integral domain that may not have an identity. Then D satisfies the *cancellation law (CL) for ideals* if for ideals A , B , and C of D with $A \neq 0$, $AB = AC$ implies $B = C$. In [14] Gilmer considered CL. He showed [14, Theorem 3] that if D has an identity, then D satisfies CL if and only if D is almost Dedekind (that is, D_M is a rank-one discrete valuation domain for each maximal ideal M of D). Suppose that D doesn't have an identity and let $D^* = D[1]$ where 1 is the identity of the quotient field of D . It is easily seen that CL holds for D if and only if CL holds for D^* . Gilmer showed that if CL holds in D , then D^*/D is finite and conversely, if J is an integral domain with identity and A is an ideal of J with J/A finite and CL holds in A , then $J = A[1]$ [14, Theorem 5]. If D is an integral domain without identity which satisfies CL, then D_M is a rank-one discrete valuation domain for each maximal ideal M of D [14, Theorem 6]. However, the converse is false. For example, let K be a finite field, $J = K[X]$, and $A = (X^2)$. Then A_M is a rank-one discrete valuation domain for each maximal ideal M of A and J/A is finite. But $A^* = K + X^2K[X]$ is not almost Dedekind and hence doesn't satisfy CL; hence neither does A .

An ideal in a commutative ring R has finite norm if R/A is finite and we then set $N(A) = |R/A|$. Also, let $L(A)$ be the length of R/A . In [17], Gilmer showed that if a commutative ring R satisfied either (1) every nonzero ideal of R has finite norm and $N(AB) = N(A)N(B)$ for nonzero ideals A and B of R , or (2) every nonzero ideal of R has finite length and $L(AB) = L(A) + L(B)$

for nonzero ideals A and B of R , then R is Noetherian and has an identity and has the property that there are no ideals properly between M and M^2 for each maximal ideal M of R . Thus if R is a domain, R is a Dedekind domain.

In [8] Butts and Gilmer considered commutative rings R that satisfy Property (α) which states that every primary ideal of R is a power of its radical or Property (δ) that states that every ideal of R is a finite intersection of powers of prime ideals. [8, Corollary 4] gave that an integral domain D with ACC on prime ideals satisfying Property (α) has D_P a valuation domain for each proper prime ideal P of D . The main result [8, Theorems 11, 13, and 14] is that a ring R satisfies Property (δ) if and only if either (1) R is a general Z.P.I.-ring with identity or (2) $R = F_1 \times \cdots \times F_k \times S$ where F_1, \dots, F_k are fields and S is either a nonzero domain in which every nonzero ideal is a power of S or S is a nonzero ring in which every ideal is a power of S . Note that in case (2), R is a general Z.P.I.-ring if and only if $k \leq 1$.

In [11] Gilmer considered primary rings. A commutative ring R is a *primary ring* if R has at most two prime ideals where R is counted as a prime ideal. So either R is the only prime ideal of R which is equivalent to every element of R being nilpotent or R has two prime ideals P and R . So in this case, if A is an ideal of R , $\sqrt{A} = P$ or $\sqrt{A} = R$, that is, $b \in R$ is either nilpotent or $\sqrt{(b)} = R$. Of course, if R has an identity, R is primary if and only if R is zero-dimensional quasi-local. However, $2\mathbb{Z}_{(2)}$ is a primary integral domain. The following realization of primary domains is given. If J is an integral domain and if M is the intersection of all nonzero prime ideals of J and D is an ideal of J contained in M , then D is a primary domain. Conversely, if D_0 is a primary domain, then there exists an integral domain J_0 with identity such that D_0 is an ideal of J_0 contained in all the nonzero prime ideals of J_0 [11, Theorem 3]. In the case where the ring is Noetherian, more can be said; see the paper for details.

In [13] Gilmer considered commutative rings in which semi-primary ideals (ideals with prime radical) are primary. Recall that R is a *u-ring* if each proper ideal has proper radical, or equivalently each proper ideal is contained in a proper prime ideal (see Section 3). The main result [13, Theorem 7] is that a commutative ring R has every semi-primary ideal primary if and only if R is of one of the following types: (A) a ring in which every element is nilpotent; (B) a primary domain, (C) a zero-dimensional *u-ring*, or (D) a one-dimensional *u-ring* with the following property: if $P \subset M$ are proper prime ideals and if $p \in P$, then $p = pm$ for some $m \in M$.

Let R be a commutative ring. If an ideal A of R has a primary decomposition, then A has a normal decomposition $A = Q_1 \cap \cdots \cap Q_n$ where Q_i is P_i -primary, the P_i 's are distinct, and $Q_i \not\supseteq \cap Q_j$ for each i . Here the length n and primes P_1, \dots, P_n are uniquely determined. If P_i is minimal over A , then Q_i is uniquely determined, but this need not be true in general. In [20] Gilmer determined the rings, which he called *W-rings*, in which every ideal has a unique normal decomposition. The main result is that an indecomposable *W-ring* is of one of the following types: (1) a primary ring with identity,

(2) a ring in which every element is nilpotent, (3) a primary domain, or (4) a one-dimensional integral domain in which the residue class ring of each maximal ideal is a field and in which every nonzero element belongs to only finitely many maximal prime ideals. Further, an arbitrary ring R is a W -ring if and only if R is a finite direct product of indecomposable W -rings, at most one of which has no identity.

Recall that a commutative ring R is an AM -ring if for each pair of ideals $A \subsetneq B$ of R , $A = BC$ for some ideal C of R and that an AM -ring R is a *multiplication ring* if $RA = A$ for each ideal A of R . For an ideal A of R , the *kernel of A* is $\bigcap\{Q \mid Q \supseteq A \text{ where } Q \text{ is } P\text{-primary, } P \text{ a minimal prime of } A\}$. The following results come from Gilmer and Mott [31]. A ring R has the property that every semiprimary ideal is primary if and only if each ideal of R is equal to its kernel [31, Theorem 4]. For a commutative ring R , the following conditions are equivalent: (1) R is an AM -ring, (2) if $A \subsetneq P$ are ideals of R with P prime, then $A = PB$ for some ideal B of R , (3) (a) every semiprimary ideal of R is primary, (b) every primary ideal of R is a prime power, and (c) if $P \neq R$ is a prime ideal and if A is an ideal of R with $A \subseteq P^n$, but $A \not\subseteq P^{n+1}$, then $P^n = (A:(y))$ for some $y \in R - P$ [31, Theorems 12 and 13]. Along the way the following interesting result is proved [31, Theorem 3]. Let R be a u -ring in which the set of prime ideals is inductive. If the zero ideal of R is a finite product of prime ideals, then R has an identity. Wood [50] and Wood and Bertholf [51] considered rings whose proper homomorphic images satisfy (a) or (b) or is a multiplication ring.

Griffin [35, 36] studied valuation rings, Prüfer rings, and multiplication rings in the context of commutative rings R that have *fixing elements* (that is, for each $a \in R$, there exists $r \in R$ with $ra = a$, Condition D in Section 3) or in which R is generated by idempotents (Condition B of Section 3). Suppose that R is generated by idempotents. For an idempotent $e \in R$, call $a \in R$, e -regular if $ae = a$ and $ax = 0$ implies $ex = 0$. Griffin defined a commutative ring K containing R to be the *total quotient ring* of R if (i) for $x \in K$, there exist $e, a, b \in R$ with $ex = x$, $e^2 = e$, and b is e -regular with $bx = a$, and (ii) if $a \in R$ is e -regular there exists $x \in K$ with $ax = e$. If R is generated by idempotents, then R has a total quotient ring unique up to isomorphism which can be constructed as follows. The set of idempotents of R is directed by the partial order $e \leq f \Leftrightarrow ef = e$. The ring Re has an identity and hence a total quotient ring K_e . If $e \leq f$, we have a natural inclusion $K_e \rightarrow K_f$. The total quotient ring of R is then the direct limit $\varinjlim K_e$ over the directed set of idempotents of R . If R has an identity, this is just the usual total quotient ring of R . Griffin extended the notions of (Manis) valuation and Prüfer rings (finitely generated regular ideals are invertible) to rings with fixing elements. The second paper gave a detailed study of multiplication rings.

5 Miscellaneous Papers

In this final section we give a brief overview of Gilmer's other papers concerning rings that may not have an identity.

For a positive integer n , let $R(n)$ be a complete set of representatives of isomorphism classes of associative rings of order n and let $\rho(n) = |R(n)|$. If $n = p_1^{e_1} \cdots p_k^{e_k}$ is the prime factorization of n , then a ring R of order n is uniquely decomposable as the direct sum of ideals I_1, \dots, I_k of orders $p_1^{e_1}, \dots, p_k^{e_k}$. Thus to determine $R(n)$ or $\rho(n)$ it suffices to determine $R(p_i^{e_i})$ or $\rho(p_i^{e_i})$ for $1 \leq i \leq k$. For a prime p , $R(p)$ and $R(p^2)$ are known. It is not hard to show that $|R(p)| = 2$ and $|R(p^2)| = 11$. In [32], Gilmer and Mott determined $R(p^3)$ and $\rho(p^3)$. (The paper contains some errors that are corrected in an unpublished Addendum.) The determination is straightforward but rather long and tedious with many cases. Unlike $\rho(p)$ and $\rho(p^2)$, $\rho(p^3)$ depends on the prime p .

In [5], Arnold and Gilmer investigated the dimension theory of rngs. Let R be a commutative ring. Then $\dim R$ is defined as usual except that R may have no proper ideals in which case we define $\dim R = -1$. Let A be an ideal of R . Then $\dim A \leq \dim R \leq \dim A + \dim R/A + 1$ [5, Corollary 3.4]. Hence if S is a unital extension of R , $\dim R \leq \dim S \leq \dim R + 2$ with $\dim S \leq \dim R + 1$ if S/R is not isomorphic to \mathbb{Z} [5, Proposition 4.1].

Let $R^{(m)} = R[X_1, \dots, X_m]$ and $n_m = \dim R^{(m)}$. Then $\{n_i\}_{i=0}^\infty$ is called the *dimension sequence for R* . If $\dim R = -1$, then each element of R is nilpotent, so each element of $R^{(m)}$ is also nilpotent, and hence $\dim R^{(m)} = -1$. Let S be a unital ring extension of R . In general the sequence $\{\dim S^{(m)} - \dim R^{(m)}\}_{m=0}^\infty$ is not well-behaved. But we do have $0 \leq \dim S^{(m)} - \dim R^{(m)} \leq m + 2$ [5, Proposition 5.13]. However, the main result of [5] is that a sequence of non-negative integers is the dimension sequence for a ring without identity if and only if it is the dimension sequence for a ring with identity [5, Theorem 5.10]. Since Arnold and Gilmer [4] had previously determined the set of dimension sequences for commutative rings with identity, the problem of finding the possible dimension sequences for rngs is solved.

Let R be a commutative ring with set of zero divisors $Z(R)$. Let $\alpha = |R|$ and $\beta = |Z(R)|$. In [23] Gilmer considered the following questions. (1) What conditions are necessary on α and β ? (2) If a pair (α, β) of cardinals satisfies these conditions, can we find a corresponding pair $(R, Z(R))$ so that $|R| = \alpha$ and $|Z(R)| = \beta$? He obtained an answer to Question 1 so that the answer to Question 2 is "yes". If $\beta = 1$, R is an integral domain and so α is either infinite (note that the ring of polynomials over \mathbb{Z} in α indeterminates has cardinality α) or is a power of a prime. So assume $\beta > 1$. Then Gilmer showed that $\alpha \leq \beta^2$. So if β is infinite, $\alpha = \beta$ (and we can take our ring to be any abelian group of cardinality α with zero products), and if β is finite, then α is finite. In the case where α is finite we can reduce to the case where R is local so $\alpha = p^t$ (p a prime) and $Z(R)$ is an ideal of R so $\beta = p^s$ where $0 \leq s < t$.

Then Gilmer [23, Theorem 3] showed that such a pair exists if and only if $t - s$ divides s .

In [22] Gilmer showed that if R is a ring (not necessarily commutative) with only finitely many subrings, then R is finite. This had previously been proved by A. Rosenfeld [45] for the case of rings with identity, but Gilmer's proof is independent of [45].

In [29] Gilmer, Lea, and O'Malley considered rings (not necessarily commutative) whose proper subrings or ideals satisfy certain properties. Let $(P1)$ be the property "has finite characteristic" and $(P2)$ be the property "has no proper zero divisor". They proved the following.

Theorem 5.1. *Let R be a ring.*

- (1) [29, Corollary 2.3] *Every proper subring of R satisfies property $(P1)$ if and only if*
 - (i) *R has finite characteristic, or*
 - (ii) *R is the zero ring on the p -quasicyclic group $\mathbb{Z}(p^\infty)$, p a prime.*
- (1) [29, Proposition 2.5 (Corollary 2.7)] *Every proper (left) ideal of R satisfies property $(P1)$ if and only if*
 - (i) *R has finite characteristic,*
 - (ii) *R is the zero ring on $\mathbb{Z}(p^\infty)$, p a prime, or*
 - (iii) *R is a simple ring having no nonzero elements of finite order (R is a division ring of characteristic zero).*
- (3) [29, Corollary 2.11] *Every proper subring of R satisfies $(P2)$ if and only if*
 - (i) *R satisfies $(P2)$,*
 - (ii) *$R \approx \mathbb{Z}/(p) \oplus \mathbb{Z}/(q)$ where p and q are primes, or*
 - (iii) *R is the zero ring on $\mathbb{Z}/(p)$ where p is prime.*
- (4) [29, Corollary 2.10 (Corollary 2.12)] *Every proper (left) ideal of R satisfies property $(P2)$ if and only if*
 - (i) *R satisfies property $(P2)$,*
 - (ii) *R is the zero ring on $\mathbb{Z}/(p)$, p a prime,*
 - (iii) *R is the direct product of two simple rings, each of which satisfies property $(P2)$ (R is a direct product of two division rings), or*
 - (iv) *R does not satisfy property $(P2)$ and R is a simple ring for which $R^2 = R$. (This case cannot occur for the case of left ideals.)*

In [30] Gilmer and O'Malley considered rings (not necessarily commutative) which satisfy property $(C1)$: R does not satisfy ACC on ideals, but each proper subring of R does or property $(C2)$: R does not satisfy ACC on left ideals, but each proper left ideal of R satisfies ACC on left ideals. They proved the following result.

Theorem 5.2. [30, Theorem 3.2] *For a ring R , the following conditions are equivalent.*

- (a) *R has property $(C1)$.*

- (b) R has property (C2).
 (c) R is the zero ring on $\mathbb{Z}(p^\infty)$, p a prime.

Let R be a commutative ring. Then R is *hereditarily Noetherian* if each subring of R is Noetherian. In [27] Gilmer and Heinzer determined the hereditarily Noetherian commutative rings. For this question they remarked that it is no restriction to consider only commutative rings with identity and subrings that contain that identity. For let R^* be the Dorroh extension of R and let e be the identity element of R^* . Let S be a subring of R . Now S is Noetherian if and only if $S[e]$ is Noetherian. Thus R is hereditarily Noetherian if and only if each subring S of R^* containing e is Noetherian. They showed [27, Theorem 2.4] that a commutative ring T is hereditarily Noetherian if and only if T is isomorphic to a subring of a ring of the form $D_1 \times \cdots \times D_n \times R$ where D_1, \dots, D_n are hereditarily Noetherian integral domains with identity and R is a unitary ring with finitely generated additive group.

In [28] Gilmer and Heinzer proved that if R is an uncountable commutative ring (with identity e) of cardinality ω , then there exists a proper subring S of R (containing e) with $|S| = \omega$.

In [18] Gilmer investigated R -automorphisms on $R[X]$. Let R be a commutative ring with 1. Then an R -endomorphism φ on $R[X]$ is completely determined by $\varphi(X)$; for if $\varphi(X) = t$, then $\varphi(g(X)) = g(t)$. Denote this map by f_t . Gilmer showed that if $t = t_0 + t_1X + \cdots + t_nX^n \in R[X]$, then f_t is an R -automorphism on $R[X]$ if and only if t_1 is a unit and t_i is nilpotent for $i \geq 2$. So each R -automorphism of $R[X]$ has this form. He then extended this result to commutative rings containing a regular element. Let R be such a ring with total quotient ring T . Gilmer showed that each R -endomorphism of $R[X]$ is induced by a T -endomorphism of $T[X]$. For $t = t_0 + t_1X + \cdots + t_nX^n \in T[X]$, $f_{t|R[X]}$ is an R -automorphism on $R[X]$ if and only if $Rt_i \subseteq R$ for each i , $Rt_1 = R$, and t_i is nilpotent for each $i \geq 2$. The paper also discussed the difficulties in extending these results to rings not containing a regular element.

By now it should be evident that sometimes the existence or lack of an identity element plays a major role and sometimes it doesn't. Gilmer has several papers concerning commutative rings that need not have an identity for the simple reason that the results don't depend on the existence of an identity. Two good examples are [26] and [44]. In fact, we quote from [44, page 97]: "On the other hand, the assumption that the rings under consideration have an identity plays no essential role, and therefore it will not be made."

We end by discussing some of Gilmer's work related to semigroup rings: [24], [33], [34], and [44]. Let R be a commutative ring and S a commutative semigroup written additively. The semigroup ring $R[X; S] = \{\sum_{s \in S} r_s X^s \mid r_s \in R\}$ has an identity if and only if both R and S do. The semigroup S is *idempotent* if $S + S = S$. It is easy to adjoin an identity to a semigroup S . Just let 0 be an element not in S and let $S^1 = S \cup \{0\}$ where $0 + 0 = 0$ and $0 + s = s + 0$ for all $s \in S$. Now if R has an identity and S is a monoid, it is not hard to prove that $R[X; S]$ is Noetherian (Artinian) if and only if

R is Noetherian (Artinian) and S is finitely generated (finite). The problem of determining when $R[S]$ is Noetherian or Artinian in the case where R or S doesn't have an identity is more complicated and cannot be solved by simply adjoining an identity. There are several reasons for this. First, while a (left) Artinian ring with identity is (left) Noetherian, this need not be the case in general; to wit, $\mathbb{Z}(p^\infty)$ with the zero product. Second, we have already remarked that in the case of $S = \mathbb{N}_0$, $R[X; \mathbb{N}_0] = R[X]$ Noetherian implies that R has an identity [16]. In [24], Gilmer determined when $R[X; S]$ is Noetherian or Artinian.

Theorem 5.3. *Let R be a commutative ring and S a commutative semigroup.*

(1) *Suppose that S is idempotent.*

(a) *If R has (doesn't have) an identity, then $R[X; S]$ is Noetherian if and only if R is Noetherian and S is finitely generated (S is finite).*

(b) *$R[X; S]$ is Artinian if and only if R is Artinian and S is finite.*

(2) *Suppose that S is not idempotent.*

(a) *If R has (doesn't have) an identity, then $R[X; S]$ is Noetherian if and only if S is finitely generated (finite) and $(R, +)$ is finitely generated.*

(b) *$R[X; S]$ is Artinian if and only if S is finite and $(R, +)$ is Artinian.*

In [44], Parker and Gilmer determined the nilradical of $R[X; S]$ for R any commutative ring and S any commutative semigroup. Here the existence of identity plays no role. Let p be a prime number. For $a, b \in S$ define $a \sim_p b$ if $p^k a = p^k b$ for all $k \geq 1$ and define $a \sim b$ if $na = nb$ for large n . Then \sim_p and \sim are congruences on S . The nilradical of $R[X; S]$ is $\bigcap_{\lambda \in A} \{P_\lambda[X; S] + I_{P_\lambda}\}$ where $\{P_\lambda\}_{\lambda \in A}$ is the set of prime ideals of R , p_λ is the characteristic of R/P_λ , and $I_{P_\lambda} = (\{rX^a - rX^b \mid r \in R \text{ and } a \sim_{p_\lambda} b\})$ if $p_\lambda > 0$ and $I_{P_\lambda} = (\{rX^a - rX^b \mid r \in R, a \sim b\})$ if $p_\lambda = 0$ [44, Theorem 3.14].

In [34], Gilmer and Teply characterized when $R[X; S]$ is von Neumann regular for R a commutative ring and S a commutative monoid (written additively). Let \sim_p and \sim be as in the preceding paragraph. They showed [34, Theorem 8] that if $\{M_\lambda\}_{\lambda \in A}$ is the set of maximal ideals of R , and $p_\lambda = \text{char } R/M_\lambda$, then R is von Neumann regular if and only if (1) R is von Neumann regular, (2) S is free of asymptotic torsion (if $x \sim y$, then $x = y$), (3) S is p -torsion-free for each prime p in $\{p_\lambda\}_{\lambda \in A}$ ($x \sim_p y \Rightarrow x = y$), and (4) S is a torsion semigroup (for each $s \in S$, there exist distinct positive integers m and n with $ms = ns$).

Let R_1 and R_2 be commutative rings and S a not necessarily commutative semigroup. In [33] Gilmer and Spiegel investigated the question of when $R_1[X; S] \approx R_2[X; S]$ implies $R_1 \approx R_2$. Of course, well-known examples show that R_1 need not be isomorphic to R_2 even if $S = \mathbb{N}_0$ or \mathbb{Z} . The main result [33, Theorem 17] of this paper is that if F and K are fields and S and T are not necessarily commutative semigroups with S containing a periodic element (an element s with $s^n = s^m$ for some $n > m > 0$) with $F[X; S] \approx K[X; T]$, then $F \approx K$.

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