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p-adic Hodge Theory

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p-adic Hodge Theory

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Preface

The basic theme of p-adic Hodge theory is to understand the relationship between various p-adic cohomology theories associated to algebraic varieties over p-adic fields. In the standard formulation, it is concerned with comparisons between algebraic de Rham cohomology, p-adic étale cohomology, and crystalline cohomology. Each of these cohomology theories carry additional structure: de Rham cohomology comes equipped with a filtration, étale cohomology with a Galois action, and crystalline cohomology with a semi-linear Frobenius operator. Comparisons between these theories shed light on each of these individual structures, and the package of all of these cohomology theories and the comparison isomorphisms between them is a very rich structure associated to algebraic varieties over p-adic fields.

In recent years, there has been a surge of activity in the area related to integral p-adic Hodge, non-Abelian phenomena, and connections to notions in algebraic topology. The basic comparison isomorphisms of p-adic Hodge theory are defined rationally and don't directly provide information about the integral structures present in the cohomology theories, and there have been recent developments in the area to understand integral and torsion phenomena. Non-abelian phenomena can be understood on several levels, but the most basic one is the development of theories with coefficients. The connections with algebraic topology arise from the strong relationship between crystalline cohomology and topological Hochschild homology. This is also closely tied to the theory of the de Rham–Witt complex.

This proceedings volume contains chapters related to the research presented at the 2017 Simons Symposium on p-adic Hodge theory. This symposium was focused on recent developments in p-adic Hodge theory, especially those concerning integral questions and their connections to notions in algebraic topology.

The volume begins with the chapter of Morrow on the $\mathbf{A}_{\text{inf}}$-cohomology theory which was introduced in the earlier fundamental paper of Bhatt, Morrow, and Scholze on integral p-adic Hodge theory. The present chapter contains a detailed presentation of the $\mathbf{A}_{\text{inf}}$-cohomology theory, largely self-contained. The author focuses, in particular, on de Rham–Witt theory and the p-adic analogue of the Cartier isomorphism.

The chapter of Colmez and Niziol is concerned with a fundamental computation of the pro-étale cohomology of the rigid analytic affine space in any dimension. Contrary to the standard results for étale cohomology of algebraic varieties, these pro-étale cohomology groups are nonzero and the authors describe them using differential forms.

The third chapter by Chung, Kim, Kim, Park, and Yoo is concerned with a certain invariant attached to representations of the fundamental group of the ring of S-integers $\mathcal{O}_F[1/S]$ of a number field F, for some finite set of primes S. The authors describe a theory of the "arithmetic Chern-Simons action", inspired by the topological theory. The main result is a formula relating an invariant of a torsor over $\mathcal{O}_F[1/S]$ to locally defined data. The authors also give several interesting applications of this formula.

Throughout the subject of p-adic Hodge theory various large rings play a central role. The chapter of Kedlaya discusses various key basic algebraic properties of the ring $\mathbf{A}_{\mathrm{inf}}$, which is the ring of Witt vectors of a perfect valuation ring in characteristic p. This ring is, in particular, fundamental for the $\mathbf{A}_{\mathrm{inf}}$-cohomology developed by Bhatt, Morrow, and Scholze, and in integral p-adic Hodge theory. This ring is quite different from the ones occurring in classical algebraic geometry: for example, it is not Noetherian. Nevertheless, the author discusses several favorable properties, e.g., those related to vector bundles.

A fundamental result in complex Hodge theory is the Simpson correspondence relating local systems and Higgs bundles. An analogue of this theory was developed in characteristic p by Ogus and Vologodsky. The chapter of Gros is concerned with the problem of lifting this characteristic p correspondence to a mixed characteristic correspondence via a q-deformation.

The final chapter of Tsuji concerns the study of integral p-adic Hodge theory with coefficients. Early in the development of p-adic Hodge theory, Faltings constructed a theory of coefficients for integral p-adic Hodge theory. The present chapter refines this theory and generalizes the work of Bhatt, Morrow, and Scholze to this context. The chapter contains a detailed exposition of the many technical aspects of the theory and contains many improvements in this regard to the existing literature as well.

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Notes on the $\mathbb{A}_{\mathbf{inf}}$-Cohomology of *Integral p-Adic Hodge Theory*

Matthew Morrow

Abstract We present a detailed overview of the construction of the $\mathbb{A}_{\text{inf}}$-cohomology theory from the preprint *Integral p-adic Hodge theory*, joint with Bhatt and Scholze. We focus particularly on the p-adic analogue of the Cartier isomorphism via relative de Rham–Witt complexes.

Keywords p-adic Hodge theory · Prismatic cohomology · Perfectoid · de Rham–Witt complex

Extended abstract

These are expanded notes of a mini-course, given at l'Institut de Mathématiques de Jussieu–Paris Rive Gauche, 15 Jan.–1 Feb. 2016, detailing some of the main results of the article

> [5] B. Bhatt, M. Morrow, P. Scholze, *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci. 128 (2018), 219–397.

More precisely, the goal of these notes is to give a detailed, and largely self-contained, presentation of the construction of the $\mathbb{A}_{\text{inf}}$-cohomology theory from [5], focussing on the p-adic analogue of the Cartier isomorphism via relative de Rham–Witt complexes. By restricting attention to this particular aspect of [5], we hope to have made the construction more accessible. However, the reader should only read these notes in conjunction with [5] itself and is strongly advised also to consult the surveys [2, 26] by the other authors, which cover complementary aspects of the theory. In particular, in these notes we do not discuss q-de Rham complexes, cotangent complex

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calculations, Breuil–Kisin(–Fargues) modules, or the crystalline and de Rham comparison theorems of [5, Sect. 12–14], as these topics are not strictly required for the construction of the $\mathbb{A}_{\mathrm{inf}}$-cohomology theory.[1] Moreover, we refer to [5] for several self-contained proofs to avoid verbatim repetition.

Section 1 is an introduction which begins by recalling some classical problems and results of p-adic Hodge theory before stating the main theorem of the course, namely the existence of a new cohomology theory for p-adic schemes which integrally interpolates étale, crystalline and de Rham cohomologies.

Section 2 introduces the décalage functor, which modifies a given complex by a small amount of torsion. This functor is absolutely essential to our constructions, as it kills the "junk torsion" which so often appears in p-adic Hodge theory and thus allows us to establish results integrally. An example of this annihilation of torsion, in the context of Faltings' almost purity theorem, is given in Sect. 2.2.

Section 3 develops the necessary elementary theory of perfectoid rings, emphasising the importance of certain maps $\theta_r, \widetilde{\theta}_r$ which generalise Fontaine's usual map θ of p-adic Hodge theory and are central to the later constructions.

Section 4 is a minimal summary of Scholze's theory of pro-étale cohomology for rigid analytic varieties. In particular, in Sect. 4.3 we explain the usual technique by which the pro-étale manifestation of the almost purity theorem allows the pro-étale cohomology of "small" rigid affinoids to be (almost) calculated in terms of group cohomology related to perfectoid rings.

Section 5 revisits the main theorem and defines the new cohomology theory as the hypercohomology of a certain complex $\mathbb{A}\Omega_{\mathfrak{X}}$. In Theorem 4 we state a *p-adic Cartier isomorphism*, which identifies the cohomology sheaves of the base change of $\mathbb{A}\Omega_{\mathfrak{X}}$ along $\widetilde{\theta}_r$ with Langer–Zink's relative de Rham–Witt complex of the p-adic scheme $\mathfrak{X}$. We then deduce all main properties of the new cohomology theory from this p-adic Cartier isomorphism.

Section 6 reviews Langer–Zink's theory of the relative de Rham–Witt complex, which may be seen as the initial object in the category of Witt complexes, i.e., families of differential graded algebras over the Witt vectors which are equipped with compatible Restriction, Frobenius, and Verschiebung maps. In Sect. 6.2 we present one of our main constructions, namely building Witt complexes from the data of a commutative algebra (in a derived sense), equipped with a Frobenius, over the infinitesimal period ring $\mathbb{A}_{\mathrm{inf}}$. In Sect. 6.3 we apply this construction to the group cohomology of a Laurent polynomial algebra and prove that the result is precisely the relative de Rham–Witt complex itself; this is the local calculation which underlies the p-adic Cartier isomorphism.

Finally, Sect. 7 sketches the proof of the p-adic Cartier isomorphism by reducing to the final calculation of the previous paragraph. This reduction is based on various technical lemmas that the décalage functor behaves well under base change and

[1] To be precise, there is one step in the construction, namely the equality ($\dim_{\mathfrak{X}}$) in the proof of Theorem 7, where we will have to assume that the p-adic scheme $\mathfrak{X}$ is defined over a discretely valued field; this assumption can be overcome using the crystalline comparison theorems of [5].

taking cohomology, and that it transforms certain almost quasi-isomorphisms into quasi-isomorphisms.

The appendices provide an introduction to Fontaine's infinitesimal period ring $\mathbb{A}_{\mathrm{inf}}$ and state a couple of lemmas about Koszul complexes which are used repeatedly in calculations.

1 Introduction

1.1 Mysterious Functor and Crystalline Comparison

Here in Sect. 1.1 we consider the following common situation:

- K a complete discrete valuation field of mixed characteristic; ring of integers $\mathcal{O}_K$; perfect residue field k.
- $\mathfrak{X}$ a proper, smooth scheme over $\mathcal{O}_K$.

For $\ell \neq p$, proper base change in étale cohomology gives a canonical isomorphism

$$H^i_{\text{ét}}(\mathfrak{X}_{\overline{k}}, \mathbb{Z}_\ell) \cong H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_\ell)$$

which is compatible with Galois actions.[2] Grothendieck's question of the mysterious functor is often now interpreted as asking what happens in the case $\ell = p$. More precisely, how are $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}) := H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_p)$ and $H^i_{\text{crys}}(\mathfrak{X}_k) := H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$ related? In other words, how does p-adic cohomology of $\mathfrak{X}$ degenerate from the generic to the special fibre?

Grothendieck's question is answered after inverting p by the Crystalline Comparison Theorem (Fontaine–Messing [15], Bloch–Kato [7], Faltings [12], Tsuji [28] Nizioł [23], ...), stating that there are natural isomorphisms

$$H^i_{\text{crys}}(\mathfrak{X}_k) \otimes_{W(k)} \mathbb{B}_{\text{crys}} \cong H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{crys}}$$

which are compatible with Galois and Frobenius actions (and filtrations after base changing to $\mathbb{B}_{\text{dR}}$), where $\mathbb{B}_{\text{crys}}$ and $\mathbb{B}_{\text{dR}}$ are Fontaine's period rings (which we emphasise contain $1/p$; they will not appear again in these notes, so we do not define them). Hence general theory of period rings implies that

$$H^i_{\text{crys}}(\mathfrak{X}_k)\left[\tfrac{1}{p}\right] = (H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{crys}})^{G_K}$$

[2]To be precise, the isomorphism depends only on a choice of specialisation of geometric points of Spec $\mathcal{O}_K$. A consequence of the compatibility with Galois actions is that the action of G_K on $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is unramified.

(i.e., the crystalline Dieudonné module of $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}})[\frac{1}{p}]$, by definition) with φ on the left induced by $1 \otimes \varphi$ on the right. In summary, $(H^n_{\text{ét}}(\mathfrak{X}_{\overline{K}})[\frac{1}{p}], G_K)$ determines $(H^n_{\text{crys}}(\mathfrak{X}_k)[\frac{1}{p}], \varphi)$. Similarly, in the other direction, $(H^n_{\text{ét}}(\mathfrak{X}_{\overline{K}})[\frac{1}{p}], G_K)$ is determined by $(H^n_{\text{crys}}(\mathfrak{X}_k)[\frac{1}{p}], \varphi, \text{Hodge fil.})$.

But what if we do not invert p? There are various partial results in the literature, including [8, 13], and a simplifying summary would be to claim that "everything seems to work integrally if $ie < p - 1$",[3] where e is the absolute ramification degree of K. With no assumptions on ramification degree, dimension, value of p, etc., we prove in [5] results of the following form:

(i) The torsion in $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}})$ is "less than" that of $H^i_{\text{crys}}(\mathfrak{X}_k)$. To be precise,

$$\text{length}_{\mathbb{Z}_p} H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}})/p^r \leq \text{length}_{W(k)} H^i_{\text{crys}}(\mathfrak{X}_k)/p^r$$

for all $r \geq 1$, as one would expect for a degenerating family of cohomologies. In particular, if $H^i_{\text{crys}}(\mathfrak{X}_k)$ is torsion-free then so is $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}})$.

(ii) If $H^*_{\text{crys}}(\mathfrak{X}_k)$ is torsion-free for $* = i, i+1$, then $(H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}), G_K)$ determines $(H^i_{\text{crys}}(\mathfrak{X}_k), \varphi)$.

It really is possible that additional torsion appears when degenerating the p-adic cohomology from the generic fibre to the special fibre, as the following example indicates (which is labeled a theorem as there seems to be no case of an $\mathfrak{X}$ as above for which $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}) \otimes_{\mathbb{Z}_p} W(k)$ and $H^i_{\text{crys}}(\mathfrak{X})$ were previously known to have non-isomorphic torsion submodules):

Theorem 0 *There exists a smooth projective relative surface $\mathfrak{X}$ over $\mathbb{Z}_2$ such that $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}})$ is torsion-free for all $i \geq 0$ but such that $H^2_{\text{crys}}(\mathfrak{X}_k)$ contains non-trivial 2-torsion.*[4]

Proof We do not reproduce the construction here; see [5, Proposition 2.2].

1.2 Statement of Main Theorem and Outline of Notes

The following notation will be used repeatedly in these notes:

- $\mathbb{C}$ is a complete, non-archimedean, algebraically closed field of mixed characteristic[5]; ring of integers $\mathcal{O}$; residue field k.

[3] Our results can presumably make this more precise.

[4] In [5, Theorem 2.10] we also give an example for which $H^2_{\text{ét}}(\mathfrak{X}_{\overline{K}})_{\text{tors}} = \mathbb{Z}/p^2\mathbb{Z}$ and $H^2_{\text{crys}}(\mathfrak{X}_k)_{\text{tors}} = k \oplus k$.

[5] More general, most of the theory which we will present works for any perfectoid field of mixed characteristic which contains all p-power roots of unity.

- $\mathcal{O}^\flat := \varprojlim_{\varphi} \mathcal{O}/p\mathcal{O}$ is the *tilt* (using Scholze's language [24]—or $\mathcal{R}_{\mathcal{O}}$ in Fontaine's original notation [14]) of $\mathcal{O}$; so $\mathcal{O}^\flat$ is a perfect ring of characteristic p which is the ring of integers of $\mathbb{C}^\flat := \operatorname{Frac} \mathcal{O}^\flat$, which is a complete, non-archimedean, algebraically closed field with residue field k.
- $\mathbb{A}_{\mathrm{inf}} := W(\mathcal{O}^\flat)$ is the first period ring of Fontaine[6]; it is equipped with the usual Witt vector Frobenius φ. There are three key specialisation maps:

$$\begin{array}{ccccc} & & W(\mathbb{C}^\flat) & & \\ & & \uparrow{\scriptstyle \text{étale}} & & \\ \mathcal{O} & \xleftarrow[\text{Fontaine's map } \theta]{\text{de Rham}} & \mathbb{A}_{\mathrm{inf}} & \xrightarrow{\text{crystalline}} & W(k) \end{array}$$

  where Fontaine's map θ will be discussed in detail, and in greater generality, in Sect. 3.

The goal of these notes is to provide a relatively detailed overview of the proof of the following theorem, establishing the existence of a cohomology theory, taking values in $\mathbb{A}_{\mathrm{inf}}$-modules, which integrally interpolates the étale, crystalline, and de Rham cohomologies of a smooth p-adic scheme:

Theorem 1 *For any proper, smooth (possibly p-adic formal) scheme $\mathfrak{X}$ over $\mathcal{O}$, there is a perfect complex $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ of $\mathbb{A}_{\mathrm{inf}}$-modules, functorial in $\mathfrak{X}$ and equipped with a φ-semi-linear endomorphism φ, with the following specialisations (which are compatible with Frobenius actions where they exist):*

(i) Étale: $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}}} W(\mathbb{C}^\flat) \simeq R\Gamma_{\mathrm{\acute{e}t}}(X, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} W(\mathbb{C}^\flat)$, where $X := \mathfrak{X}_{\mathbb{C}}$ is the generic fibre of $\mathfrak{X}$ (viewed as a rigid analytic variety over $\mathbb{C}$ in the case that $\mathfrak{X}$ is a formal scheme)
(ii) Crystalline: $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}}} W(k) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$.
(iii) de Rham: $R\Gamma_{\mathbb{A}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathbb{A}_{\mathrm{inf}}} \mathcal{O} \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O})$.

The individual cohomology groups

$$H^i_{\mathbb{A}}(\mathfrak{X}) := H^i(R\Gamma_{\mathbb{A}}(\mathfrak{X}))$$

have the following properties:

(iv) $H^i_{\mathbb{A}}(\mathfrak{X})$ is a finitely presented $\mathbb{A}_{\mathrm{inf}}$-module;
(v) $H^i_{\mathbb{A}}(\mathfrak{X})[\frac{1}{p}]$ is finite free over $\mathbb{A}_{\mathrm{inf}}[\frac{1}{p}]$;
(vi) $H^i_{\mathbb{A}}(\mathfrak{X})$ is equipped with a Frobenius-semi-linear endomorphism φ which becomes an isomorphism after inverting any generator $\xi \in \mathbb{A}_{\mathrm{inf}}$ of Ker *θ, i.e., $\varphi : H^i_{\mathbb{A}}(\mathfrak{X})[\frac{1}{\xi}] \xrightarrow{\simeq} H^i_{\mathbb{A}}(\mathfrak{X})[\frac{1}{\varphi(\xi)}]$;*

[6] A brief introduction to $\mathcal{O}^\flat$ and $\mathbb{A}_{\mathrm{inf}}$ may be found at the beginning of Appendix 1.

(vii) *Étale:* $H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} W(\mathbb{C}^\flat) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(\mathbb{C}^\flat)$, *whence*

$$(H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} W(\mathbb{C}^\flat))^{\varphi=1} \cong H^i_{\text{ét}}(X, \mathbb{Z}_p).$$

(viii) *Crystalline: there is a short exact sequence*

$$0 \longrightarrow H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} W(k) \to H^i_{\text{crys}}(\mathfrak{X}_k/W(k)) \longrightarrow \operatorname{Tor}_1^{\mathbb{A}_{\inf}}(H^{i+1}_{\mathbb{A}}(\mathfrak{X}), W(k)) \longrightarrow 0,$$

where the Tor_1 *term is killed by a power of* p.

(ix) *de Rham: there is a short exact sequence*

$$0 \longrightarrow H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} \mathcal{O} \to H^i_{\text{dR}}(\mathfrak{X}/\mathcal{O}) \longrightarrow H^{i+1}_{\mathbb{A}}(\mathfrak{X})[\xi] \longrightarrow 0,$$

where the third term is again killed by a power of p.

(x) *If* $H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$ *or* $H^i_{\text{dR}}(\mathfrak{X}/\mathcal{O})$ *is torsion-free, then* $H^i_{\mathbb{A}}(\mathfrak{X})$ *is a finite free* $\mathbb{A}_{\inf}$*-module.*

Corollary 1 *Let* $\mathfrak{X}$ *be as in the previous theorem, fix* $i \geq 0$, *and assume* $H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$ *is torsion-free. Then* $H^i_{\text{ét}}(X, \mathbb{Z}_p)$ *is also torsion-free. If we assume further that* $H^{i+1}_{\text{crys}}(\mathfrak{X}_k/W(k))$ *is torsion-free, then*

$$H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} W(k) = H^i_{\text{crys}}(\mathfrak{X}_k/W(k)) \quad \text{and} \quad H^i_{\mathbb{A}}(\mathfrak{X}) \otimes_{\mathbb{A}_{\inf}} \mathcal{O} = H^i_{\text{dR}}(\mathfrak{X}/\mathcal{O}).$$

Proof We first mention that the "whence" assertion of part (vii) of the previous theorem is the following general, well-known assertion: if M is a finitely generated $\mathbb{Z}_p$-module and F is any field of characteristic p, then $(M \otimes_{\mathbb{Z}_p} W(F))^{\varphi=1} = M$ (where φ really means $1 \otimes \varphi$).

Now assume $H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$ is torsion-free. Then part (x) of the previous theorem implies that $H^i_{\mathbb{A}}(\mathfrak{X})$ is finite free; so from part (vii) we see that $H^i_{\text{ét}}(X, \mathbb{Z}_p)$ cannot have torsion. If we also assume $H^{i+1}_{\text{crys}}(\mathfrak{X}_k/W(k))$ is torsion-free, then $H^{i+1}_{\mathbb{A}}(\mathfrak{X})$ is again finite free by (x), and so no torsion terms appear in the short exact sequences in parts (viii) and (ix) of the previous theorem.

Having stated the main theorem, we now give a very brief outline of the ideas which will be used to construct the $\mathbb{A}_{\inf}$-cohomology theory.

(i) We will define $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ to be the Zariski hypercohomology of the following complex of sheaves of $\mathbb{A}_{\inf}$-modules on the formal scheme $\mathfrak{X}$:

$$\mathbb{A}\Omega_{\mathfrak{X}} := L\eta_\mu\big(R\nu_*(\widehat{\mathbb{A}_{\inf,X}})\big)$$

where:

- $\mathbb{A}_{\inf,X}$ is a certain period sheaf of $\mathbb{A}_{\inf}$-modules on the pro-étale site $X_{\text{proét}}$ of the rigid analytic variety X (note that even if $\mathfrak{X}$ is an honest scheme over $\mathcal{O}$, we must view its generic fibre as a rigid analytic variety);

- $\nu : X_{\text{proét}} \to \mathfrak{X}_{\text{Zar}}$ is the projection map to the Zariski site of $\mathfrak{X}$;
- the hat indicates the derived p-adic completion of $R\nu_*(\mathbb{A}_{\text{inf},X})$ (see also the end of item (iv));
- $L\eta$ is the décalage functor which modifies a given complex by a small amount of torsion (in this case with respect to a prescribed element $\mu \in \mathbb{A}_{\text{inf}}$).

(ii) Parts (ii) and (iii) of Theorem 1 are proved simultaneously by relating $\mathbb{A}\Omega_{\mathfrak{X}}$ to Langer–Zink's relative de Rham–Witt complex $W_r\Omega^\bullet_{\mathfrak{X}/\mathcal{O}}$; indeed, this equals $\Omega^\bullet_{\mathfrak{X}/\mathcal{O}}$ if $r = 1$ (which computes de Rham cohomology of $\mathfrak{X}$) and satisfies $W_r\Omega^\bullet_{\mathfrak{X}/\mathcal{O}} \otimes_{W_r(\mathcal{O})} W_r(k) = W_r\Omega^\bullet_{\mathfrak{X}_k/k}$ (where $W_r\Omega^\bullet_{\mathfrak{X}_k/k}$ is the classical de Rham–Witt complex of Bloch–Deligne–Illusie computing crystalline cohomology of $\mathfrak{X}_k$).

(iii) If Spf R is an affine open of $\mathfrak{X}$ (so R is a p-adically complete, formally smooth $\mathcal{O}$-algebra[7]) which is *small*, i.e., formally étale over $\mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle$ (:= the p-adic completion of $\mathcal{O}[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$), then we will use the almost purity theorem to explicitly calculate $R\Gamma_{\text{Zar}}(\text{Spf } R, \mathbb{A}\Omega_{\mathfrak{X}})$ in terms of group cohomology and Koszul complexes. These calculations can be rephrased using "q-de Rham complexes" over $\mathbb{A}_{\text{inf}}$ (=deformations of the de Rham complex), but we do not do so in these notes.

(iv) Some remarks on the history and development of the results:

- Early motivation for the existence of $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ (e.g., as discussed by Scholze at Harris' 2014 MSRI birthday conference) came from topological cyclic homology. These notes say nothing about that point of view, which may now be found in [6].
- At the time of writing the announcement of our results [4], we only knew that the definition of $R\Gamma_{\mathbb{A}}(\mathfrak{X})$ in part (i) of the remark almost (in the precise sense of Faltings' almost mathematics) had the desired properties of Theorem 1, so it was necessary to modify the definition slightly; this modification is no longer necessary.
- Further simplifications of some of the proofs were explained in [2], some of which are also taken into account in these notes.
- The definition of $\mathbb{A}\Omega_{\mathfrak{X}}$ continues to make sense for any p-adic formal $\mathcal{O}$-scheme $\mathfrak{X}$, not necessarily smooth, and in particular the comparison isomorphisms of Theorem 1 have been extended to case of semi-stable reduction by Česnavičus and Koshikawa [9].
- In late 2018 the authors of [5] realised that the period sheaf $\mathbb{A}_{\text{inf},X}$ on $X_{\text{proét}}$ might not be derived p-adically complete, though this had been implicitly used in the construction. This is easily fixed, without changing any of the

[7] Throughout these notes we follow the convention that *formally smooth/étale* includes the condition of being topologically finitely presented, i.e., a quotient of $\mathcal{O}\langle T_1, \dots, T_N\rangle$ by a finitely generated ideal. Under this convention formal smoothness implies flatness. In fact, according to a result of Elkik [11, Theorem 7] (see Rmq. 2 on p. 587 for elimination of the Noetherian hypothesis), a p-adically complete $\mathcal{O}$-algebra is formally smooth if and only if it is the p-adic completion of a smooth $\mathcal{O}$-algebra.

ensuing arguments, either by replacing $\mathbb{A}_{\mathrm{inf},X}$ by its derived p-adic completion (which is then a complex of sheaves) or else by derived p-adically completing all occurrences of $R\nu_*(\mathbb{A}_{\mathrm{inf},X})$ and $R\Gamma_{\mathrm{pro\acute{e}t}}(-, \mathbb{A}_{\mathrm{inf},X})$ in the theory. In the published version of [5] the former approach is adopted, but in these notes we will follow the latter route which has the conceptual advantage that $\mathbb{A}_{\mathrm{inf},X}$ remains an honest sheaf of rings. Unfortunately this leads to a notation inconsistency: the $\mathbb{A}_{\mathrm{inf},X}$ of these notes is $\mathcal{H}^0(-)$ of the complex of sheaves $\mathbb{A}_{\mathrm{inf},X}$ of [5].

- Most recently, a site theoretic definition of the $\mathbb{A}_{\mathrm{inf}}$-cohomology is now available through the prismatic theory of Bhatt–Scholze [3].

2 The décalage Functor $L\eta$: Modifying Torsion

For a ring A and non-zero divisor $f \in A$, we define the *décalage functor* which was introduced first by Berthelot–Ogus [1, Chap. 8] following a suggestion of Deligne. It will play a fundamental role in our constructions.

Definition 1 Suppose that C is a cochain complex of f-torsion-free A-modules. Then we denote by $\eta_f C$ the subcomplex of $C[\frac{1}{f}]$ defined as

$$(\eta_f C)^i := \{x \in f^i C^i : dx \in f^{i+1} C^{i+1}\}$$

i.e., $\eta_f C$ is the largest subcomplex of $C[\frac{1}{f}]$ which in degree i is contained in $f^i C^i$ for all $i \in \mathbb{Z}$.

It is easy to compute the cohomology of $\eta_f C$:

Lemma 1 *The map on cocycles $Z^i C \to Z^i(\eta_f C)$ given by $m \to f^i m$ induces a natural isomorphism*

$$H^i(C)/H^i(C)[f] \xrightarrow{\simeq} H^i(\eta_f C).$$

Proof It is easy to see that the map induces $H^i(C) \to H^i(\eta_f C)$, and the kernel corresponds to those $x \in C^i$ such that $dx = 0$ and $fx \in d(C^{i-1})$, i.e., $H^i(C)[f]$.

Corollary 2 *If $C \xrightarrow{\sim} C'$ is a quasi-isomorphism of complexes of f-torsion-free A-modules, then the induced map $\eta_f C \to \eta_f C'$ is also a quasi-isomorphism.*

Proof Immediate from the previous lemma.

We may now derive η_f. There is a well-defined endofunctor $L\eta_f$ of the derived category $D(A)$ defined as follows: if $D \in D(A)$ then pick a quasi-isomorphism $C \xrightarrow{\sim} D$ where C is a cochain complex of f-torsion-free A-modules (e.g., pick a projective resolution, at least if D is bounded above) and set

$$L\eta_f D := \eta_f C.$$

This is well-defined by the previous corollary and standard formalism of derived categories.

Warning: $L\eta_f$ does not preserve distinguished triangles! For example, if $A = \mathbb{Z}$ then $L\eta_p(\mathbb{Z}/p\mathbb{Z}) = 0$ but $L\eta_p(\mathbb{Z}/p^2\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.

The general theory of the functor $L\eta_f$ will be spread out through the notes (see especially Remarks 7 and 9); now we proceed to two important examples.

2.1 Example 1: Crystalline Cohomology

The following proposition is the origin of the décalage functor, in which $A = W(k)$ and $f = p$; it is closely related to the Cartier isomorphism for the de Rham–Witt complex.

Proposition 1 *Let k be a perfect field of characteristic p and R a smooth k-algebra. Then*

(i) (Illusie 1979) The absolute Frobenius $\varphi : W\Omega^\bullet_{R/k} \to W\Omega^\bullet_{R/k}$ is injective and has image $\eta_p W\Omega^\bullet_{R/k}$, thus inducing a Frobenius-semi-linear isomorphism

$$\Phi : W\Omega^\bullet_{R/k} \xrightarrow{\simeq} \eta_p W\Omega^\bullet_{R/k}.$$

(ii) (Berthelot–Ogus 1978) There exists a Frobenius-semi-linear quasi-isomorphism

$$\Phi : R\Gamma_{\mathrm{crys}}(R/W(k)) \xrightarrow{\sim} L\eta_p R\Gamma_{\mathrm{crys}}(R/W(k)).$$

Proof Obviously (i)⇒(ii), but (ii) was proved earlier and is the historical origin of $L\eta$: see [1, Theorem 8.20] (with the zero gauge). Berthelot–Ogus applied it to study the relation between the Newton and Hodge polygons associated to a proper, smooth variety over k.

(i) is a consequence of the following standard de Rham–Witt identities:

- φ has image in $\eta_p W\Omega^\bullet_{R/k}$ since $\varphi = p^i F$ on $W\Omega^i_{R/k}$ and $d\varphi = \varphi d$.
- φ is injective since $FV = VF = p$.
- the image of φ is exactly $\eta_p W\Omega^\bullet_{R/k}$ since $d^{-1}(pW\Omega^{i+1}_{R/k}) = F(W\Omega^i_{R/k})$ [18, Equation I.3.21.1.5].

2.2 "Example 2": An Integral Form of Faltings' Almost Purity Theorem

We now present an integral form of (the main consequence of) Faltings' almost purity theorem; we do not need this precise result, but we will make use of Lemma 2 and the "goodness" of the group cohomology established in the course of the proof of Theorem 2. Moreover, readers familiar with Faltings' approach to p-adic Hodge theory may find this result motivating. To recall Faltings' almost purity theorem we consider the following situation:

- $\mathbb{C}$ is a complete, non-archimedean, algebraically closed field of mixed characteristic; ring of integers $\mathcal{O}$.
- R is a p-adically complete, formally smooth $\mathcal{O}$-algebra, which we further assume is connected and *small*, i.e., formally étale over $\mathcal{O}\langle \underline{T}^{\pm 1}\rangle := \mathcal{O}\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle$. As usual in Faltings' theory, we associate to this the following two rings:
- $R_\infty := R\widehat{\otimes}_{\mathcal{O}\langle \underline{T}^{\pm 1}\rangle}\mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle$—this is acted on by $\Gamma := \mathbb{Z}_p(1)^d$ via R-algebra automorphisms in the usual way: given $\gamma \in \Gamma = \mathrm{Hom}_{\mathbb{Z}_p}((\mathbb{Q}_p/\mathbb{Z}_p)^d, \mu_{p^\infty})$ and $k_1, \dots, k_d \in \mathbb{Z}[\frac{1}{p}]$, the action is $\gamma \cdot T_1^{k_1}\dots T_d^{k_d} := \gamma(k_1, \dots, k_d)T_1^{k_1}\dots T_d^{k_d}$.
- $\overline{R}$:= the p-adic completion of the normalisation of R in the maximal (ind)étale extension of $R[\frac{1}{p}]$—this is acted on by $\Delta := \mathrm{Gal}(R[\frac{1}{p}])$ via R-algebra automorphisms, and its restriction to R_∞ gives the Γ-action there.

Faltings' almost purity theorem states $\overline{R}$ is an "almost étale" R_∞-algebra, and the main consequence of this is that the resulting map on continuous group cohomology

$$R\Gamma_{\mathrm{cont}}(\Gamma, R_\infty) \longrightarrow R\Gamma_{\mathrm{cont}}(\Delta, \overline{R})$$

is an almost quasi-isomorphism (i.e., all cohomology groups of the cone are killed by the maximal ideal $\mathfrak{m} \subset \mathcal{O}$). This is his key to calculating étale cohomology in terms of de Rham cohomology; indeed, $R\Gamma_{\mathrm{cont}}(\Delta, \overline{R})$ is a priori hard to calculate and encodes Galois/étale cohomology, while $R\Gamma_{\mathrm{cont}}(\Gamma, R_\infty)$ is easy to calculate using Koszul complexes (as we will see in the proof of Theorem 2) and differential forms.

The following is our integral form of this result, in which we apply $L\eta$ with respect to any element $f \in \mathfrak{m} \subset \mathcal{O}$:

Theorem 2 *Under the above set-up, the induced map*

$$L\eta_f R\Gamma_{\mathrm{cont}}(\Gamma, R_\infty) \longrightarrow L\eta_f R\Gamma_{\mathrm{cont}}(\Delta, \overline{R})$$

is a quasi-isomorphism (not just an almost quasi-isomorphism!) for any non-zero $f \in \mathfrak{m}$.

Remark 1 (i) The proof of Theorem 2 requires knowing nothing new about $R\Gamma_{\mathrm{cont}}(\Delta, \overline{R})$: a key remarkable property of $L\eta$ is that it can transform almost quasi-isomorphisms into actual quasi-isomorphisms, having only imposed hypotheses on the domain, not the codomain, of the morphism; this will be explained in the next lemma.

(ii) The theorem implies that the kernel and cokernel of $H^i_{\text{cont}}(\Gamma, R_\infty) \to H^i_{\text{cont}}(\Delta, \overline{R})$ are killed by f; since f is any element of $\mathfrak{m}$, the kernel and cokernel are killed by $\mathfrak{m}$. Thus Theorem 2 is a family of on-the-nose integral results which recovers Faltings' almost quasi-isomorphism $R\Gamma_{\text{cont}}(\Gamma, R_\infty) \to R\Gamma_{\text{cont}}(\Delta, \overline{R})$.

Lemma 2 *Let $\mathfrak{M} \subseteq A$ be an ideal of a ring and $f \in \mathfrak{M}$ a non-zero-divisor. Say that an A-module M is "good" if and only if both M and M/fM contain no non-zero elements killed by $\mathfrak{M}$. Then the following statements hold:*

(i) If $M \to N$ is a homomorphism of A-modules with kernel and cokernel killed by $\mathfrak{M}$, and M is good, then $M/M[f] \to N/N[f]$ is an isomorphism.
(ii) If $C \to D$ is a morphism of complexes of A-modules whose cone is killed by $\mathfrak{M}$, and all cohomology groups of C are good, then $L\eta_f C \to L\eta_f D$ is a quasi-isomorphism.

Proof Clearly (ii) is a consequence of (i) and Lemma 1. So we must prove (i).

Since the kernel of M is killed by $\mathfrak{M}$, but M contains no non-zero elements killed by $\mathfrak{M}$, we see that $M \to N$ is injective, and we will henceforth identify M with a submodule of N. Then $M[f] = M \cap N[f]$ and so $M/M[f] \to N/N[f]$ is also injective.

Since the quotient N/M is killed by $\mathfrak{M}$, there is a chain of inclusions $\mathfrak{M}fN \subseteq fM \subseteq fN \subseteq M$. But M/fM contains no non-zero elements killed by $\mathfrak{M}$, so $fM = fN$, and this completes the proof: any $n \in N$ satisfies $fn = fm$ for some $m \in M$, whence $n \equiv m \bmod N[f]$.

Proof (*Proof of Theorem* 2). To prove Theorem 2 we use Faltings' almost purity theorem and Lemma 2 (in the context $A = \mathcal{O}$, $f \in \mathfrak{M} = \mathfrak{m}$): so it is enough to show that $H^i_{\text{cont}}(\Gamma, R_\infty)$ is "good" for all $i \geq 0$. This is a standard type of explicit calculation of $H^i_{\text{cont}}(\Gamma, R_\infty)$ in terms of Koszul complexes. For the sake of the reader unfamiliar with this type of calculation, the special case that $R = \mathcal{O}\langle T^{\pm 1}\rangle$ is presented in a footnote[8]; here in the main text we will prove the general case. Both there and

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[8]In this footnote we carry out the calculation of the proof of Theorem 2 when $R = \mathcal{O}\langle T^{\pm 1}\rangle$, in which case $R_\infty = \mathcal{O}\langle T^{\pm 1/p^\infty}\rangle$. To reiterate, we must show that $H^i_{\text{cont}}(\Gamma, R_\infty)$ is good for all $i \geq 0$.

First note that R_∞ admits a Γ-equivariant decomposition into $\mathcal{O}$-submodules

$$R_\infty = \widehat{\bigoplus}_{k \in \mathbb{Z}\left[\frac{1}{p}\right]} \mathcal{O}T^k$$

(where the hat denotes p-adic completion of the sum), with the generator $\gamma \in \Gamma$ acting on the rank-one free $\mathcal{O}$-module $\mathcal{O}T^k$ as multiplication by ζ^k. Thus $R\Gamma_{\text{cont}}(\mathbb{Z}_p, \mathcal{O}T^k) \simeq [\mathcal{O} \xrightarrow{\zeta^k - 1} \mathcal{O}]$ (since the group cohomology of an infinite cyclic group with generator γ is computed by the invariants and coinvariants of γ, and similarly in the case of continuous group cohomology), and so

$$R\Gamma_{\text{cont}}(\mathbb{Z}_p, R_\infty) \simeq \widehat{\bigoplus}_{k \in \mathbb{Z}\left[\frac{1}{p}\right]} [\mathcal{O} \xrightarrow{\zeta^k - 1} \mathcal{O}]$$

here we pick a compatible sequence $\zeta_p, \zeta_{p^2}, \dots, \in \mathcal{O}$ of p-power roots of unity to get a generator $\gamma \in \mathbb{Z}_p(1)$ and an identification $\Gamma \cong \mathbb{Z}_p^d$; as a convenient abuse of notation, we write $\zeta^k := \zeta_{p^j}^a$ when $k = a/p^j \in \mathbb{Z}[\frac{1}{p}]$.

First note that $\mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle$ admits a Γ-equivariant decomposition into $\mathcal{O}\langle \underline{T}^{\pm 1}\rangle$-modules:

$$\mathcal{O}\langle \underline{T}^{\pm 1/p^\infty}\rangle = \mathcal{O}\langle \underline{T}^{\pm 1}\rangle \oplus \mathcal{O}\langle \underline{T}^{\pm 1}\rangle^{\text{non-int}},$$

where

$$\mathcal{O}\langle \underline{T}^{\pm 1}\rangle^{\text{non-int}} := \widehat{\bigoplus_{\substack{k_1,\dots,k_d \in \mathbb{Z}\left[\frac{1}{p}\right]\cap[0,1) \\ \text{not all zero}}}} \mathcal{O}\langle \underline{T}^{\pm 1}\rangle T_1^{k_1}\dots T_d^{k_d}$$

(where the hat denotes p-adic completion of the sum), with the generators $\gamma_1, \dots, \gamma_d \in \Gamma$ acting on the rank-one free $\mathcal{O}$-module $\mathcal{O}T_1^{k_1}\dots T_d^{k_d}$ respectively as multiplication by $\zeta^{k_1}, \dots, \zeta^{k_d}$.

Base changing to R we obtain a similar Γ-equivariant decomposition of R_∞ into R-modules

$$R_\infty = R \oplus R_\infty^{\text{non-int}}, \qquad R_\infty^{\text{non-int}} := \widehat{\bigoplus_{\substack{k_1,\dots,k_d \in \mathbb{Z}\left[\frac{1}{p}\right]\cap[0,1) \\ \text{not all zero}}}} RT_1^{k_1}\dots T_d^{k_d},$$

and so $R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, R_\infty) \simeq R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, R) \oplus R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, R_\infty^{\text{non-int}})$, where

$$R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, R_\infty^{\text{non-int}}) \simeq \widehat{\bigoplus_{\substack{k_1,\dots,k_d \in \mathbb{Z}\left[\frac{1}{p}\right]\cap[0,1) \\ \text{not all zero}}}} R\Gamma_{\text{cont}}(\mathbb{Z}_p^d, RT_1^{k_1}\dots T_d^{k_d})$$

(where the hat now denotes the derived p-adic completion of the sum of complexes). Now we must calculate $H^i_{\text{cont}}(\mathbb{Z}_p, ?)$ for $? = R$ or $RT_1^{k_1}\dots T_d^{k_d}$.

In the first case, the action of $\mathbb{Z}_p^d$ on R is trivial and so a standard group cohomology fact says that $H^i_{\text{cont}}(\mathbb{Z}_p^d, R) \cong \bigwedge_R^i R^d$. In the second case, another standard group

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(where the hat now denotes the derived p-adic completion of the sum of complexes), which has cohomology groups

$$H^0_{\text{cont}}(\mathbb{Z}_p, R_\infty) \cong \widehat{\bigoplus}_{k\in\mathbb{Z}}\mathcal{O} \oplus 0, \qquad H^1_{\text{cont}}(\mathbb{Z}_p, R_\infty) \cong \widehat{\bigoplus}_{k\in\mathbb{Z}}\mathcal{O} \oplus \bigoplus_{k\in\mathbb{Z}\left[\frac{1}{p}\right]\setminus\mathbb{Z}} \mathcal{O}/(\zeta^k - 1)\mathcal{O}$$

(once some care is taken regarding the p-adic completions: see footnote 9).

We claim that both cohomology groups are good. Since $\mathcal{O}$ has no non-zero elements killed by $\mathfrak{m}$, it remains only to prove that the same is true of $\mathcal{O}/a\mathcal{O}$, where $a = f$ or $\zeta^k - 1$ for some $k \in \mathbb{Z}[\frac{1}{p}] \setminus \mathbb{Z}$. But this is an easy argument with valuations: if $x \in \mathcal{O}$ is almost a multiple of a, then $\nu_p(x) + \varepsilon \geq \nu_p(a)$ for all $\varepsilon > 0$, whence $\nu_p(x) \geq \nu_p(a)$ and so x is actually a multiple of a.

cohomology fact says that $R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p^d, RT_1^{k_1}\dots T_d^{k_d})$ can be calculated by the Koszul complex $K_R(\zeta^{k_1}-1,\dots,\zeta^{k_d}-1)$; then Lemma 23 reveals (crucially using that not all k_i are zero) that

$$H^i_{\mathrm{cont}}(\mathbb{Z}_p^d, RT_1^{k_1}\dots T_d^{k_d}) \cong R/(\zeta_{p^r}-1)R^{\binom{d-1}{i-1}}$$

where $r := -\min_{1\le i\le d}\nu_p(k_i)\ge 1$ is the smallest integer such that $\zeta_{p^r}-1|\zeta^{k_i}-1$ for all $i=1,\dots,d$.

Assembling[9] these calculations yields isomorphisms

$$H^i_{\mathrm{cont}}(\Gamma, R_\infty)\cong \bigwedge_R^i R^d \oplus \bigoplus_{\substack{k_1,\dots,k_d\in\mathbb{Z}\left[\frac{1}{p}\right]\cap[0,1)\\ \text{not all zero}}} R/(\zeta_{p^{-\min_{1\le i\le d}\nu_p(k_i)}}-1)R^{\binom{d-1}{i-1}},$$

which we claim is good for each $i\ge 0$. That is, we must show that R, R/fR, and $R/(\zeta_{p^r}-1)R$, for $r\ge 1$, contain no non-zero elements killed by $\mathfrak{m}$. This is trivial for R itself since it is a torsion-free $\mathcal{O}$-algebra, so it remains to show, for each non-zero $a\in\mathfrak{m}$, that R/aR contains no non-zero elements killed by $\mathfrak{m}$; but R is a topologically free $\mathcal{O}$-module [5, Lemma 8.10] and so R/aR is a free $\mathcal{O}/a\mathcal{O}$-module, thereby reducing the problem to the analogous assertion for $\mathcal{O}/a\mathcal{O}$, which was proved in the final paragraph of footnote 8.

[9]This step requires some care about p-adic completions: the following straightforward result is sufficient. Suppose $(C_\lambda)_\lambda$ is a family of complexes satisfying the following for all $i\in\mathbb{Z}$: the group $H^i(C_\lambda)$ is p-adically complete and separated for all λ, with a bound on its p-power-torsion which is independent of λ. Then $H^i(\widehat{\bigoplus}_\lambda C_\lambda)=\widehat{\bigoplus}_\lambda H^i(C_\lambda)$, where the left hat is the derived p-adic completion of the sum of complexes, and the right hat is the usual p-adic completion of the sum of cohomology groups. *Proof*. Set $C_{\mathrm{disc}} := \bigoplus_\lambda C_\lambda$ and $C=\widehat{C_{\mathrm{disc}}}$ (derived p-adic completion); then the usual short exact sequences associated to a derived p-adic completion are

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \varprojlim^1_r H^i(C_{\mathrm{disc}})[p^r] & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, H^i(C_{\mathrm{disc}})) & \longrightarrow & H^i(C) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, H^{i+1}(C_{\mathrm{disc}})) & \longrightarrow & 0\\
 & & \downarrow & & & & & & \\
 & & \widehat{H^i(C_{\mathrm{disc}})}=\widehat{\bigoplus}_\lambda H^i(C_\lambda) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Our assumption that $\bigoplus_\lambda H^i(C_\lambda)$ has bounded p-power-torsion implies that the right and top terms vanish.

3 Algebraic Preliminaries on Perfectoid Rings

Fix a prime number p, and let A be a commutative ring which is π-adically complete (and separated) for some element $\pi \in A$ dividing p. Denoting by $\varphi : A/pA \to A/pA$ the absolute Frobenius, we have:

- the *tilt* $A^\flat := \varprojlim_\varphi A/pA$ of A, which is a perfect $\mathbb{F}_p$-algebra, on which we also denote the absolute Frobenius by φ. We sometimes write elements of $A^\flat$ as $x = (x_0, x_1, \dots)$, where $x_i \in A/pA$ and $x_i^p = x_{i-1}$ for all $i \geq 1$, and unless indicated otherwise the "projection $A^\flat \to A/pA$" refers to the map $x \mapsto x_0$.
- the associated "infinitesimal period ring" $W(A^\flat)$ of Fontaine, which is denoted by $\mathbb{A}_{\text{inf}}(A)$ in [5]. Note that, since $A^\flat$ is a perfect ring, $W(A^\flat)$ behaves just like the ring of Witt vectors of a perfect field of characteristic p: in particular p is a non-zero divisor of $W(A^\flat)$, each element has a unique expansion of the form $[x] + p[y] + p^2[z] + \cdots$, and $W(A^\flat)/p^r = W_r(A^\flat)$ for any $r \geq 1$.

The goal of this section is to study these constructions in more detail, in particular to introduce ring homomorphisms

$$\widetilde{\theta}_r, \theta_r : W(A^\flat) \longrightarrow W_r(A)$$

which play a fundamental role in the paper, and to define perfectoid rings.

3.1 The Maps θ_r, $\widetilde{\theta}_r$

The following lemma is helpful in understanding $A^\flat$ and will be used several times; we omit the proof since it is relatively well-known and based on standard p-adic or π-adic approximations:

Lemma 3 *The canonical maps*

$$\varprojlim_{x \mapsto x^p} A \longrightarrow A^\flat = \varprojlim_{\varphi} A/pA \longrightarrow \varprojlim_{\varphi} A/\pi A$$

are isomorphisms of monoids (resp. rings).

Before stating the main lemma which permits us to define the maps θ_r, we recall that if B is any ring, then the associated rings of Witt vectors $W_r(B)$ are equipped with three operators:

$$R, F : W_{r+1}(B) \to W_r(B) \qquad V : W_r(B) \to W_{r+1}(B),$$

where R, F are ring homomorphisms, and V is merely additive. Therefore we can take the limit over r in two ways (of which the second is probably more familiar):

$$\varprojlim_{r \text{ wrt } F} W_r(B) \quad \text{or} \quad W(B) = \varprojlim_{r \text{ wrt } R} W_r(B).$$

Lemma 4 *Let A be as above, i.e., a ring which is π-adically complete with respect to some element $\pi \in A$ dividing p. Then the following three ring homomorphisms are isomorphisms:*

$$\begin{array}{ccc} W(A^\flat) = \varprojlim_{r \text{ wrt } R} W_r(A^\flat) & \xleftarrow[(i)]{\varphi^\infty} & \varprojlim_{r \text{ wrt } F} W_r(A^\flat) \\ & & \downarrow (ii) \\ \varprojlim_{r \text{ wrt } F} W_r(A) & \xrightarrow[(iii)]{} & \varprojlim_{r \text{ wrt } F} W_r(A/\pi A) \end{array}$$

where

(i) φ^∞ is induced by the homomorphisms $\varphi^r : W_r(A^\flat) \to W_r(A^\flat)$ for $r \geq 1$;
(ii) the right vertical arrow is induced by the projection $A^\flat \to A/pA \to A/\pi A$;
(iii) the bottom horizontal arrow is induced by the projection $A \to A/\pi A$.

There is therefore an induced isomorphism

$$W(A^\flat) \xrightarrow{\simeq} \varprojlim_{r \text{ wrt } F} W_r(A)$$

making the diagram commute.

Proof We refer the reader to [5, Lemma 3.2] for the elementary proofs of the isomorphisms.

Definition 2 Continue to let A be as in the previous lemma, and $r \geq 1$. Define $\widetilde{\theta}_r : W(A^\flat) \to W_r(A)$ to be the composition

$$\boxed{\widetilde{\theta}_r : W(A^\flat) \xrightarrow{\simeq} \varprojlim_{r \text{ wrt } F} W_r(A) \longrightarrow W_r(A),}$$

where the first map is the isomorphism of the previous lemma, and the second map is the canonical projection. Also define

$$\boxed{\theta_r := \widetilde{\theta}_r \circ \varphi^r : W(A^\flat) \longrightarrow W_r(A).}$$

We stress that the Frobenius maps $F : W_{r+1}(A) \to W_r(A)$ need not be surjective, and thus $\theta_r, \widetilde{\theta}_r$ need not be surjective; indeed, such surjectivity will be part of the definition of a perfectoid ring (see Lemma 7).

To explicitly describe the maps θ_r and $\widetilde{\theta}_r$, we follow the usual convention of exploiting the isomorphism of monoids of Lemma 3 to denote an element $x \in A^\flat$ either as $x = (x_0, x_1, \ldots) \in \varprojlim_{\varphi} A/pA$ or $x = (x^{(0)}, x^{(1)}, \ldots) \in \varprojlim_{x \mapsto x^p} A$:

Lemma 5 *For any $x \in A^\flat$ we have $\theta_r([x]) = [x^{(0)}] \in W_r(A)$ and $\widetilde{\theta}_r([x]) = [x^{(r)}]$ for $r \geq 1$.*

Proof The formula for $\widetilde{\theta}_r$ follows from a straightforward chase through the above isomorphisms, and the corresponding formula for θ_r is an immediate consequence.

In particular, Lemma 5 implies that $\theta := \theta_1 : W(A^\flat) \to A$ is the usual map of p-adic Hodge theory as defined by Fontaine [14, Sect. 1.2], and also shows that the diagram

$$\begin{array}{ccc} W(A^\flat) & \xrightarrow{\theta_r} & W_r(A) \\ \downarrow & & \downarrow \\ W_r(A^\flat) & \longrightarrow & W_r(A/pA) \end{array}$$

commutes, where the left arrow is the canonical restriction map and the bottom arrow is induced by the projection $A^\flat \to A/pA$.

The following records the compatibility of the maps θ_r and $\widetilde{\theta}_r$ with the usual operators on the Witt groups; though it is probably the first set of diagrams which initially appears more natural, it is the second set which we we will use when constructing Witt complexes:

Lemma 6 *Continue to let A be as in the previous two lemmas. Then the following diagrams commute:*

$$\begin{array}{ccc} W(A^\flat) & \xrightarrow{\theta_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle R} \\ W(A^\flat) & \xrightarrow{\theta_r} & W_r(A) \end{array} \qquad \begin{array}{ccc} W(A^\flat) & \xrightarrow{\theta_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle F} \\ W(A^\flat) & \xrightarrow{\theta_r} & W_r(A) \end{array} \qquad \begin{array}{ccc} W(A^\flat) & \xrightarrow{\theta_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \lambda_{r+1}\varphi^{-1}}\uparrow & & \uparrow{\scriptstyle V} \\ W(A^\flat) & \xrightarrow{\theta_r} & W_r(A) \end{array}$$

where the third diagram requires an element $\lambda_{r+1} \in W(A^\flat)$ satisfying $\theta_{r+1}(\lambda_{r+1}) = V(1)$ in $W_{r+1}(A)$. Equivalently, the following diagrams commute:

$$\begin{array}{ccc} W(A^\flat) & \xrightarrow{\widetilde{\theta}_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \varphi^{-1}}\downarrow & & \downarrow{\scriptstyle R} \\ W(A^\flat) & \xrightarrow{\widetilde{\theta}_r} & W_r(A) \end{array} \qquad \begin{array}{ccc} W(A^\flat) & \xrightarrow{\widetilde{\theta}_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle F} \\ W(A^\flat) & \xrightarrow{\widetilde{\theta}_r} & W_r(A) \end{array} \qquad \begin{array}{ccc} W(A^\flat) & \xrightarrow{\widetilde{\theta}_{r+1}} & W_{r+1}(A) \\ {\scriptstyle \times\varphi^{r+1}(\lambda_{r+1})}\uparrow & & \uparrow{\scriptstyle V} \\ W(A^\flat) & \xrightarrow{\widetilde{\theta}_r} & W_r(A) \end{array}$$

Proof See [5, Lemma 3.4] for the short verification.

3.2 Perfectoid Rings

The next goal is to define what it means for A to be perfectoid, which requires discussing surjectivity and injectivity of the Frobenius on A/pA. We do this in greater generality than we require, but this greater generality reveals the intimate relation to the map θ and its generalisations $\theta_r, \widetilde{\theta}_r$.

Lemma 7 *Let A be a ring which is π-adically complete with respect to some element $\pi \in A$ such that π^p divides p. Then the following are equivalent:*

(i) Every element of $A/\pi pA$ is a p^{th}-power.
(ii) Every element of A/pA is a p^{th}-power.
(iii) Every element of $A/\pi^p A$ is a p^{th}-power.
(iv) The Witt vector Frobenius $F : W_{r+1}(A) \to W_r(A)$ is surjective for all $r \geq 1$.
(v) $\theta_r : W(A^\flat) \to W_r(A)$ is surjective for all $r \geq 1$.
(vi) $\theta : W(A^\flat) \to A$ is surjective.

Moreover, if these equivalent conditions hold then there exist $u, v \in A^\times$ such that $u\pi$ and vp admit systems of p-power roots in A.

Proof The implications (i)⇒(ii)⇒(iii) are trivial since $\pi pA \subseteq pA \subseteq \pi^p A$. (v)⇒(vi) is also trivial since $\theta = \theta_1$.

(iii)⇒(i): a simple inductive argument allows us to write any given element $x \in A$ as an infinite sum $x = \sum_{i=0}^\infty x_i^p \pi^{pi}$ for some $x_i \in A$; but then $x \equiv (\sum_{i=0}^\infty x_i \pi^i)^p \bmod p\pi A$.

(iv)⇒(ii): Clear from the fact that the Frobenius $F : W_2(A) \to W_1(A) = A$ is explicitly given by $(\alpha_0, \alpha_1) \mapsto \alpha_0^p + p\alpha_1$.

(iv)⇒(v): The hypothesis states that the transition maps in the inverse system $\varprojlim_{r \text{ wrt } F} W_r(A)$ are surjective, which implies that each map $\widetilde{\theta}_r$ is surjective, and hence that each map θ_r is surjective.

(vi)⇒(ii): Clear since any element of A in the image of θ is a p^{th}-power mod p.

It remains to show that (ii)⇒(iv), but we will first prove the "moreover" assertion using only (i) (which we have shown is equivalent to (ii)). Applying Lemma 3 to both A and $A/\pi pA$ implies that the canonical map $\varprojlim_{x \mapsto x^p} A \to \varprojlim_{x \mapsto x^p} A/\pi pA$ is an isomorphism. Applying (i) repeatedly, there therefore exists $\omega \in \varprojlim_{x \mapsto x^p} A$ such that $\omega^{(0)} \equiv \pi \bmod \pi pA$ (resp. $\equiv p \bmod \pi pA$). Writing $\omega^{(0)} = \pi + \pi px$ (resp. $\omega^{(0)} = p + \pi px$) for some $x \in A$, the proof of the "moreover" assertion is completed by noting that $1 + px \in A^\times$ (resp. $1 + \pi x \in A^\times$).

(ii)⇒(iv): By the "moreover" assertion, there exist $\pi' \in A$ and $v \in A^\times$ satisfying $\pi'^p = vp$. Note that A is π'-adically complete, and so we may apply the implication (ii)⇒(i) for the element π' to deduce that every element of $A/\pi' pA$ is a p^{th}-power; it follows that every element of A/Ip is a p^{th}-power, where I is the ideal $\{a \in A : a^p \in pA\}$. Now apply implication "(xiv)$'$ ⇒(ii)" of Davis–Kedlaya [10].

Lemma 8 *Let A be a ring which is π-adically complete with respect to some element $\pi \in A$ such that π^p divides p, and assume that the equivalent conditions of the previous lemma are true.*

(i) If Ker θ *is a principal ideal of* $W(A^\flat)$*, then*

(a) $\Phi : A/\pi A \to A/\pi^p A$*,* $a \mapsto a^p$*, is an isomorphism;*
(b) any generator of Ker θ *is a non-zero-divisor*[10]*;*
(c) an element $\xi \in$ Ker θ *is a generator if and only if it is "distinguished", i.e., its Witt vector expansion* $\xi = (\xi_0, \xi_1, \dots)$ *has the property that* ξ_1 *is a unit of* $A^\flat$.
(d) any element $\xi \in$ Ker θ *satisfying* $\theta_r(\xi) = V(1) \in W_r(A)$ *for some* $r > 1$ *is distinguished (and such an element exists for any given* $r > 1$*).*

(ii) Conversely, if π *is a non-zero-divisor and* $\Phi : A/\pi A \to A/\pi^p A$ *is an isomorphism (which is automatic if* A *is integrally closed in* $A[\frac{1}{\pi}]$*), then* Ker θ *is a principal ideal.*

Proof Rather than copying the proof here, we refer the reader to Lemma 3.10 and Remark 3.11 of [5]. The only assertion which is not proved there is the parenthetical assertion in (ii), for which we just note that if A is integrally closed in $A[\frac{1}{\pi}]$, then Φ is automatically injective: indeed, if a^p divides π^p, then $(a/\pi)^p \in A$ and so $a/\pi \in A$.

We can now define a perfectoid ring[11]:

Definition 3 A ring A is *perfectoid* if and only if the following three conditions hold:

- A is π-adically complete for some element $\pi \in A$ such that π^p divides p;
- the Frobenius map $\varphi : A/pA \to A/pA$ is surjective (equivalently, $\theta : W(A^\flat) \to A$ is surjective);
- the kernel of $\theta : W(A^\flat) \to A$ is principal.

Remark 2 The first condition of the definition could be replaced by the seemingly stronger, but actually equivalent and perhaps more natural, condition that "A is p-adically complete and there exists a unit $u \in A^\times$ such that pu is a p^{th}-power." Indeed, this follows from the final assertion of Lemma 7.

We return to the maps θ_r, describing their kernels in the case of a perfectoid ring:

Lemma 9 *Suppose that* A *is a perfectoid ring, and let* $\xi \in W(A)$ *be any element generating* Ker θ *(this exists by Lemma 7). Then* Ker θ_r *is generated by the non-zero-divisor*

[10]In all our cases of interest the ring A will be an integral domain, in which case it may be psychologically comforting to note that $A^\flat$ and $W(A^\flat)$ are also integral domains. *Proof.* The ring $W(A^\flat)$ is p-adically separated, satisfies $W(A^\flat)/p = A^\flat$, and p is a non-zero-divisor in it (these properties all follow simply from $A^\flat$ being perfect). So, once we show that $A^\flat$ is an integral domain, it will easily follow that $W(A^\flat)$ is also an integral domain. But the fact that $A^\flat$ is an integral domain follows at once from the same property of A using the isomorphism of monoids $\varprojlim_{x \mapsto x^p} A \xrightarrow{\simeq} A^\flat$ which already appeared in Lemma 4.

[11]Perhaps "*integral* perfectoid ring" would be better terminology to avoid conflict with the more common notion of perfectoid algebras in which p is invertible.

$$\xi_r := \xi\varphi^{-1}(\xi)\ldots\varphi^{-(r-1)}(\xi)$$

for any $r \geq 1$*, and so* Ker $\widetilde{\theta}_r$ *is generated by the non-zero-divisor*

$$\widetilde{\xi}_r := \varphi^r(\xi_r) = \varphi(\xi)\ldots\varphi^r(\xi).$$

Proof It is enough to prove the claim about ξ_r, since the claim about $\widetilde{\xi}_r$ then follows by applying φ^r. The proof is by induction on $r \geq 1$, using the diagrams of Lemma 6 for the inductive step; we refer to [5, Lemma 3.12] for the details.

We finish this introduction to perfectoid rings with some examples:

Example 1 (*Perfect rings of characteristic p*) Suppose that A is a ring of characteristic p. Then A is perfectoid if and only if it is perfect. Indeed, if A is perfect, then it is 0-adically complete, the Frobenius is surjective, and the kernel of $\theta : W(A) \to A$ is generated by p. Conversely, if A is perfectoid, then Lemma 8(i)(c) implies that the distinguished element $p \in \mathrm{Ker}(\theta : W(A^\flat) \to A)$ must be a generator, whence $W(A^\flat)/p \cong A$; but $W(A^\flat)/p = A^\flat$ is perfect.

In particular, in this case $A^\flat = A$ and the maps $\theta_r : W(A^\flat) \to W_r(A)$ are the canonical Witt vector restriction maps.

Example 2 If $\mathbb{C}$ is a complete, non-archimedean algebraic closed field of residue characteristic $p > 0$, then its ring of integers $\mathcal{O}$ is a perfectoid ring. Indeed, if $\mathbb{C}$ has equal characteristic p then $\mathcal{O}$ is perfect and we may appeal to the previous lemma. If $\mathbb{C}$ has mixed characteristic (our main case of interest), then $\mathcal{O}$ is $p^{1/p}$-adically complete, integrally closed in $\mathcal{O}[\frac{1}{p^{1/p}}] = \mathbb{C}$, and every element of $\mathcal{O}/p\mathcal{O}$ is a p^{th}-power since $\mathbb{C}$ is algebraically closed, so we may appeal to Lemma 8(ii); in this situation the ring $W(\mathcal{O}^\flat)$ will always be denoted by $\mathbb{A}_{\inf}$.

Example 3 Let A be a perfectoid ring which is π-adically complete with respect to some non-zero-divisor $\pi \in A$ such that π^p divides p. Here we offer some constructions of new perfectoid rings from A:

(i) The rings $A\langle T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}\rangle$ and $A\langle T_1^{\pm 1/p^\infty}, \ldots, T_d^{\pm 1/p^\infty}\rangle$, which are by definition the π-adic completions of $A[T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}]$ and $A[T_1^{\pm 1/p^\infty}, \ldots, T_d^{\pm 1/p^\infty}]$ respectively, are also perfectoid.
(ii) Any π-adically complete, formally étale A-algebra is also perfectoid.

Proof Since the π-adic completeness of the given ring is tautological in each case, we only need to check that $\Phi : B/\pi B \to B/\pi^p B, b \mapsto b^p$ is an isomorphism in each case. This is clear for $B = A\langle \underline{T}^{\pm 1/p^\infty}\rangle$ and $A\langle \underline{T}^{1/p^\infty}\rangle$, and it hold for and A-algebra B as in (ii) since the square

$$\begin{array}{ccc} B/\pi & \xrightarrow{\varphi} & B/\pi \\ \uparrow & & \uparrow \\ A/\pi & \xrightarrow[\varphi]{} & A/\pi \end{array}$$

is a pushout diagram (the base change of the Frobenius along an étale morphism in characteristic p is again the Frobenius).

3.3 *Main Example: Perfectoid Rings Containing Enough Roots of Unity*

Here in Sect. 3.3 we fix a perfectoid ring A which has no p-torsion and which contains a compatible system $\zeta_p, \zeta_{p^2}, \ldots$ of primitive p-power roots of unity (to be precise, since A is not necessarily an integral domain, this means that ζ_{p^r} is a root of the $p^{r\text{th}}$ cyclotomic polynomial), which we fix. The simplest example is $\mathcal{O}$ itself, but we also need the theory for perfectoid algebras containing $\mathcal{O}$ such as $\mathcal{O}\langle T_1^{\pm 1/p^\infty}, \ldots, T_d^{\pm 1/p^\infty}\rangle$.

In particular we define particular elements $\varepsilon, \xi, \mu, \ldots$, which will be used repeatedly in our main constructions, and so we highlight (or rather box) the primary definitions and relations. Firstly, set

$$\boxed{\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \in A^\flat, \qquad \mu := [\varepsilon] - 1 \in W(A^\flat),}$$

and

$$\boxed{\xi := 1 + [\varepsilon^{1/p}] + [\varepsilon^{1/p}]^2 + \cdots + [\varepsilon^{1/p}]^{p-1} \in W(A^\flat).}$$

Lemma 10 *ξ is a generator of* Ker θ *satisfying $\theta_r(\xi) = V(1)$ for all $r \geq 1$.*

Proof By Lemma 8(i)(d) it is sufficient to show that $\theta_r(\xi) = V(1)$ for all $r \geq 1$. The ghost map $\mathrm{gh} : W_r(A) \to A^r$ is injective since A is p-torsion-free, and so it is sufficient to prove that $\mathrm{gh}(\theta_r(\xi)) = \mathrm{gh}(V(1))$. But it follows easily from Lemma 5 that the composition $\mathrm{gh} \circ \theta_r : W(A^\flat) \to A^r$ is given by $(\theta, \theta\varphi, \ldots, \theta\varphi^{r-1})$, and so in particular that

$$\mathrm{gh}(\theta_r(\xi)) = (\theta(\xi), \theta\varphi(\xi), \ldots, \theta\varphi^{r-1}(\xi)).$$

Since $\theta(\xi) = 0$ and $\mathrm{gh}(V(1)) = (0, p, p, p, \ldots)$, it remains only to check that $\theta\varphi^i(\xi) = p$ for all $i \geq 1$, which is straightforward:

$$\theta\varphi^i(\xi) = \theta(1 + [\varepsilon^{p^{i-1}}] + [\varepsilon^{p^{i-1}}]^2 + \cdots + [\varepsilon^{p^{i-1}}]^{p-1}) = 1 + 1 + \cdots + 1 = p.$$

It now follows from Lemma 9 that Ker θ_r is generated by

$$\boxed{\xi_r := \xi\varphi^{-1}(\xi) \ldots \varphi^{-(r-1)}(\xi) = \sum_{i=0}^{p^r-1} [\varepsilon^{1/p^r}]^i,}$$

and that Ker $\widetilde{\theta}_r$ is generated by

$$\boxed{\widetilde{\xi}_r := \varphi^r(\xi_r) = \varphi(\xi)\dots\varphi^r(\xi)}.$$

Proposition 2 *μ is a non-zero divisor of $W(A^\flat)$ which satisfies*

$$\boxed{\mu = \xi_r\varphi^{-r}(\mu), \qquad \varphi^r(\mu) = \widetilde{\xi}_r\mu, \qquad \widetilde{\theta}_r(\mu) = [\zeta_{p^r}] - 1 \in W_r(A)}$$

for all $r \geq 1$.

Proof The final identity is immediate from Lemma 5. It is clear that $\mu = \xi\varphi^{-1}(\mu)$, whence the identity $\mu = \xi_r\varphi^{-r}(\mu)$ follows by a trivial induction on r, and the central identity then follows by applying φ^r. To prove that μ is a non-zero-divisor, it suffices to show that $\widetilde{\theta}_r(\mu) = [\zeta_{p^r}] - 1$ is a non-zero-divisor of $W_r(A)$ for all $r \geq 1$ (since $W(A^\flat) = \varprojlim_{r \text{ wrt } F} W_r(A)$). Since A is p-torsion-free the ghost map is injective and so we may check this by proving that

$$\mathrm{gh}([\zeta_{p^r}] - 1) = (\zeta_{p^r} - 1, \zeta_{p^{r-1}} - 1, \dots, \zeta_p - 1)$$

is a non-zero-divisor of A^r; i.e., we must show that $\zeta_{p^r} - 1$ is a non-zero-divisor in A for all $r \geq 1$. But $\zeta_{p^r} - 1$ divides p, and A is assumed to be p-torsion-free.

Remark 3 The reader may wish to note that the Teichmüller lifts $[\zeta_p], [\zeta_{p^2}], \dots$ are not primitive p-power roots unity in $W_r(A)$ in any reasonable sense. Indeed, it follows from its ghost components $\mathrm{gh}([\zeta_p]) = (\zeta_p, 1, 1, \dots, 1)$ that $[\zeta_p]$ is not a root of $X^{p-1} + \dots + X + 1$ when $r > 1$.

However, the element $[\zeta_{p^r}] - 1 \in W_r(A)$ will play a distinguished role in our constructions and so we point out that it is a non-zero-divisor whose powers define the p-adic topology. Indeed, it follows from the ghost component calculation of the previous proposition that $[\zeta_{p^r}] - 1$ is a root of the polynomial

$$((X + 1)^{p^r} - 1)/X = X^{p^r - 1} + pX(\cdots) + p^r,$$

whence p divides $([\zeta_{p^r}] - 1)^{p^r-1}$, and $[\zeta_{p^r}] - 1$ divides p^r. A particularly important consequence of this is that $L\eta_{[\zeta_{p^r}]-1}$ commutes with derived p-adic completion, by [5, Lemma 6.20].

4 The Pro-étale Site and Its Sheaves

In this section we review aspects of pro-étale cohomology following [25, Sects. 3–4], working under the following set-up:

- $\mathbb{C}$ is a complete, non-archimedean, algebraically closed field of mixed characteristic; ring of integers $\mathcal{O}$ with maximal ideal $\mathfrak{m}$; residue field k.
- X is a quasi-separated rigid analytic variety over $\mathbb{C}$.

In particular, we will introduce various pro-étale sheaves on X which will play an essential role in our constructions, and explain how to calculate their cohomology via affinoid perfectoids and almost purity theorems.

4.1 The Pro-étale Site $X_{\text{proét}}$

We will take for granted that the reader is either familiar with, or can reasonably imagine, étale morphisms and coverings of rigid analytic varieties, and we let $X_{\text{ét}}$ denote the associated étale site of X. To define coverings in $X_{\text{ét}}$ (and soon in $X_{\text{proét}}$) it is useful to view X as an adic space,[12] and we therefore denote by $|X|$ the underlying topological space of its associated adic space X^{ad}: for example, if T is an affinoid $\mathbb{C}$-algebra, then $|\operatorname{Sp} T|$ denotes the topological space of (equivalences classes of) all continuous valuations on T, not merely those factoring through a maximal ideal (which correspond to the closed points of the adic space).

We now define (a countable version of) Scholze's pro-étale site $X_{\text{proét}}$ in several steps:

- An object of $X_{\text{proét}}$ is simply a formal inverse system $\mathcal{U} = \text{“}\varprojlim\text{”}_i U_i$ in $X_{\text{ét}}$ of the form

$$\begin{array}{c} \vdots \\ \downarrow \\ U_3 \\ \downarrow{\scriptstyle \text{fin. ét. surj.}} \\ U_2 \\ \downarrow{\scriptstyle \text{fin. ét. surj.}} \\ U_1 \\ \downarrow{\scriptstyle \text{ét.}} \\ X \end{array}$$

  In other words, $\mathcal{U}$ is the data of a tower of finite étale covers of U_1, which is étale over X. The underlying topological space of $\mathcal{U}$ is by definition $|\mathcal{U}| := \varprojlim_i |U_i|$.

[12]There is an equivalence of categories between quasi-separated rigid analytic varieties over $\mathbb{C}$ and those adic spaces over $\operatorname{Spa}(\mathbb{C}, \mathcal{O})$ whose structure map is quasi-separated and locally of finite-type [16, Proposition 4.5]. A collection of étale maps $\{f_\lambda : U_\lambda \to U\}$ in $X_{\text{ét}}$ is a cover if and only if it is jointly “strongly surjective”, which is equivalent to being jointly surjective at the level of adic points [17, Sect. 2.1].