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# Preface

Fuzzy set (FS) theory has long been introduced to handle inexact and imprecise data by Zadeh (1965). The drawback of using the single membership value in FS theory is that it cannot be used to express the evidences of support and objection simultaneously in many practical situations. In order to tackle this problem, Atanassov (1986) proposed the intuitionistic fuzzy set (IFS) using two characteristic functions expressing the degree of membership and the degree of non-membership of elements of the universal set to the IFS. It can cope with the presence of vagueness and hesitancy originating from imprecise knowledge or information. Atanassov and Gargov (1989) further generalized the IFS in the spirit of the ordinary interval-valued fuzzy set (IVFS) and defined the concept of an interval-valued intuitionistic fuzzy set (IVIFS), which enhances greatly the representation ability of uncertainty than IFS.

Over the last few decades, the theories of IFSs and IVIFSs have received extensive attention from researchers and practitioners, and have been widely applied to various fields. The theories of IFSs and IVIFSs are undergoing continuous in-depth study as well as continuous expansion of the scope of their applications. As such, it has been found that intuitionistic fuzzy decision making, interval-valued intuitionistic fuzzy decision making, intuitionistic fuzzy preference relations, and interval-valued intuitionistic fuzzy preference relations become significantly important. Both IFSs and IVIFSs have broad prospects of application in the fields of multi-attribute decision making (MADM) and multi-attribute group decision making (MAGDM), but pose many interesting yet challenging topics for research.

In this book, we will give a thorough and systematic introduction to the latest research achievements on the theories of IVIFSs and their applications to MADM and MAGDM. The main theoretical results obtained in this book include the possibility degree of comparison between two interval-valued intuitionistic fuzzy numbers (IVIFNs), the ordered weighted average operator and hybrid weighted average operator for IVIFNs based on the Karnik–Mendel algorithms, the asymptotic property of the Atanassov's interval-valued intuitionistic fuzzy (AIVIF) matrix, the weight of each decision maker (DM) with respect to every attribute, the attribute weights, a novel interval-valued intuitionistic fuzzy (IVIF) mathematical programming method for hybrid multi-criteria group decision making (MCGDM) considering alternative comparisons with hesitancy degrees, aggregating decision information into interval-valued intuitionistic fuzzy numbers (IVIFNs), a new consistency index of an Atanassov intuitionistic fuzzy preference relation (AIFPR), a new iterative algorithm to repair the consistency of an interval-valued Atanassov intuitionistic fuzzy preference relation (IVAIFPR) with unacceptable consistency, a linear program to derive interval-valued Atanassov intuitionistic fuzzy (IVAIF) priority weights of alternatives, a TOPSIS (technique for order performance by similarity to an ideal solution) based approach to rank such IVAIF priority weights, the likelihood of IVAIF values (IVAIFVs), the additive consistency of an IVAIFPR, the IVAIF priority weights of an IVAIFPR, the mean and variance of IVIF values (IVIFVs), a ranking method for IVIFVs considering the risk attitude of the expert, the group consensus, an iteration algorithm to improve the group consensus, a new multiplicative consistency of IVIFPR, and a new method for the group decision making (GDM) with IVIFPRs. At the same time, some real decision making problems are analyzed, such as supply chain collaboration, cooperative alliance and stock selection, cloud computing service evaluation, etc.

This book is organized into nine chapters that deal with nine different but related issues and are listed below:

Chapter 1 defines the possibility degree of comparison between two interval-valued intuitionistic fuzzy numbers (IVIFNs) by using the notion of 2-dimensional random vector, and a new method is then developed to rank IVIFNs. Hereby the ordered weighted average operator and hybrid weighted average operator for IVIFNs are defined based on the Karnik–Mendel algorithms and employed to solve multi-attribute group decision making problems with IVIFNs. The individual overall attribute values of alternatives are obtained by using the weighted average operator of IVIFNs. By using the hybrid weighted average operator for IVIFNs, we can obtain the collective overall attribute values of alternatives, which are used to rank the alternatives. A numerical example is examined to illustrate the effectiveness and flexibility of the proposed method in this chapter.

Chapter 2 develops a new method for solving multiple attribute group decision-making (MAGDM) problems with Atanassov's interval-valued intuitionistic fuzzy values (AIVIFVs) and incomplete attribute weight information. Firstly, we investigate the asymptotic property of the Atanassov's interval-valued intuitionistic fuzzy (AIVIF) matrix. It is demonstrated that after applying weights an infinite number of times, all elements in an AIVIF matrix will approach the same AIVIFV without regard to the initial values of elements. Then, the weight of each decision maker (DM) with respect to every attribute is determined by considering the similarity degree and proximity degree simultaneously. To avoid weighting an AIVIF matrix too many times, the collective decision matrix is transformed into an interval matrix using the risk coefficient of DMs. Subsequently, to derive the attribute weights objectively, we construct a multi-objective interval-programming model which is solved by transforming it into a linear program. The ranking order of alternatives is generated by the comprehensive interval values of alternatives.

Finally, an example of a research and development (R&D) project selection problem is provided to illustrate the implementation process and applicability of the method developed in this chapter.

Chapter 3 develops a novel interval-valued intuitionistic fuzzy (IVIF) mathematical programming method for hybrid multi-criteria group decision making (MCGDM) considering alternative comparisons with hesitancy degrees. The subjective preference relations between alternatives given by each decision maker (DM) are formulated as an IVIF set (IVIFS). The IVIFSs, intuitionistic fuzzy sets (IFSs), trapezoidal fuzzy numbers (TrFNs), linguistic variables, intervals, and real numbers are used to represent the multiple types of criteria values. The information of criteria weights is incomplete. The IVIFS-type consistency and inconsistency indices are defined through considering the fuzzy positive and negative ideal solutions simultaneously. To determine the criteria weights, we construct a novel bi-objective IVIF mathematical programming of minimizing the inconsistency index and meanwhile maximizing the consistency index, which is solved by the technically developed linear goal programming approach. The individual ranking order of alternatives furnished by each DM is subsequently obtained according to the comprehensive relative closeness degrees of alternatives to the fuzzy positive ideal solution. The collective ranking order of alternatives is derived through establishing a new multi-objective assignment model. A real example of critical infrastructure evaluation is provided to demonstrate the applicability and effectiveness of the proposed method.

Chapter 4 puts forward a new selection method based on MAGDM with interval-valued intuitionistic fuzzy sets and applies to cloud service selection problem. As the cloud computing develops rapidly, more and more cloud services appear. Many enterprises tend to utilize cloud service to achieve better flexibility and react faster to market demands. In the cloud service selection, several experts may be invited and many attributes (indicators or goals) should be considered. Therefore, the cloud service selection can be regarded as a kind of multi-attribute group decision making (MAGDM) problems. The aim of this paper is to develop a new method for solving such MAGDM problems. In this method, the ratings of the alternatives on attributes in individual decision matrix given by each expert are in the form of Interval-valued Intuitionistic Fuzzy Sets (IVIFSs) which can precisely describe the preferences of experts on qualitative attributes. First, the weights of experts with respect to each attribute are determined by extending the classical Gray Relational Analysis (GRA) into IVIF environment. Then, based on the collective decision matrix obtained by aggregating the individual matrices, the score (profit) matrix, accuracy matrix and uncertainty (risk) matrix are derived. A multi-objective programming model is constructed to determine the attribute weights. Finally, the alternatives are ranked by employing the overall scores and uncertainties of alternatives. The feasibility and effectiveness of the proposed methods are illustrated by a cloud service selection problem and the comparison analysis is constructed.

Chapter 5 aggregates decision information into interval-valued intuitionistic fuzzy numbers (IVIFNs) to solve heterogeneous MAGDM problem in which the decision information involves real numbers, interval numbers, triangular fuzzy numbers (TFNs) and trapezoidal fuzzy numbers (TrFNs). There are three issues being addressed in this chapter. The first is to propose a new general method to aggregate the attribute value vector into IVIFNs under heterogeneous MAGDM environment utilizing the relative closeness in technique for order preference by similarity to ideal solution (TOPSIS). The second is to construct a multiple objective intuitionistic fuzzy programming model to determine the attribute weights. Borrowing the results of the former two issues, the last is to present a new method to solve heterogeneous MAGDM problem. A comparison analysis with existing method is conducted to demonstrate the advantages of the proposed method. Two examples are provided to verify the practicality and effectiveness of the proposed method.

Chapter 6 investigates the group decision making (GDM) problems with interval-valued Atanassov intuitionistic fuzzy preference relations (IV-AIFPRs) and develops a novel method for solving such problems. A new consistency index of an Atanassov intuitionistic fuzzy preference relation (AIFPR) is introduced to judge the consistency of an AIFPR, and then a convergent iterative Algorithm I is designed to repair the consistency of an AIFPR with unacceptable consistency. Subsequently, the consistency and acceptable consistency of an IV-AIFPR are defined through separating an IV-AIFPR into two AIFPRs. Based on Algorithm I, a new iterative Algorithm II is devised to repair the consistency of an IV-AIFPR with unacceptable consistency. Afterward, to determine decision makers' (DMs') weights objectively, an optimization model is established by minimizing the deviations between each individual IV-AIFPR and the collective one. This model is skillfully transformed into a linear goal program to resolve sufficiently considering different principles of decision making. A linear programming model is built to derive interval-valued Atanassov intuitionistic fuzzy (IVAIF) priority weights of alternatives. Then, a TOPSIS (technique for order performance by similarity to an ideal solution) based approach is proposed to rank such IVAIF priority weights. Thereby, a method for GDM with IV-AIFPRs is put forward. At length, a practical example of a virtual enterprise partner selection is provided to illustrate the feasibility and validity of the proposed method.

Chapter 7 investigates a group decision making (GDM) method based on additive consistent interval-valued Atanassov intuitionistic fuzzy (IVAIF) preference relations (IVAIFPRs) and likelihood comparison algorithm. Firstly, the likelihood of IVAIF values (IVAIFVs) is defined by the likelihood of intervals. Then a likelihood comparison algorithm is designed to rank IVAIFVs. According to the additive consistent interval fuzzy preference relation, we define the additive consistency of an IVAIFPR. Two special interval fuzzy preference relations are extracted from an IVAIFPR. They can be regarded as the lowest and highest preferred matrices of the IVAIFPR, respectively. Using a parametric linear program, the IVAIF priority weights of an IVAIFPR are generated from these two extracted special interval fuzzy preference relations. For the GDM with IVAIFPRs, the group consensus is defined by the distances between the individual IVAIFPRs and the collective one. To derive decision makers' weights, an optimization model is constructed by maximizing the group consensus and transformed into a linear program to resolve. Subsequently, utilizing the IVAIF weighted averaging operator, the collective IVAIFPR is obtained and applied to derive the IVAIF priority weights. The order of alternatives is generated by ranking the IVAIF priority weights. At length, an enterprise resource planning system selection example is analyzed to verify the effectiveness of the proposed method.

Chapter 8 develops a new method for solving group decision making (GDM) problems with interval-valued intuitionistic fuzzy preference relations (IVIFPRs). First, an additive consistency of an IVIFPR is defined by the additive consistency of intuitionistic fuzzy preference relation (IFPR). Based on the additive consistency definition of IVIFPR, two linear programming models are established to extract the most optimistic and pessimistic consistent IFPRs from an IVIFPR, respectively. Especially, if the feasible regions of these two models are empty, two adjusted programming models are constructed. Afterward, a risk attitudinal-based consistent IFPR is determined considering decision maker's (DM's) risk attitude. To derive the intuitionistic fuzzy priority weights from the risk attitudinal-based consistent IFPR, a multi-objective programming model is established and transformed into a linear goal program to resolve. Subsequently, combining DMs' subjective and objective importance degrees, the comprehensive importance degrees of DMs are generated. Using comprehensive importance degrees as order inducing variables, a new comprehensive importance interval-valued intuitionistic fuzzy induced ordered weighted averaging (CI-IVIF-IOWA) operator is defined to aggregate the individual IVIFPRs into a collective one. Thereby, a three-phase method is proposed for GDM with IVIFPRs. An example of network system selection is examined to illustrate the practicability and effectiveness of the proposed method.

Chapter 9 investigates a group decision-making (GDM) method that considers group consensus and multiplicative consistency of interval-valued intuitionistic fuzzy (IVIF) preference relations (IVIFPRs). First, the mean and variance of IVIF values (IVIFVs) are defined and a ranking method for IVIFVs is proposed considering the risk attitude of the expert. Then, the group consensus is presented by the individual similarity between experts. An iteration algorithm is designed to improve the group consensus. A statistical comparative analysis validates this algorithm. Subsequently, a new multiplicative consistency of IVIFPR is defined based on the multiplicative consistency of interval fuzzy preference relation. Two single-objective programming models are established to extract the most optimistic and pessimistic interval priority weight vectors from an IVIFPR, respectively. In particular, if the feasible domains of these two models are empty, two adjusted programs are constructed to replace the originals. Combining the most optimistic and pessimistic interval priority weights, the IVIF priority weights are generated. Further, expert weights are derived from Markov model and used to derive the collective IVIFPR for generating the IVIF priority weights. Therefore, a new method is proposed to solve the GDM with IVIFPRs. Finally, two cases are analyzed to verify the effectiveness of the proposed method.

This book is suitable for practitioners and researchers working in the fields of fuzzy mathematics, operations research, information science and management science and engineering, etc. It can help practitioners making scientific and reasonable decision in practice or help researchers swiftly access into the intuitionistic fuzzy decision making field. It can also be used as a textbook for postgraduate and senior undergraduate students.

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# Contents

# Chapter 1
# A Possibility Degree Method for Interval-Valued Intuitionistic Fuzzy Multi-attribute Group Decision Making

**Abstract** An interval-valued intuitionistic fuzzy set (IVIFS) is the extension of an intuitionistic fuzzy set (IFS). The ranking of IVIFSs is very important for the interval-valued intuitionistic fuzzy decision making. From a viewpoint of probability, the possibility degree of comparison between two interval-valued intuitionistic fuzzy numbers (IVIFNs) is defined by using the notion of 2-dimensional random vector, and a new method is then developed to rank IVIFNs. Hereby the ordered weighted average operator and hybrid weighted average operator for IVIFNs are defined based on the Karnik-Mendel algorithms and employed to solve multi-attribute group decision making problems with IVIFNs. In the proposed decision method, the individual overall attribute values of alternatives are obtained by using the weighted average operator of IVIFNs. The collective overall attribute values of alternatives are integrated by using the hybrid weighted average operator of IVIFNs. The ranking orders of alternatives are generated according to the collective overall attribute values. A numerical example is examined to illustrate the proposed method and the comparison analysis demonstrates the effectiveness and flexibility of the proposed method in this chapter.

**Keywords** Multi-attribute group decision making · Interval-valued intuitionistic fuzzy set · Aggregation operator · Karnik-Mendel algorithm

## 1.1 Introduction

Fuzzy set (FS) theory has long been introduced to handle inexact and imprecise data by Zadeh [56]. The drawback of using the single membership value in FS theory is that the evidence for $x \in X$ and the evidence against $x \in X$ are in fact mixed together (Here $X$ is the universe of discourse). In order to tackle this problem, Atanassov [1] proposed the intuitionistic fuzzy set (IFS) using two characteristic functions expressing the degree of membership and the degree of non-membership of elements of the universal set to the IFS. It can cope with the presence of vagueness and hesitancy originating from imprecise knowledge or information.



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IFS has been widely applied to the multi-attribute decision making (MADM) [3, 5, 6, 9–13, 19, 22, 23, 25–30, 32, 33, 36, 47, 50–52, 54, 57] and multi-attribute group decision making (MAGDM) [17, 18, 20, 27, 35, 37, 39, 41]. These achievements can be roughly classified into four types: aggregation operators, similarity (or distance) measures, extension of classic decision making methods and judgment matrix, which are respectively reviewed as follows.

In the aspect of aggregation operators, Li [11, 12] and Li et al. [18] proposed the generalized OWA operators with IFSs. Liu and Wang [23] proposed the intuitionistic fuzzy point operators. Xu [39] developed the intuitionistic fuzzy power aggregation operators. Xu and Yager [48] developed some geometric aggregation operators based on IFS, such as the intuitionistic fuzzy weighted geometric operator, the intuitionistic fuzzy ordered weighted geometric operator, and the intuitionistic fuzzy hybrid geometric (IFHG) operator, and applied them to MADM. Xu [35] used the IFHG operator to propose a MAGDM method with incomplete weight information under intuitionistic fuzzy environment. Yang and Chen [52] defined the quasi-arithmetic intuitionistic fuzzy OWA operators. Zeng and Su [57] proposed the intuitionistic fuzzy ordered weighted distance operator. Wei [27] proposed the induced intuitionistic fuzzy ordered weighted geometric (I-IFOWG) operator and applied to MAGDM with intuitionistic fuzzy numbers (IFNs [48]). Zhao et al. [58] developed some new generalized aggregation operators, such as generalized intuitionistic fuzzy weighted averaging operator, generalized intuitionistic fuzzy ordered weighted averaging operator, generalized intuitionistic fuzzy hybrid averaging operator, and applied to MADM with intuitionistic fuzzy information. These aggregation operators for IFNs may be viewed as the extension of the ones for the real numbers, which mainly involve the arithmetic aggregation operators, geometric aggregation operators, power aggregation operators, generalized average operators and induced aggregation operators.

In the aspect of similarity (or distance) measures, Li [9] discussed some measures of dissimilarity in intuitionistic fuzzy structures. Xu [32] defined the normalized Hamming and distance between two IFNs and proposed the intuitionistic fuzzy MADM method. Xu [33] defined similarity measures of IFSs based on the geometric distance model, the set-theoretic approach and the matching function, respectively, and applied the similarity measures to the MADM under intuitionistic fuzzy environment. Xu and Yager [49] also defined some new similarity measures of IFSs based on the distance measures. These similarity (or distance) measures of IFSs also can be applied to pattern recognitions and approximate reasoning.

In the aspect of extension of classic decision making methods, Li [13] and Li et al. [17] extended the classic LINMAP method to the intuitionistic fuzzy environments. Wu and Chen [28] proposed the ELECTRE multicriteria analysis approach based on IFSs. Xu and Hu [47] constructed the projection models for intuitionistic fuzzy MADM. These decision making methods under intuitionistic fuzzy environment generalize the classic decision making methods, such as TOPSIS, ELECTRE and LINMAP.

In the aspect of judgment matrix, Xu [36] defined some concepts, such as consistent intuitionistic preference relation, incomplete intuitionistic preference relation and acceptable intuitionistic preference relation, etc. He also developed an approach to group decision making based on intuitionistic preference relations and an approach to group decision making based on incomplete intuitionistic preference relations, respectively, in which the intuitionistic fuzzy arithmetic averaging operator and intuitionistic fuzzy weighted arithmetic averaging operator are used to aggregate intuitionistic preference information, and the score function and accuracy function are applied to the ranking and selection of alternatives. Xu and Yager [49] investigated some intuitionistic fuzzy preference relations and their measures of similarity for the evaluation of agreement within a group. The intuitionistic fuzzy preference relations enrich the research content of intuitionistic fuzzy decision making theory.

Atanassov and Gargov [2] further generalized the IFS in the spirit of the ordinary interval-valued fuzzy set (IVFS) and defined the concept of an interval-valued intuitionistic fuzzy set (IVIFS), which enhances greatly the representation ability of uncertainty than IFS. Recently, IVIFSs were also used in the fields of MADM [4, 14–16, 26, 32, 42, 53] and MAGDM [27, 31, 34, 38, 40, 43–45, 49, 58]. Similar to the IFS, these achievements on IVIFSs mainly focused on the aggregation operators, similarity (or distance) measures, extension of classic decision making methods and judgment matrix, which are respectively reviewed as follows.

In the aspect of aggregation operators, Xu [31] defined some operational laws of interval-valued intuitionistic fuzzy numbers (IVIFNs) and then proposed the interval-valued intuitionistic fuzzy weighted arithmetic aggregation operator and interval-valued intuitionistic fuzzy weighted geometric aggregation operator. Xu [38] used the Choquet integral to propose the interval-valued intuitionistic fuzzy correlated averaging operator and the interval-valued intuitionistic fuzzy correlated geometric operator to aggregate interval-valued intuitionistic fuzzy information. Xu and Chen [44] developed some geometric aggregation operators for IVIFNs. Wei [27] proposed the induced interval-valued intuitionistic fuzzy ordered weighted geometric (I-IIFOWG) operator and applied to MAGDM with IVIFNs. Xu and Chen [45] proposed the interval-valued intuitionistic fuzzy Bonferroni means. Zhao et al. [58] developed generalized interval-valued intuitionistic fuzzy weighted averaging operator, generalized interval-valued intuitionistic fuzzy ordered weighted averaging operator, generalized interval-valued intuitionistic fuzzy hybrid average operator, and applied to MADM with interval-valued intuitionistic fuzzy information. These aggregation operators for IVIFNs may be roughly divided into two kinds. One is the aggregation operators with independent attributes, including arithmetic aggregation operators and geometric aggregation operators, the other is the aggregation operators with dependent attributes, such as the interval-valued intuitionistic fuzzy correlated averaging operator and the interval-valued intuitionistic fuzzy correlated geometric operator [38].

In the aspect of similarity (or distance) measures, Xu [32] defined the normalized Hamming distance between two IVIFNs and proposed the interval-valued fuzzy MADM method. Xu [34] defined the Hamming and Euclidean distances, the Hamming and Euclidean distances based on the Hausdorff metric for IVIFSs. The corresponding similarity measures for IVIFSs are also defined and applied to pattern recognitions. The similarity measures for IVIFSs also can be applied to MADM with interval-valued fuzzy assessment information.

In the aspect of extension of classic decision making methods, Li [14] developed the closeness coefficient based nonlinear programming method for interval-valued intuitionistic fuzzy MADM with incomplete preference information. Li [15] proposed the TOPSIS-based nonlinear-programming methodology for MADM with IVIFSs. Li [16] proposed the linear programming method for MADM with IVIFSs. These decision methods under interval-valued intuitionistic fuzzy environment also generalize the classic decision making methods, such as TOPSIS and LINMAP. The other methods, such as ELECTRE and PROMETHEE, are expected to be applied to MADM and MAGDM with IVIFSs.

In the aspect of judgment matrix, Xu and Cai [42] investigated the incomplete interval-valued intuitionistic fuzzy preference relations. Xu and Chen [43] developed the ordered weighted aggregation operator and hybrid aggregation operator for aggregating interval-valued intuitionistic preference information. Interval-valued intuitionistic judgment matrix and its score matrix and accuracy matrix were defined. Some of their desirable properties were investigated in detail. The relationships among interval-valued intuitionistic judgment matrix, intuitionistic judgment matrix and complement judgment matrix, were discussed. Xu and Yager [49] investigated some interval-valued intuitionistic fuzzy preference relations and their measures of similarity for the evaluation of agreement within a group.

In most the existing aggregation operators for IFSs and IVIFSs, the weighted vector of the aggregated arguments usually takes the form of real numbers. However, with ever increasing complexity in many real decision situations, there are often some challenges for the decision maker (DM) to provide precise weight information due to time pressure, lack of knowledge (or data) and the DM's limited expertise about the problem domain. In other words, usually weights are not necessary the real numbers and may be the intervals, IFNs or IVIFNs. Recently, Chen et al. [4] proposed the intuitionistic fuzzy weighted average operator and interval-valued intuitionistic fuzzy weighted average operator based on the traditional weighted average method and the Karnik-Mendel algorithms [8]. Although they sufficiently considered the different forms of the weighted vector, such as real numbers, intervals, IFNs and IVIFNs, the proposed intuitionistic fuzzy weighted average operator and interval-valued intuitionistic fuzzy weighted average operator can only weight the arguments themselves rather than the ordered positions of the arguments. As we all know, both the arguments themselves and the ordered positions of the arguments are very important in the process of information aggregation. To overcome this drawback, this chapter first proposes a new ranking method of IVIFNs based on the possibility degree from the probability viewpoint and then develops the ordered weighted average operator and hybrid weighted average operator for IVIFNs. Finally, combining the interval-valued intuitionistic fuzzy weighted average operator and hybrid weighted average operator, a new MAGDM method with IVIFSs is proposed.

The rest of the paper is arranged as follows. Section 1.2 develops a new ranking method of IVIFNs based on the possibility degree. Section 1.3 proposes the ordered weighted average operator and hybrid weighted average operator for IVIFNs. Section 1.4 constructs the MAGDM model with IVIFSs and proposes the corresponding decision making method. An air-condition system selection example is examined to illustrate the proposed method in Sect. 1.5. The comparison analysis with other method is also conducted in Sect. 1.5. Concluding remark is made in Sect. 1.6.

## 1.2 A New Ranking Method of IVIFS from the Probability Viewpoint

Atanassov and Gargov [2] gave the definition of IVIFS as follows:

**Definition 1.1** [2] An IVIFS $A$ in the universe of discourse $X$ is defined as

$$A = \{ <x, \mu_A(x), v_A(x) > | x \in X\},$$

where $\mu_A(x) \subseteq [0, 1]$ and $v_A(x) \subseteq [0, 1]$ denote respectively the membership degree interval and the non-membership degree interval of $x$ to $A$, with the condition:

$$\sup \mu_A(x) + \sup v_A(x) \leq 1, \forall x \in X.$$

Since IVIFS is composed of two ordered interval pairs, Xu [31, 32] called them interval-valued intuitionistic fuzzy numbers (IVIFNs) and simply denoted by $G = ([a, b], [c, d])$, where $[a, b] \subseteq [0, 1]$, $[c, d] \subseteq [0, 1]$ and $a + b \leq 1$.

### *1.2.1 New Ranking Method for Intervals from the Probability Viewpoint*

Since the IVIFN consists of two intervals, we first give the ranking method for two intervals.

Let $x = [a, b] \subseteq [0, 1]$ and $y = [c, d] \subseteq [0, 1]$ be two intervals. Let $X$ be a random variable with uniform probability distribution, which is defined on the closed interval $[a, b]$. Since the arbitrary element $z \in [a, b]$ has the equal possibility for the interval $x = [a, b]$, the arbitrary basic random event has equal occurrence probability for the random variable $X$. Thus, we can view the intervals $x$ and $y$ as the random variables with uniform distribution defined on the corresponding intervals. The occurrence possibility for the fuzzy event $\{x \geq y\}$ can be transformed into the occurrence probability for the random event $\{x \geq y\}$ . Maybe there is a constant deviation or constant multiplier between the occurrence possibility and probability. The main reason and explanation come from [55].

Yoon [55] proposed two kinds of methods for the transformation between the membership function and probability density function. One is the proportional probability distribution as follows:

$$f_1(x) = c_1 \mu_A(x)$$

where $f_1(x)$ is the occurrence probability for the random event $A$, $\mu_A(x)$ is the value of membership function, $c_1$ is a proportional constant satisfying the condition that the area under the continuous probability function is equal to one.

The other is the uniform probability distribution as follows:

$$f_2(x) = \mu_A(x) + c_2$$

where $c_2$ is a uniform constant satisfying the probability function requirement.

The probability is based on probability distribution while the possibility is based on possibility distribution (membership function). There only exists a difference of constant multiplier $c_1$ or constant $c_2$ for the transformation between the membership function and probability density function. The difference of constant multiplier $c_1$ or constant $c_2$ does not influence the ranking order for the intervals. Therefore, we can use the occurrence probability for the random event $\{x \geq y\}$ to define the possibility of $\{x \geq y\}$ .

Without loss of generality and for simplicity, the random variable $x$ is supposed to be independent of $y$. Then, the jointly density function for the 2-dimensional random vector $(x, y)$ is

$$f(x,y) = \begin{cases} \frac{1}{(b-a)(d-c)}, & (x,y) \in [a,b] \times [c,d], \\ 0, & \text{else.} \end{cases}$$

The marginal density functions for $x$ and $y$ are as follows:

$$f_x(x) = \begin{cases} \frac{1}{(b-a)}, & x \in [a,b], \\ 0, & \text{else} \end{cases}$$

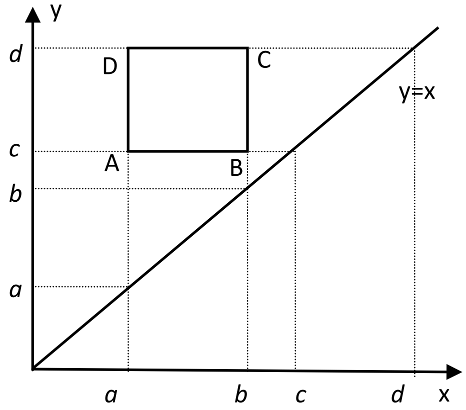
and

$$f_y(y) = \begin{cases} \frac{1}{(d-c)}, & y \in [c,d], \\ 0, & \text{else,} \end{cases}$$

respectively.

Because the location relations between $x = [a, b]$ and $y = [c, d]$ include the following six cases, we can calculate the occurrence probability for the fuzzy (or random) event $\{x \geq y\}$ denoted by $P(x \geq y)$ under different cases.

**Fig. 1.1** $a < b \le c < d$




(1) Case 1: $a < b \le c < d$ as Fig. 1.1 shown. The intersection between rectangle $[a, b] \times [c, d]$ and $x \ge y$ is null, so

$$P(x \ge y) = 0. \tag{1.1}$$

(2) Case 2: $a \le c < b < d$ or $a < c < b \le d$ as Fig. 1.2 shown. The intersection between rectangle $[a, b] \times [c, d]$ and $x \ge y$ is triangle ABC, so

$$P(x \ge y) = \int_c^b f_x(x) \left[ \int_c^x f_y(y) dy \right] dx = \frac{(b-c)^2}{2(b-a)(d-c)}. \tag{1.2}$$

(3) Case 3: $a \le c < d < b$ or $a < c < d \le b$ or $a \le c < d \le b$ as Fig. 1.3 shown. The intersection between rectangle $[a, b] \times [c, d]$ and $x \ge y$ is trapezoid ABCD, so

**Fig. 1.2** $a \le c < b < d$ or $a < c < b \le d$

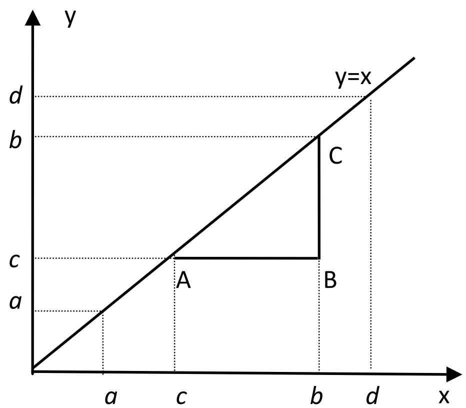
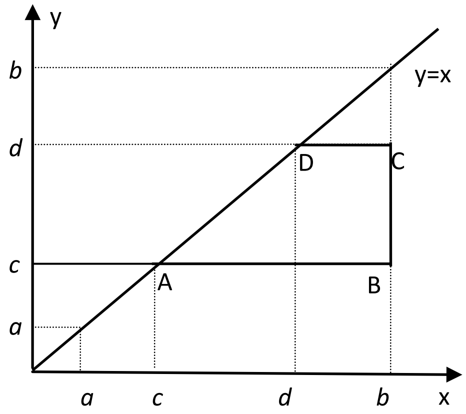


**Fig. 1.3** $a \le c < d < b$ or $a < c < d \le b$ or $a \le c < d \le b$



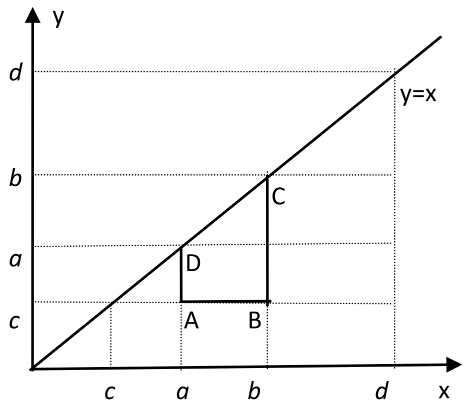

$$P(x \ge y) = \int_c^d f_y(y)\left[\int_y^b f_x(x)dx\right]dy = \frac{2b-d-c}{2(b-a)}. \tag{1.3}$$

(4) Case 4: $c \le a < b < d$ or $c < a < b \le d$ as Fig. 1.4 shown. The intersection between rectangle $[a,b] \times [c,d]$ and $x \ge y$ is trapezoid ABCD, so

$$P(x \ge y) = \int_a^b \left[\int_c^x f_y(y)dy\right]f_x(x)dx = \frac{b+a-2c}{2(d-c)}. \tag{1.4}$$

(5) Case 5: $c \le a < d < b$ or $c < a < d \le b$ as Fig. 1.5 shown. The intersection between rectangle $[a,b] \times [c,d]$ and $x \ge y$ is pentagon ABCDE, so

**Fig. 1.4** $c \le a < b < d$ or $c < a < b \le d$



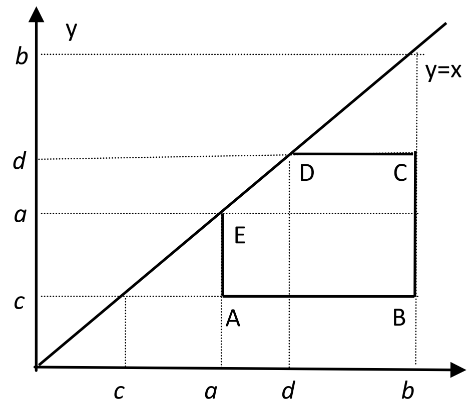



**Fig. 1.5** $c \leq a < d < b$ or $c < a < d \leq b$

$$P(x \geq y) = \int_c^a f_y(y) \int_a^b f_x(x)dxdy + \int_a^d f_y(y) \int_y^b f_x(x)dxdy$$
$$= \frac{2bd + 2ac - 2bc - a^2 - d^2}{2(b-a)(d-c)}. \tag{1.5}$$

(6) Case 6: $c < d \leq a < b$, as Fig. 1.6 shown. The intersection between rectangle $[a, b] \times [c, d]$ and $x \geq y$ is rectangle ABCD, so

$$P(x \geq y) = \int_c^d f_y(y) \left[ \int_a^b f_x(x)dx \right] dy = 1. \tag{1.6}$$

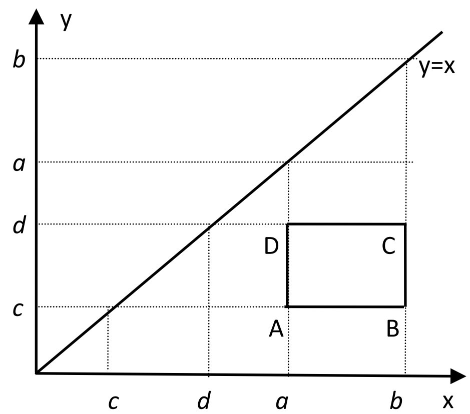



**Fig. 1.6** $c < d \leq a < b$

**Theorem 1.1** Let $x = [a, b]$ and $y = [c, d]$ be two intervals. Define the possibility degree of $x \geq y$ as $P(x \geq y)$. Then, the following properties hold:

(i) $0 \leq P(x \geq y) \leq 1$, $0 \leq P(y \geq x) \leq 1$;
(ii) $P(x \geq y) = 1$ if and only if $d \leq a$. Similarly, $P(y \geq x) = 1$ if and only if $b \leq c$;
(iii) $P(x \geq y) = 0$ if and only if $b \leq c$. Similarly, $P(y \geq x) = 0$ if and only if $d \leq a$;
(iv) $P(x \geq x) = 0.5$;
(v) $P(x \geq y) + P(y \geq x) = 1$;
(vi) $P(x \geq y) \geq 0.5$ if and only if $a + b \geq c + d$. Especially, $P(x \geq y) = 0.5$ if and only if $a + b = c + d$;
(vii) Let $z = [e, f]$ be an interval. If $P(x \geq y) \geq 0.5$ and $P(y \geq z) \geq 0.5$, then $P(x \geq z) \geq 0.5$.

***Proof*** Obviously, the properties (i)–(v) are right. We prove (vi) in the following. If $P(x \geq y) \geq 0.5$, then the location relations between $x = [a, b]$ and $y = [c, d]$ must be Case 3 or Case 4. For Case 3, by Eq. (1.3), we have $P(x \geq y) = \frac{2b-d-c}{2(b-a)} \geq 0.5$, so $a + b \geq c + d$. For Case 4, by Eq. (1.4), we have $P(x \geq y) = \frac{b+a-2c}{2(d-c)} \geq 0.5$, so $a + b \geq c + d$.

If $a + b \geq c + d$, then the location relations between $x = [a, b]$ and $y = [c, d]$ must be Case 3 or Case 4. For Case 3, by Eq. (1.3), we have $P(x \geq y) = \frac{2b-d-c}{2(b-a)} \geq 0.5$. For Case 4, by Eq. (1.4), we have $P(x \geq y) = \frac{b+a-2c}{2(d-c)} \geq 0.5$.

Thus, the property (vi) holds. In a similar way, the property (vii) can be proved, which completes the proof of Theorem 1.1.

In order to rank intervals $\tilde{a}_i = [a_i, b_i] (i = 1, 2, \ldots, n)$, similar to Xu [29], we can construct the matrix of possibility degree as $P = (P_{ij})_{n \times n}$, where $P_{ij} = P(\tilde{a}_i \geq \tilde{a}_j)(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$. Then, the ranking vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is derived as follows:

$$\omega_i = \left( \sum_{j=1}^{n} P_{ij} + \frac{n}{2} - 1 \right) / (n(n-1)) (i = 1, 2, \ldots, n). \tag{1.7}$$

The larger the value of $\omega_i$, the bigger the corresponding interval $\tilde{a}_i = [a_i, b_i]$.

Xu and Da [46] defined the possibility degree of $x \geq y$ as follows (see Definition 2.3 of [46]):

$$\mathrm{P}(x \geq y) = \max\left\{ 1 - \max\left( \frac{d-a}{b-a+d-c}, 0 \right), 0 \right\}.$$

Then, they discussed the properties about the possibility degree and formulated these properties as Theorem 2.1.

Theorem 2.1 of [46] contains seven properties. It is easy to see that Theorem 1.1 of this chapter is similar to Theorem 2.1 of [46]. In other words, the possibility degrees proposed in this chapter and [46] have the identical properties. However, the former is based on the 2-dimensional random vector. The possibility degree $\mathrm{P}(x \geq y)$ is explained as the occurrence probability for the random event $\{x \geq y\}$, which has the intuitive meaning from the probability viewpoint, whereas the latter did not give the corresponding explanation.

**Example 1.1** [46] Let $\tilde{a}_1 = [3,5]$, $\tilde{a}_2 = [4,6]$, $\tilde{a}_3 = [4,7]$ and $\tilde{a}_4 = [3,6]$ be four intervals. Using the ranking method proposed in this chapter, the matrix of possibility degree is obtained as follows:

$$P = \begin{pmatrix} 0.5 & \frac{1}{8} & \frac{1}{12} & \frac{1}{3} \\ \frac{7}{8} & 0.5 & \frac{1}{3} & \frac{2}{3} \\ \frac{11}{12} & \frac{2}{3} & 0.5 & \frac{7}{9} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{9} & 0.5 \end{pmatrix}.$$

By Eq. (1.7), the ranking vector is calculated as follows:

$$\omega_1 = 0.1701,\ \omega_2 = 0.2813,\ \omega_3 = 0.3218,\ \omega_4 = 0.2269$$

Hence, the ranking order is $\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_4 \succ \tilde{a}_1$. The ranking order given by [46] is identical, i.e., $\tilde{a}_3 \succ \tilde{a}_2 \succ \tilde{a}_4 \succ \tilde{a}_1$. This example shows the effectiveness of the ranking method proposed in this chapter.

### 1.2.2 New Ranking Method for IVIFNs Based on the Possibility Degree

**Definition 1.2** Let $G_1 = ([a_1, b_1], [c_1, d_1])$ and $G_2 = ([a_2, b_2], [c_2, d_2])$ be two IVIFNs. The possibility degree of $G_1 \geq G_2$ can be defined as

$$P(G_1 \geq G_2) = \frac{1}{2}\{P([a_1, b_1] \geq [a_2, b_2]) + P([c_2, d_2] \geq [c_1, d_1])\}. \tag{1.8}$$

It is easy to verify that the above definition satisfies the following properties:

(i) $0 \leq P(G_1 \geq G_2) \leq 1$;
(ii) $P(G_1 \geq G_1) = 0.5$;
(iii) $P(G_1 \geq G_2) + P(G_2 \geq G_1) = 1$.

In order to rank IVIFNs $G_i = ([a_i, b_i], [c_i, d_i])(i = 1, 2, \ldots, n)$, similar to Xu [29], we can construct the matrix of possibility degree as $P = (P_{ij})_{n \times n}$, where $P_{ij} = P(G_i \geq G_j)(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$. Then, the ranking vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ is derived as follows:

$$\omega_i = \left( \sum_{j=1}^{n} P_{ij} + \frac{n}{2} - 1 \right) / (n(n-1))(i = 1, 2, \ldots, n). \tag{1.9}$$

Thus, the IVIFNs $G_i (i = 1, 2, \ldots, n)$ can be ranked in descending order in accordance with the values of $\omega_i (i = 1, 2, \ldots, n)$.

**Example 1.2** Let $G_1 = ([0.1, 0.2], [0.3, 0.5])$, $G_2 = ([0.4, 0.6], [0.2, 0.4])$, $G_3 = ([0.3, 0.5], [0.1, 0.2])$ and $G_4 = ([0.6, 0.8], [0.0, 0.1])$ be four IVIFNs. Using the ranking method proposed in this chapter, the matrix of possibility degree is obtained as follows:

$$P = \begin{pmatrix} 0.5 & \frac{1}{16} & 0 & 0 \\ \frac{15}{16} & 0.5 & \frac{7}{16} & 0 \\ 1 & \frac{9}{16} & 0.5 & 0 \\ 1 & 1 & 1 & 0.5 \end{pmatrix}. \tag{1.10}$$

By Eq. (1.9), the ranking vector is calculated as follows:

$$\omega_1 = 0.1302, \ \omega_2 = 0.2396, \ \omega_3 = 0.2552, \ \omega_4 = 0.3750.$$

Hence, the ranking order is $G_4 > G_3 > G_2 > G_1$ .

### 1.2.3 Comparative Analysis with Score and Accuracy Functions for IVIFNs

Xu [31] and Xu and Chen [43] defined the score function and accuracy function of an IVIFN $G = ([a, b], [c, d])$ as follows:

$$S(G) = \frac{1}{2}(a + b - c - d), \tag{1.11}$$

$$H(G) = \frac{1}{2}(a + b + c + d). \tag{1.12}$$

Let $G_1 = ([a_1, b_1], [c_1, d_1])$ and $G_2 = ([a_2, b_2], [c_2, d_2])$ be two IVIFNs. Xu [31] and Xu and Chen [43] gave the ranking method as follows:

(1) If $S(G_1) < S(G_2)$, then $G_1 < G_2$;
(2) If $S(G_1) = S(G_2)$, then

if $H(G_1) < H(G_2)$, then $G_1 < G_2$;
if $H(G_1) = H(G_2)$, then $G_1 = G_2$.

Applying the above ranking method to Example 2, the score functions for the four IVIFNs are respectively computed by Eq. (1.11) as follows:

$$S(G_1) = -0.25,\ S(G_2) = 0.20,\ S(G_3) = 0.25,\ S(G_4) = 0.65.$$

Therefore, the ranking order obtained by [31, 43] is $G_4 > G_3 > G_2 > G_1$, which coincides with that obtained by this chapter.

Nevertheless, it should be pointed out that the ranking method proposed in this chapter has an outstanding advantage, namely, the possibility degrees between any adjacent IVIFNs for the ranking order can also be obtained by the matrix of possibility degree (i.e., Eq. (1.10)) as follows:

$$G_4 >_1 G_3 >_{9/16} G_2 >_{15/16} G_1.$$

That is to say, the possibility degree of $G_4 > G_3$ is 1, the possibility degree of $G_3 > G_2$ is $\frac{9}{16}$ and the possibility degree of $G_2 > G_1$ is $\frac{15}{16}$, which is very useful for the DM to make decision reasonably in some real decision making problems. However, this advantage cannot be reflected in the ranking method of [31, 43].

## 1.3 Ordered Weighted Average Operator and Hybrid Weighted Average Operator for IVIFNs

Karnik-Mendel algorithms are iterative procedures and converge monotonically and super-exponentially fast [8, 21, 24]. They have successfully been used to deal with the fuzzy weighted average and the linguistic weighted average [4]. Chen et al. [4] proposed the intuitionistic fuzzy weighted average operator and interval-valued intuitionistic fuzzy weighted average operator based on the traditional weighted average method and the Karnik-Mendel algorithms [8]. As stated earlier, these two operators did not consider the ordered positions of the arguments. Consequently, based on Karnik-Mendel algorithms, this section develops the ordered weighted average operator and hybrid weighted average operator for IVIFNs to overcome this drawback.

### *1.3.1 Weighted Average Operator for IVIFNs*

**Definition 1.3** (See [4]) Let $G_i (i = 1, 2, \ldots, n)$ be a collection of the IVIFNs, where $G_i = ([a_i, b_i], [c_i, d_i]) = [[a_i, b_i], [1 - d_i, 1 - c_i]]$. If