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Leavitt Path Algebras and Classical *K*-Theory

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Preface

This volume is an outcome of the International Workshop on Leavitt Path Algebras and K-Theory held at the Department of Mathematics, Cochin University of Science and Technology, Kerala, during July 1–3, 2017. This workshop intended to give an introduction of the newly developing subject Leavitt path algebras (LPAs for short) and the classical K-theory. It consists of articles on several aspects of Leavitt path algebras, on the one hand, and related K-theory, on the other. The articles on LPAs are mostly of an expository nature. A number of articles dealing with K-theory give new proofs of old results and are accessible to and of interest to students and beginners.

The subject of Leavitt path algebras was born about sixty years ago out of a construction by William Leavitt to showcase counterexamples to the invariant basis number problem. Leavitt path algebras were then introduced about fifteen years ago, associating certain algebras to directed graphs. The algebra associated with one vertex and n loops retrieves Leavitt's algebra. The initial impetus came from the theory of C^*-algebras which was already well developed and analogues came to be discovered in the theory of LPAs. A recent book by G. Abrams, P. Ara, and M. Siles Molina titled *Leavitt Path Algebras* has appeared in Springer's *Lecture Notes in Mathematics* series in 2017. It details several aspects of the main thrusts in this subject until 2015, but there has been a mushrooming of questions and ideas in the last five years. At least three conferences have been held recently, and it has been mentioned by several interested mathematicians that it ought to be very useful to have the proceedings of this CUSAT workshop published. In order to introduce the vast possibilities of this subject to graduate students and mathematicians working in somewhat allied areas, we have included surveys on topics that have not been covered in the above text. Development of K-theory of LPAs with initial impetus from that of graph C^*-algebras has also begun. The volume also contains articles on K-theoretic aspects apart from LPAs.

K. M. Rangaswamy has substantially contributed to this subject for several decades. In his survey, he concentrates on various algebraic aspects and describes some of his very recent results. Especially, since the theory of modules—questions on Morita equivalence, etc.—over these algebras is still in its infancy, the results obtained by Rangaswamy and others are described in clear detail by him. As a sample, we state one result in his article—every one-sided ideal over a Leavitt path algebra $L_K(E)$ is graded, if and only if every simple module over it is graded, and these happen only when $L_K(E)$ is a von Neumann regular ring.

In a lucid, self-contained exposition, Simon Rigby elucidates the groupoid approach to the LPAs. It was noticed a few years back that a new approach using topological groupoids can assist in the study of hitherto difficult questions on LPAs. The key fact here is that the LPA of a graph is graded isomorphic to the Steinberg algebra of the boundary path groupoid. This survey is expected to be useful to all levels of interested mathematicians. The author proves some results in more generality than have appeared in publications so far. One such instance is the uniqueness theorems for LPAs.

Very recently, étale groupoids have shown up in the forefront of several areas of mathematics. Important algebras such as the Cuntz algebra are known to arise as the convolution algebras arising from étale groupoids. The realization that invariants long studied in topological dynamics can be modeled on étale groupoids permits an interaction between analysis and algebra. Lisa Orloff Clark and Roozbeh Hazrat describe in complete detail how the LPAs allow us to treat all these algebras systematically and uniformly.

For a certain finite graph E and for the corresponding finite-dimensional algebra A with a square of the radical equal to zero, Huanhuan Li had constructed a compact generator of the homotopy category of acyclic complexes of injective modules over A—the so-called injective Leavitt complex of E. She gives an overview of the connection between the injective or projective Leavitt complex and the Leavitt path algebra of E.

Müge Kanuni and Suat Sert give an overview of results on the ideal theory of LPAs. In recent times, there has been a large body of work on graded, non-graded, prime, primitive, and maximal ideals of LPAs. Their survey is at an introductory level and narrates the correspondence between the lattice of ideals and the lattice of hereditary and saturated subsets of the graph over which the LPA is constructed.

In their article, Fatemeh Bagherzadeh and Murray Bremner recount the connection with the theory of operads. Gröbner bases for operads had been introduced by Dotsenko, Vallette, and others. The authors consider certain nonsymmetric operads for which they construct Gröbner bases and thereby compute their dimension formulae.

About 50 years ago, Stewart Priddy introduced Koszul algebras and Koszul duality partly in order to construct examples of algebras for which the Peter May spectral sequence is easy to compute and stops early. Steenrod algebra is a classic example. Koszul duality has been generalized to the operad setting also. In an expository article, Neeraj Kumar traces the notions and results on Koszul algebras developed in the last decade or so. These involve connections with combinatorics, geometry of monomial curves, Stanley–Reisner ring, Polya frequency sequence, etc. He also recalls older results and gives modern proofs for some of them like the theorem of Tate on the Poincaré–Betti series of quadratic complete intersection ring.

There are open questions mentioned which will be helpful to young researchers entering this area.

The spectacular work of Quillen and Suslin in solving Serre's conjecture on projective modules led to an explosion of sorts with new approaches designed to study more general problems of a similar nature. The technique of completion of unimodular rows developed by Suslin and Vaserstein led to the theorem on the normality of the elementary subgroup in $SL(n)$ for $n > 2$ over any commutative ring. Led by many experts, including Bak and Bass, more general notions of classical-like groups were defined and analogous questions were posed.

An article written by Ravi A. Rao and Ram Shila establishes an elementary symplectic analogue of Karoubi's linearization process of a polynomial matrix. They prove that one can stably linearize an alternating polynomial matrix by conjugating it with an elementary symplectic matrix.

Bhatoa Joginder Singh and Selby Jose study the action of SL_n on alternating matrices over a commutative ring A and prove (an analogue of the isomorphism of A_3 and D_3 over fields) that there is an injection from $SL_4(A)/E_4(A)$ to $SO_6(A)/EO_6(A)$.

Leonid Vaserstein had shown for two-dimensional rings that the unimodular rows of length three up to elementary transformations have the structure of a Witt group. The Vaserstein symbol mapping these classes of unimodular rows of length three to the elementary symplectic group has been studied in recent times. The non-injectivity of this symbol map for the coordinate ring of the 3-sphere has produced intense interest in the question of injectivity for more general rings. Neena Gupta and Dhvanita Rao had even produced an uncountable family of rings of dimension three over $\mathbb{R}$ for which the symbol is not injective. Neena Gupta, Dhvanita Rao, and Sagar Kolte survey these results as well as related works of Ravi A. Rao, van der Kallen, Richard Swan, and Jean Fasel.

In another paper, Ravi A. Rao and Selby Jose provide two possible approaches to the famous Bass–Suslin conjecture on the completability of unimodular polynomial rows over local rings.

Rabeya Basu, Reema Khanna, and Ravi A. Rao show for a commutative ring that the normality of the relative elementary subgroup is equivalent to the relative Quillen–Suslin local–global principle. They also obtain a relative local–global principle for the transvection subgroups. They use the concept of a Noetherian excision ring.

Reema Khanna, Selby Jose, Sampat Sharma, and Ravi A. Rao study the so-called special unimodular vector group and its elementary unimodular vector subgroup. They use certain ideas of Anthony Bak to deduce that the quotient vector group is nilpotent of class at most the dimension of the ring.

Raja Sridharan and Sunil Yadav reprove some classical theorems in a novel manner. They use the theory of Euler classes to deduce Seshadri's old theorem of the freeness of finitely generated projective modules over $k[X,Y]$. In another article, they deduce Suslin's $n!$ theorem on unimodular rows using Quillen's splitting principle. Along with Sumit Kumar Upadhyay in an article, they describe an algebraic analogue of the Mayer–Vietoris sequence for the part of the sequence that corresponds to the zeroth and the first cohomology.

In yet another article demonstrating the utility of Euler classes in algebra, Anjan Gupta, Raja Sridharan, and Sunil Yadav use it to give a group structure on the equivalence classes of unimodular rows of length three over a two-dimensional ring.

Finally, in an article dealing with Quillen–Suslin's foundational principles, Ravi A. Rao and Sunil Yadav demonstrate that monic inversion is equivalent to the local–global principle as well as to the normality of the elementary subgroup.

Bangalore, India
June 2019

B. Sury
A. A. Ambily

Acknowledgements

This volume is an outcome of the International Workshop on Leavitt Path Algebras and K-Theory held at the Department of Mathematics, Cochin University of Science and Technology, Kerala, during July 1–3, 2017. This workshop intended to give an introduction of the newly developing subject Leavitt path algebras and the subject classical K-theory.The administrative and logistic support for the workshop was given by the Cochin University of Science and Technology. The organizers gratefully acknowledge this support.

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Cochin, India — A. A. Ambily
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Part I
Leavitt Path Algebras

The theory of Leavitt path algebras has created an astonishingly large amount of recent activities in ring theory. Besides a beautiful subject in its own right, it is closely related to several other areas in mathematics, which might explain the burst of activity in the subject. The first part of this volume exclusively deals with Leavitt path algebras and the related areas.

Chapter 1
A Survey of Some of the Recent Developments in Leavitt Path Algebras

Kulumani M. Rangaswamy

1.1 Introduction

Leavitt path algebras are algebraic analogues of graph C^*-algebras and, ever since they were introduced in 2004, have become an active area of research. Many of the initial developments during the 2004–2014 period have been nicely described in the recent book [2] and in the excellent survey article [1]. Our goal in this article is to report on some of the recent developments in the investigation of the algebraic aspects of Leavitt path algebras not included in [1, 2]. Because the Leavitt path algebras grew as algebraic analogues of graph C^*-algebras, their initial investigation involved mostly the ideas and techniques used in the study of graph C^*-algebras such as the graph properties of Conditions (K) and (L), and the ring properties of being simple, purely infinite simple, prime/primitive, etc. An important starting goal in this initial study was to work out the algebraic analogue of the deep and powerful Kirchberg Phillips theorem to classify purely infinite simple Leavitt path algebras $L := L_K(E)$ up to isomorphism or up to Morita equivalence by means of the Grothendieck groups $K_0(L)$ and the sign of the determinant $\det(I - A_E)$ where A_E is the adjacent matrix of the graph E. After such initial progress, there has been an explosion of articles dealing with not only the various different aspects of Leavitt path algebras, but also many natural generalizations such as Leavitt path algebras over commutative rings, of separated graphs, of high-rank graphs, Steinberg algebras and groupoids etc. In the background of many of these investigations is the special feature that every Leavitt path algebra L is endowed with three mutually compatible structures: L is a K-algebra, L is a $\mathbb{Z}$-graded ring and L is a ring with involution *. Our focus in this survey is to describe a selection of recent graded and non-graded ring-theoretic and module-theoretic investigations of Leavitt path algebras. My apologies

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to authors whose work has not been included due to the constrained focus, limitation of time and length of the paper.

In the first part of this survey, we describe graphical conditions on E under which the corresponding Leavitt path algebra $L_K(E)$ belongs to well-known classes of rings. The interesting fact is that often a single graph property of E seems to imply multiple ring properties of $L_K(E)$ and these properties for general rings are usually independent of each other. The poster child of such a phenomenon is the graph property for a finite graph E that no cycle in E has an exit. In this case, $L_K(E)$ possesses at least nine completely different ring properties! (see Theorem 1.5). Because of such connections between E and $L_K(E)$, Leavitt path algebras can be effective tools in the construction of examples of rings with various desired properties. If we do not impose any graphical conditions on E and just look at $L_K(E)$ as a $\mathbb{Z}$-graded ring, a really interesting result by Hazrat [23] states that $L_K(E)$ is a graded von Neumann regular ring. Because of this, the graded one-sided and two-sided ideals of $L_K(E)$ possess many desirable properties.

The module theory over Leavitt path algebras is still at an infant stage. The second part of this survey gives an account of some of the recent advances in this theory. Naturally, the initial investigations focussed on the simplest of the modules, namely, the simple modules over $L_K(E)$. We begin with outlining a few methods of constructing graded and non-graded simple left/right $L_K(E)$-modules. A special type of simple modules, called Chen simple modules introduced by Chen [19], play an important role. This is followed by characterizing Leavitt path algebras over which all the simple modules possess some special properties, such as, when all the simple modules are flat, or injective, or finitely presented or graded etc. For example, very recently, Ambily, Hazrat and Li [10] have proved that every simple left/right $L_K(E)$-module is flat if and only if $L_K(E)$ is von Neumann regular, thus showing, in the case of Leavitt path algebras, an open question in ring theory has an affirmative answer. Likewise, it was shown in [5] that $Ext^1_{L_K(E)}(S, S) \neq 0$ for a Chen simple module S induced by a cycle. It can then be shown that if all the simple left $L_K(E)$-modules are injective, then $L_K(E)$ is von Neumann regular. The converse easily holds if E is a finite graph, since it that case $L_K(E)$ is semi-simple artinian. In contrast, if R is an arbitrary non-commutative ring, the injectivity of all simple left R-modules need not imply von Neumann regularity of R (see [20]). Our next result in this section describes Leavitt path algebras of finite graphs having only finitely many isomorphism classes of simple modules. Interestingly, this class of algebras turns out to be precisely the class of Leavitt path algebras of finite graphs having finite Gelfand–Kirillov dimension.

The last section deals with one-sided ideals of a Leavitt path algebra L. Four years ago it was shown in [36] that finitely generated two-sided ideals of L are principal ideals. Recently, Abrams, Mantese and Tonolo [6] generalized this by showing that every finitely generated one-sided ideal of L is a principal ideal. Such rings are called Bézout rings. Using a deep theorem of Bergman, Ara and Goodearl [12] showed that one-sided ideals of L are projective. From these two results, it follows that the sum and the intersection of principal one-sided ideals of L are again principal. Thus, the principal one-sided ideals of L form a sublattice of the lattice of all one-sided ideals of L. A well-known theorem, proved originally for graph C^*-algebras and later for Leavitt path algebras $L_K(E)$, states that every two-sided ideal of $L_K(E)$ is a graded ideal if and only if E satisfies Condition (K), equivalently $L_K(E)$ is a weakly regular ring. What happens when every one-sided ideal of $L_K(E)$ is graded? The last theorem of this section answers this question, namely, every one-sided ideal of $L_K(E)$ is graded if and only if every simple $L_K(E)$-module is graded if and only if $L_K(E)$ is a von Neumann regular ring (see [25]).

In summary, this survey is intended to showcase a small sample of some of the recent research on the algebraic aspects of Leavitt path algebras. Hopefully, this provides the reader with some insights into this theory and generates further interest in this exciting and growing field of algebra.

1.2 Preliminaries

For the general notation, terminology and results in Leavitt path algebras, we refer to [1, 2]. We give below an outline of some of the needed basic concepts and results.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. A vertex v is called a *sink* if it emits no edges and a vertex v is called a *regular vertex* if it emits a non-empty finite set of edges. An *infinite emitter* is a vertex which emits infinitely many edges. For each $e \in E^1$, we call e^* a *ghost* edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. A *path* μ of length $n > 0$ is a finite sequence of edges $\mu = e_1 e_2 \cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \ldots, n-1$. In this case $\mu^* = e_n^* \cdots e_2^* e_1^*$ is the corresponding ghost path. A vertex is considered a path of length 0.

A path $\mu = e_1 \cdots e_n$ in E is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be *based at the vertex* $s(e_1)$. A closed path μ as above is called *simple* provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, ..., n$. The closed path μ is called a *cycle* if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$. An *exit* for a path $\mu = e_1 \cdots e_n$ is an edge e such that $s(e) = s(e_i)$ for some i and $e \neq e_i$.

For any vertex v, the *tree* of v is $T_E(v) = \{w \in E^0 : v \geq w\}$. We say there is a *bifurcation* at a vertex v or v is a *bifurcation vertex*, if v emits more than one edge. In a graph E, a vertex v is called a *line point* if there is no bifurcation or a cycle based at any vertex in $T_E(v)$. Thus, if v is a line point, the vertices in $T_E(v)$ arrange themselves on a straight line path μ starting at v (μ could just be v) such as $\bullet_v \to \bullet \cdots \bullet \to \bullet \cdots$ which could be finite or infinite.

If p is an infinite path in E, say, $p = e_1 \cdots e_n e_{n+1} \ldots$, we follow Chen [19] to define, for each $n \geq 1$, $\tau^{\leq n}(p) = e_1 \cdots e_n$ and $\tau^{>n}(p) = e_{n+1} e_{n+2} \cdots$. Two infinite paths p, q are said to be *tail-equivalent* if there are positive integers m, n such that $\tau^{>m}(p) = \tau^{>n}(q)$. This defines an equivalence relation among the infinite paths in E and the equivalence class containing the path p is denoted by $[p]$. An infinite path

p is said to be a *rational path* if it is tail-equivalent to an infinite path $q = ccc\cdots$, where c is a closed path.

Given an arbitrary graph E and a field K, the *Leavitt path algebra* $L_K(E)$ is defined to be the K-algebra generated by a set $\{v : v \in E^0\}$ of pair-wise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$.
(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
(3) (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.
(4) (The "CK-2 relations") For every regular vertex $v \in E^0$,

$$v = \sum_{e \in E^1, s(e)=v} ee^*.$$

An arbitrary element $a \in L := L_K(E)$ can be written as $a = \sum_{i=1}^{n} k_i \alpha_i \beta_i^*$ where α_i, β_i are paths and $k_i \in K$. Here $r(\alpha_i) = s(\beta_i^*) = r(\beta_i)$.

Every Leavitt path algebra $L_K(E)$ is a $\mathbb{Z}$*-graded algebra*, namely, $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_n$ induced by defining, for all $v \in E^0$ and $e \in E^1$, $\deg(v) = 0$, $\deg(e) = 1$, $\deg(e^*) = -1$. Here the L_n are abelian subgroups satisfying $L_m L_n \subseteq L_{m+n}$ for all $m, n \in \mathbb{Z}$. Further, for each $n \in \mathbb{Z}$, the homogeneous component L_n is given by $L_n = \{\sum k_i \alpha_i \beta_i^* \in L : \alpha, \beta \in Path(E), |\alpha_i| - |\beta_i| = n\}$. (For details, see Sect. 2.1 in [2]). An ideal I of $L_K(E)$ is said to be a *graded ideal* if $I = \bigoplus_{n \in \mathbb{Z}} (I \cap L_n)$.

Throughout this paper, E will denote an arbitrary graph (with no restriction on the number of vertices or on the number of edges emitted by each vertex) and K will denote an arbitrary field. For convenience in notation, we will denote, most of the times, the Leavitt path algebra $L_K(E)$ by L.

We shall first recall the definition of the Gelfand–Kirillov dimension of associative algebras over a field.

Let A be a finitely generated K-algebra, generated by a finite dimensional subspace $V = Ka_1 \oplus \cdots \oplus Ka_m$. Let $V^0 = K$ and, for each $n \geq 1$, let V^n denote the K-subspace of A spanned by all the monomials of length n in $a_1, \ldots, a_m$. Set $V_n = \sum_{i=0}^{n} V^i$. Then the **Gelfand–Kirillov dimension** of A (for short, the **GK-dimension** of A) is defined by

$$\text{GK-dim}(A) := \limsup_{n \to \infty} \log_n(\dim V_n).$$

It is known that the GK-dim(A) is independent of the choice of the generating subspace V.

If A is an infinitely generated K-algebra, then the GK-dimension of A is defined as

$$\text{GK-dim}(A) := \sup_{B} \text{GK-dim}(B)$$

where B runs over all the finitely generated K-subalgebras of A.

The GK-dimension of an algebra A measures the growth of the algebra A and might be considered the non-commutative analogue of the classical Krull dimension for commutative algebras. Some useful examples (see [28]) are: The GK-dimension the matrix ring $M_\Lambda(K)$ is 0 and the GK-dimension of the matrix ring $M_{\Lambda'}(K[x, x^{-1}])$ is 1, where Λ, Λ' are arbitrary possibly infinite index sets.

Grading of a matrix ring over a $\mathbb{Z}$-graded ring: We wish to recall the grading of matrices of finite order and then indicate how to extend this to the case of infinite matrices in which at most finitely many entries are non-zero (see [22, 31]). This will be used in Sect. 1.4. In the following, the length of a path p will be denoted by $|p|$.

Let Γ be an additive abelian group, A be a Γ-graded ring and $(\delta_1, \ldots, \delta_n)$ an n-tuple where $\delta_i \in \Gamma$. Then $M_n(A)$ is a Γ-graded ring where, for each $\lambda \in \Gamma$, its λ-homogeneous component consists of $n \times n$ matrices

$$M_n(A)(\delta_1, \ldots, \delta_n)_\lambda = \begin{pmatrix} A_{\lambda+\delta_1-\delta_1} & A_{\lambda+\delta_2-\delta_1} & \cdots & A_{\lambda+\delta_n-\delta_1} \\ A_{\lambda+\delta_1-\delta_2} & A_{\lambda+\delta_2-\delta_2} & & A_{\lambda+\delta_n-\delta_2} \\ & & & \\ A_{\lambda+\delta_1-\delta_n} & A_{\lambda+\delta_2-\delta_n} & & A_{\lambda+\delta_n-\delta_n} \end{pmatrix}. \tag{1.1}$$

This shows that for each homogeneous element $x \in A$,

$$\deg(e_{ij}(x)) = \deg(x) + \delta_i - \delta_j, \tag{1.2}$$

where $e_{ij}(x)$ is a matrix with x in the ij-position and with every other entry 0.

Now let A be a Γ-graded ring and let I be an arbitrary infinite index set. Denote by $M_I(A)$ the matrix with entries indexed by $I \times I$ having all except finitely many entries non-zero and for each $(i, j) \in I \times I$, the ij-position is denoted by $e_{ij}(a)$ where $a \in A$. Considering a "vector" $\bar{\delta} := (\delta_i)_{i \in I}$ where $\delta_i \in \Gamma$ and following the usual grading on the matrix ring (see (1.1), (1.2)), define, for each homogeneous element a,

$$\deg(e_{ij}(a)) = \deg(a) + \delta_i - \delta_j. \tag{1.3}$$

This makes $M_I(A)$ a Γ-graded ring, which we denote by $M_I(A)(\bar{\delta})$. Clearly, if I is finite with $|I| = n$, then the graded ring coincides (after a suitable permutation) with $M_n(A)(\delta_1, \ldots, \delta_n)$.

Suppose E is a finite acyclic graph consisting of exactly one sink v. Let $\{p_i : 1 \leq i \leq n\}$ be the set of all paths ending at v. Then it was shown in (Lemma 3.4, [4])

$$L_K(E) \cong M_n(K) \tag{1.4}$$

under the map $p_i p_j^* \longmapsto e_{ij}$. Now taking into account the grading of $M_n(K)$, it was further shown in (Theorem 4.14, [22]) that the same map induces a graded isomorphism

$$L_K(E) \longrightarrow M_n(K)(|p_1|, \ldots, |p_n|) \tag{1.5}$$

$$p_i p_j^* \longmapsto e_{ij}.$$

In the case of a comet graph E (that is, a finite graph E with a cycle c without exits and a vertex v on c such that every path in E which does not include all the edges in c ends at v), it was shown in (Lemma 2.7.1, [2]) that the map

$$L_K(E) \longrightarrow M_n(K[x, x^{-1}]) \tag{1.6}$$

given by

$$p_i c^k p_j^* \longmapsto e_{ij}(x^k)$$

where the e_{ij} are matrix units, induces an isomorphism. Again taking into account the grading, it was shown in (Theorem 4.20, [22]) that the map

$$L_K(E) \longrightarrow M_n(K[x^{|c|}, x^{-|c|}])(|p_1|, \ldots, |p_n|) \tag{1.7}$$

given by

$$p_i c^k p_j^* \longmapsto e_{ij}(x^{k|c|})$$

induces a graded isomorphism. Later in the paper [3], the isomorphisms (1.4) and (1.6) were extended to infinite acyclic and infinite comet graphs, respectively (see Proposition 3.6 [3]). The same isomorphisms with the grading adjustments will induce graded isomorphisms for Leavitt path algebras of such graphs. We now describe this extension below.

Let E be a graph such that no cycle in E has an exit and such that every infinite path contains a line point or is tail-equivalent to a rational path $ccc\cdots$. where c is a cycle (without exits). Define an equivalence relation in the set of all line points in E by setting $u \sim v$ if $T_E(u) \cap T_E(v) \neq \emptyset$. Let X be the set of representatives of distinct equivalence classes of line points in E, so that for any two line points $u, v \in X$ with $u \neq v$, $T_E(u) \cap T_E(v) = \emptyset$. For each vertex $v_i \in X$, let $\overline{p^{v_i}} := \{p_s^{v_i} : s \in \Lambda_i\}$ be the set of all paths that end at v_i, where Λ_i is an index set which could possibly be infinite. Denote by $|\overline{p^{v_i}}| = \{|p_s^{v_i}| : s \in \Lambda_i\}$.

Let Y be the set of all distinct cycles in E. As before, for each cycle $c_j \in Y$ based at a vertex w_j, let $\overline{q^{w_j}} := \{q_r^{w_j} : r \in \Upsilon_j\}$ be the set of all paths that end at w_j but do not include all the edges of c_j, where Υ_j is an index set which could possibly be infinite. Let $|\overline{q^{w_j}}| := \{|q_r^{w_j}| : r \in \Upsilon_j\}$. Then the isomorphisms (1.5) and (1.7) extend to a $\mathbb{Z}$-graded isomorphism

$$L_K(E) \cong_{gr} \bigoplus_{v_i \in X} M_{\Lambda_i}(K)(|\overline{p^{v_i}}|) \oplus \bigoplus_{w_j \in Y} M_{\Upsilon_j}(K[x^{|c_j|}, x^{-|c_j|}])(|\overline{q^{w_j}}|) \qquad (1.8)$$

where the grading is as in (1.3).

1.3 Leavitt Path Algebras Satisfying a Polynomial Identity

Observe that Leavitt path algebras in general are highly non-commutative. For instance, if the graph E contains an edge e with $u = s(e) \neq r(e) = v$, then $ue = e$, but $eu = 0$. Indeed, it is an easy exercise to conclude that if E is a connected graph, then $L_K(E)$ is a commutative ring if and only if the graph E consists of just a single vertex v or is a loop e, that is a single edge e with $s(e) = r(e) = v$. In this case, $L_K(E)$ is isomorphic to K or $K[x, x^{-1}]$.

Note that to say a ring R is commutative is equivalent to saying that R satisfies the polynomial identity $xy - yx = 0$. An algebra A over a field K is said to **satisfy a polynomial identity** (or simply, a **PI-algebra**), if there is a polynomial $p(x_1, \ldots, x_n)$ in finitely many non-commuting variable $x_1, \ldots, x_n$ with coefficients in K such that $p(a_1, \ldots, a_n) = 0$ for all choices of elements $a_1, \ldots, a_n \in A$. For example, the Amitsur-Levitzky theorem (see [33]) states that the ring $M_n(R)$ of $n \times n$ matrices over a commutative ring R satisfies the so called standard polynomial identity $P_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \epsilon_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ where S_n is the symmetric group of $n!$ permutations of the set $\{1, \ldots, n\}$ and $\epsilon_\sigma = 1$ or -1 according as σ is even or odd. A natural question is to characterize the Leavitt path algebras which satisfy a polynomial identity. This is completely answered in the next theorem.

Theorem 1.1 ([17]) *Let E be an arbitrary graph. Then the following properties are equivalent for $L_K(E)$:*

(a) *$L_K(E)$ satisfies a polynomial identity;*
(b) *No cycle in E has an exit, there is a fixed positive integer d such that the number of distinct paths that end at any vertex v is $\leq d$ and the only infinite paths in E are paths that are eventually of the form $ggg \cdots$, for some cycle g;*
(c) *There is a fixed positive integer d such that $L_K(E)$ is a subdirect product of matrix rings over K or $K[x, x^{-1}]$ of order at most d.*

If the graph E is row-finite, then the Leavitt path algebra $L_K(E)$ in Theorem 1.1 actually decomposes as a ring direct sum of matrix rings over K or $K[x, x^{-1}]$ of order at most a fixed positive integer d. This shows that satisfying a polynomial identity imposes a serious restriction on the structure of Leavitt path algebras.

1.4 Four Important Graphical Conditions

In this section, we shall illustrate how specific graphical conditions on the graph E give rise to various algebraic properties of $L_K(E)$. We illustrate this by choosing four different graph properties of E. Interestingly, a single graph theoretical property of E often implies several different ring properties for $L_K(E)$. It is amazing that a single property that no cycle in a finite graph E has an exit implies that the corresponding Leavitt path algebra $L_K(E)$ possesses several different ring properties such as being directly finite, self-injective, having bounded index of nilpotence, a Baer ring, satisfying a polynomial identity, having GK-dimension ≤ 1, etc. (see Theorem 1.5 below). Consequently, Leavitt path algebras turn out to be useful tools in the construction of various examples of rings. We will also describe the interesting history behind the terms Condition (K) and Condition (L) which play an important role in the investigation of both the graph C^*-algebras and Leavitt path algebras (see [2, 40, 42]).

Recall, a ring R is said to be von Neumann regular if to each element $a \in R$ there is an element $b \in R$ such that $a = aba$. The ring R is said to be π-regular (strongly left or right π-regular) if to each element $a \in R$, there is a $b \in R$ and an integer $n \geq 1$ such that $a^n = a^n b a^n$ ($a^n = a^{n+1}b$ or $a^n = ba^{n+1}$). In general, these ring properties are not equivalent. But as the next theorem shows, they all coincide for Leavitt path algebras.

A graph E is said to be **acyclic** if E contains no cycles. The next theorem characterizes the von Neumann regular Leavitt path algebras.

Theorem 1.2 ([7]) *For an arbitrary graph E, the following conditions are equivalent for $L := L_K(E)$:*

(a) *The graph E is acyclic;*
(b) *L is von Neumann regular;*
(c) *L is π-regular;*
(d) *L is strongly left/right π-regular.*

Another important graph property is Condition (K). In some sense this property is diametrically opposite of being acyclic.

Definition 1.1 A graph E satisfies **Condition (K)** if whenever a vertex v lies on a simple closed path α, v also lies on another simple closed path β distinct from α.

The Condition (K) implies a number of ring properties;

Definition 1.2 (i) A ring R is said to be left/right **weakly regular** if for every left/right ideal I of R, $I^2 = I$;

(ii) A ring R is said to be an **exchange ring** if given any left/right R-module M and two direct decompositions of M as $M = M' \oplus A$ and $M = \bigoplus_{i=1}^{n} A_i$, where $M' \cong R$, there exist submodules $B_i \subseteqq A_i$ such that $M = M' \oplus \bigoplus_{i=1}^{n} B_i$.

Theorem 1.3 ([15, 16, 40]) *Let E be an arbitrary graph. Then the following conditions are equivalent for $L := L_K(E)$:*

(i) *The graph E satisfies Condition (K);*
(ii) *L is an exchange ring;*
(iii) *L is left/right weakly regular;*
(iv) *Every two-sided ideal of L is a graded ideal.*

Definition 1.3 A graph E is said to satisfy **Condition (L)**, if every cycle in E has an exit.

Theorem 1.4 ([35]) *Let E be an arbitrary graph. Then the following are equivalent for $L_K(E)$:*

(i) *E satisfies Condition (L);*
(ii) *L is a **Zorn ring**, that is, every (non-nil) right/left ideal I contains a non-zero idempotent.;*
(iii) *Every element $a \in L$ is the von Neumann inverse of another element $b \in L$; that is, to each $a \in L$, there is an element $b \in L$ such that $bab = b$.*

An interesting history of Conditions (K) and (L): One may wonder about the choice of the letters K and L in the terms Condition (K) and Condition (L). There is an interesting narrative about the origins of these terms. I am grateful to Mark Tomforde for outlining this history to me which he will also be including in his forthcoming book on Graph Algebras [42]. Both these two graph conditions were originally introduced by graph C^*-algebraists. It all started when Cuntz and Krieger (whom some consider the founders of graph C^*-algebras), introduced in their original paper [21] a condition on matrices with entries in {0, 1} and called it Condition (I). Assuming that the 'I' is the English letter I and not the Roman numeral one, Pask and Raeburn introduced in 1996 a Condition (J) in their paper [32], as J is the letter that follows I in the English alphabet. (They apparently did not recognize that Cüntz and Krieger also introduced a follow-up Condition (II), thus indicating, in their view, I and II stand for Roman numerals.) Conforming to this pattern, when Kumjian, Pask, Raeburn and Renault introduced a new condition in their 1997 paper [29], they chose the letter K to denote this new condition and called it Condition (K). Continuing this pattern yet again, Kumjian, Pask and Raeburn introduced Condition (L) in 1998 [30]. Actually, Astrid an Huef later showed that Condition (L) coincides with Condition (I) for graphs of finite matrices. Moreover, Condition (K) is considered analogous to

Condition (II) for Cuntz-Krieger algebras. In all the investigations that followed in graph C^*-algebras and also in Leavitt path algebras, Conditions (K) and (L) emerged as important graph conditions. Poor Condition (J) remains neglected!

Recall, Condition (L) requires every cycle to have an exit. We next consider a graph property that is diametrically opposite to Condition (L), namely, no cycle in the graph has an exit. This implies several interesting ring/module properties.

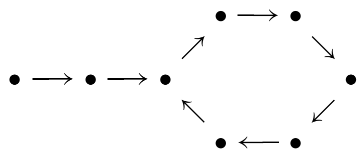
First, consider a finite graph E in which no cycle has an exit. In this case, $L_K(E)$ is a ring with identity. We begin recalling a number of ring properties.

A ring R with identity 1 is said to be **directly finite** if for any two elements x, y, $xy = 1$ implies $yx = 1$. This is equivalent to R being not isomorphic to any proper direct summand of R as a left or a right R-module. A ring R with identity is called a **Baer ring** if the left/right annihilator of every subset X of R is generated by an idempotent. A Γ-graded ring R is said to be a **graded Baer ring**, if the left/right annihilator of every subset X of homogeneous elements is generated by a homogeneous idempotent. A ring R is said to have **bounded index of nilpotence** if there is a positive integer n which is such that $a^n = 0$ for every nilpotent element $a \in R$.

Theorem 1.5 ([9, 17, 25, 27, 39, 43]) *For a finite graph E, the following conditions are equivalent for $L := L_K(E)$:*

(i) *No cycle in E has an exit;*
(ii) *L is directly finite;*
(iii) *L is a Baer ring;*
(iv) *L is a graded Baer ring;*
(v) *L is a graded left/right self-injective ring;*
(vi) *L satisfies a polynomial identity;*
(vii) *L has bounded index of nilpotence;*
(viii) *L is graded semi-simple;*
(ix) *L has GK-dimension* ≤ 1;
(x) *L is finite over its center.*

Thus if E is the following graph,



then $L_K(E)$ will possess all the stated nine ring properties.

For a finite graph E, if $L_K(E)$ satisfies any of the equivalent conditions in the preceding theorem, $L_K(E)$ decomposes as a graded direct sum of finitely many matrix rings of finite order over K and/or $K[x, x^{-1}]$ which are given the matrix gradings indicated in Eqs. (1.5) and (1.7) in the Preliminary section.

For a ring R without identity, but with local units, R is said to be directly finite if for every $x, y \in R$ and an idempotent $u \in R$ satisfying $ux = x = xu$, $uy = y = yu$, we have $xy = u$ implies $yx = u$. Every commutative ring is trivially directly finite.

If R is a ring without identity, R is called a **locally Baer ring** (**locally graded Baer** ring) if for every idempotent (homogeneous idempotent) e, the corner eRe is a Baer (graded Baer) ring.

Theorem 1.6 ([25, 27]) *Let E be an arbitrary graph. Then the following conditions are equivalent for $L := L_K(E)$:*

(i) *No cycle in E has an exit, E is row-finite and every infinite path ends at a sink or a cycle.*

(i) *L is a locally Baer ring;*
(ii) *L is a graded locally Baer ring;*
(iii) *L is a graded left/right self-injective ring;*
(iv) *L is graded isomorphic to a ring direct sum of matrix rings*

$$L_K(E) \cong_{gr} \bigoplus_{v_i \in X} M_{\Lambda_i}(K)(|\overline{p^{v_i}}|) \oplus \bigoplus_{w_j \in Y} M_{\Upsilon_j}(K[x^{t_j}, x^{-t_j}])(|\overline{q^{w_j}}|)$$

where Λ_i,Υ_j are suitable index sets, the t_j are positive integers, X is the set of representatives of distinct equivalence classes of line points in E and Y is the set of all distinct cycles (without exits) in E.

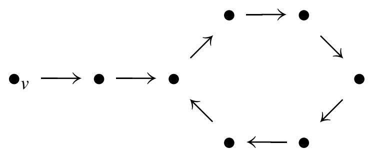
1.5 Simple Modules over Leavitt Path Algebras

In this section, we shall indicate the methods of constructing simple modules over Leavitt path algebras by graphical means.

As noted in [2], every element a of a Leavitt path algebra $L_K(E)$ of a graph E can be written in the form $a = \sum_{i=1}^{n} \alpha_i \beta_i^*$ and that the map $\sum_{i=1}^{n} \alpha_i \beta_i^* \longrightarrow \sum_{i=1}^{n} \beta_i \alpha_i^*$ induces an isomorphism $L_K(E) \longrightarrow (L_K(E))^{op}$. Consequently, $L_K(E)$ is left-right symmetric. So in this and the next section, we shall only be stating results on left ideals and left modules over $L_K(E)$. The corresponding results on right ideals and right modules hold by symmetry.

Definition 1.4 A vertex v is called a **Laurent vertex** if $T_E(v)$ consists of the set of all vertices on a single path $\gamma = \mu c$ where μ is a path without bifurcations starting at v and c is a cycle without exits based on a vertex $u = r(\mu)$.

An easy example of a Laurent vertex is the vertex v in the following graph:



The next theorem gives conditions under which a vertex in the graph E generates a simple left ideal/graded simple left ideal of $L_K(E)$.

Theorem 1.7 ([2, 25]) *Let E be an arbitrary graph and let v be a vertex. Then*

(a) *The left ideal $L_K(E)v$ is a simple/minimal left ideal of $L_K(E)$ if and only if v is a line point;*
(b) *The left ideal $L_K(E)v$ is a graded simple/minimal left ideal of $L_K(E)$ if and only if v is either a line point or a Laurent vertex.*

Next, we shall describe the general methodology used by Chen [19] and extended in [25, 37] to construct left simple and graded simple modules over $L_K(E)$ by using special vertices or cycles in the graph E.

(I) **Definition of the module** A_u: Let u be a vertex in a graph E which is either a sink or an infinite emitter. Let A_u be the K-vector space having as a basis the set $B = \{p : p \text{ is a path in } E \text{ with } r(p) = u\}$. We make A_u a left $L_K(E)$-module as follows: Define, for each vertex v and each edge e in E, linear transformations P_v, S_e and S_{e^*} on A by defining their actions on the basis B are as follows:

For all $p \in B$,

(I) $P_v(p) = \begin{cases} p, \text{ if } v = s(p) \\ 0, \text{ otherwise} \end{cases}$

(II) $S_e(p) = \begin{cases} ep, \text{ if } r(e) = s(p) \\ 0, \text{ otherwise} \end{cases}$

(III) $S_{e^*}(p) = \begin{cases} p', \text{ if} p = ep' \\ 0, \text{ otherwise} \end{cases}$

(IV) $S_{e^*}(u) = 0$.

The endomorphisms $\{P_v, S_e, S_{e^*} : v \in E^0, e \in E^1\}$ satisfy the defining relations (1.1)–(1.4) of the Leavitt path algebra $L_K(E)$. This induces an algebra homomorphism ϕ from $L_K(E)$ to $End_K(A_u)$ mapping v to P_v, e to S_e and e^* to S_{e^*}. Then A_u can be made a left module over $L_K(E)$ via the homomorphism ϕ. We denote this $L_K(E)$-module operation on A_u by $\cdot$.

Theorem 1.8 ([19, 37]) *If the vertex u is either a sink or an infinite emitter, then A_u is a simple left $L_K(E)$-module.*

If the vertex u lies on a cycle without exits, then in the Definition of A_u, define the basis $B = \{pq^* : p, q \text{ path in } E \text{ with } r(q^*) = s(q) = u\}$. We then get the following result.

Theorem 1.9 ([25]) *If a vertex $u \in E$ lies on a cycle without exits, then A_u is a graded simple left $L_K(E)$-module graded isomorphic to the graded minimal left ideal $L_K(E)u$ and A_u is not a simple left $L_K(E)$-module.*

Remark 1.1 In [25], the module A_u is defined by using an algebraic branching system and is denoted as N_{vc}. Here we have defined the module A_u differently, but the proof of the above theorem is just the proof of Theorem 3.5(1) of [25].

With a slight modification of the definition of A_u, Chen [19] shows one more way of constructing simple modules by using the infinite paths that are tail-equivalent to a fixed infinite path in E. Recall, two infinite paths $p = e_1 \cdots e_r \cdots$ and $q = f_1 \cdots f_s \cdots$ are said to be **tail-equivalent** if there exist fixed positive integers m, n such that $e_{n+k} = f_{m+k}$ for all $k \geq 1$. Let $[p]$ denote the tail-equivalence class of all infinite paths equivalent to p. Let $A_{[p]}$ denote the K-vector space having $[p]$ as a basis. As in the definition of A_u, for each vertex v and each edge e in E, define the linear transformations P_v, S_e and S_{e^*} on A by defining their actions on the basis $[p]$ satisfying the conditions (I), (II), (III), but not (IV) above. As before, they satisfy the defining relations of a Leavitt path algebra and thus induce a homomorphism $\varphi : L_K(E) \to A_{[p]}$. The vector space $A_{[p]}$ then becomes a left $L_K(E)$-module via the map φ.

Theorem 1.10 ([19]) *The module $A_{[p]}$ is a simple left $L_K(E)$-module and for two infinite paths p, q, $A_{[p]} \cong A_{[q]}$ if and only if $[p] = [q]$.*

It can be shown (see [37]) that the simple modules A_u, A_v and $A_{[p]}$ corresponding, respectively, to a sink u, an infinite emitter v and an infinite path p, are all non-isomorphic.

A special infinite path is the so-called a **rational infinite path** induced by a simple closed path (and in particular, a cycle) c. This is the infinite path $ccc \cdots$. We denote this path by c^∞. We shall be using the corresponding simple $L_K(E)$-module A_{c^∞} subsequently.

1.6 Leavitt Path Algebras with Simple Modules Having Special Properties

We shall describe when all the simple modules over a Leavitt path algebra are flat or injective or finitely presented or graded etc.

An open problem, raised by Ramamuthi [34] some forty years ago, asks whether a non-commutative ring R with 1 is von Neumann regular if all the simple left R-modules are flat. Using a more general approach of Steinberg algebras, Ambily, Hazrat and Li [10] obtain the following theorem which shows that Ramamurthi's question has an affirmative answer in the case of Leavitt path algebras.

Theorem 1.11 ([10]) *Let E be an arbitrary graph. Then every simple left $L_K(E)$-module is flat if and only if $L_K(E)$ is von Neumann regular.*

Next, we consider the case when $L_K(E)$ is a left V-ring, that is, when every simple left $L_K(E)$-module is injective. Kaplansky showed that if R is a commutative ring, then every simple R-module is injective if and only if R is von Neumann regular. A natural question is under what conditions every simple left $L_K(E)$-module is injective. It was shown in [26] that, in this case, $L_K(E)$ becomes a weakly regular ring. However, recently Abrams, Mantese and Tonolo [5] showed that, if c is a cycle in a graph E, then the corresponding simple left $L_K(E)$-module A_{c^∞} satisfies $Ext^1_{L_K(E)}(A_{c^\infty}, A_{c^\infty}) \neq 0$. This implies that the module A_{c^∞} cannot be an injective module. So, if every simple left $L_K(E)$-module is injective, then necessarily E contains no cycles. Then by [7] $L_K(E)$ must be von Neumann regular. Thus, we obtain the following new result and its corollary.

Theorem 1.12 *Let E be an arbitrary graph. If every simple left $L_K(E)$-module is injective, then $L_K(E)$ is a von Neumann regular ring.*

Conversely, if $L_K(E)$ is a von Neumann regular ring then the graph E contains no cycles [7] and, if E is further a finite graph, then $L_K(E)$ is a direct sum of finitely many matrix rings of finite order over K (Theorem 2.6.17, [2]). In this case, $L_K(E)$ is a direct sum of left/right simple modules and hence every simple left/right $L_K(E)$-module is injective. This leads to the following corollary.

Corollary 1.1 *Let E be a finite graph. Then every simple left/right $L_K(E)$-module is injective if and only if $L_K(E)$ is a von Neumann regular ring.*

When E is an arbitrary graph, it is an open question whether the von Neumann regularity of $L_K(E)$ implies that every simple left/right $L_K(E)$-module is injective.

Next, we consider Leavitt path algebras $L_K(E)$ whose simple modules are all finitely presented. When E is a finite graph, $L_K(E)$ possesses a number of interesting properties as noted in the following theorem.

Theorem 1.13 ([13]) *For any finite graph E, the following properties of the Leavitt path algebra $L := L_K(E)$ are equivalent:*

(i) *Every simple left L-module is finitely presented;*
(ii) *No two cycles in E have a common vertex;*
(iii) *There is a one-to-one correspondence between isomorphism classes of simple L-modules and primitive ideals of L;*
(iv) *The Gelfand–Kirillov dimension of L is finite.*

The preceding theorem has been generalized in [38] to the case when E is an arbitrary graph with several similar equivalent conditions.

It is easy to observe that every simple left module over a Leavitt path algebra $L := L_K(E)$ is of the form Lv/N for some vertex v and a maximal left submodule N of Lv. A natural question is, given a vertex u, can we estimate the number κ_u of distinct maximal left L-submodules M of Lu such that Lu/M is isomorphic to Lv/N? In [38] it is shown that $\kappa_u \leq |uLv \backslash N|$ and consequently the cardinality of the