Sio-Iong Ao · Len Gelman
Haeng Kon Kim *Editors*

Transactions on Engineering Technologies

World Congress on Engineering 2018

Springer

Transactions on Engineering Technologies

Sio-Iong Ao • Len Gelman • Haeng Kon Kim
Editors

Transactions on Engineering Technologies

World Congress on Engineering 2018

Springer

Editors
Sio-Iong Ao
IAENG Secretariat
Hong Kong, Hong Kong

Haeng Kon Kim
Department of Computer and Communication
Daegu Catholic University
Daegu, Korea (Republic of)

Len Gelman
Department of Engineering and Technology, Centre for Efficiency and Performance Engineering, School of Computing and Engineering
University of Huddersfield
Huddersfield, UK

ISBN 978-981-32-9530-8 ISBN 978-981-32-9531-5 (eBook)
https://doi.org/10.1007/978-981-32-9531-5

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

A large international conference on Advances in Engineering Technologies and Physical Science was held in London, UK, 4–6 July 2018, under the World Congress on Engineering 2018 (WCE 2018). The WCE 2018 is organized by the International Association of Engineers (IAENG); the Congress details are available at http://www.iaeng.org/WCE2018. IAENG is a nonprofit international association for engineers and computer scientists, which was founded originally in 1968. The World Congress on Engineering serves as good platforms for the engineering community to meet with each other and to exchange ideas. The conferences have also struck a balance between theoretical and application developments. The conference committees have been formed with over 300 committee members who are mainly research center heads, faculty deans, department heads, professors, and research scientists from over 30 countries. The congress is truly a global international event with a high level of participation from many countries. The response to the Congress has been excellent. There have been more than 500 manuscript submissions for the WCE 2018. All submitted papers have gone through the peer-review process, and the overall acceptance rate is 51%.

This volume contains 26 revised and extended research articles written by prominent researchers participating in the conference. Topics covered include mechanical engineering, engineering mathematics, computer science, knowledge engineering, electrical engineering, wireless networks, and industrial applications. The book offers the state of the art of tremendous advances in engineering technologies and physical science and applications and also serves as an excellent reference work for researchers and graduate students working on engineering technologies and physical science and applications.

Hong Kong, Hong Kong — Sio-Iong Ao
Huddersfield, UK — Len Gelman
Daegu, Republic of Korea — Haeng Kon Kim

Contents

Chapter 1
A New Mathematical Model for a Membrane MEMS Device

Luisa Fattorusso and Mario Versaci

Abstract The membrane MEMSs represent a good design solution for the industry requirements about the construction of micro-dimensional devices, because easily constructible and extremely versatile. In this domain, the experience of the authors in the modeling of membrane MEMS devices has matured. In this chapter, they present a formalization of stationary $1D$-membrane MEMS in which the electric field magnitude, $|\mathbf{E}|$, is proportional to the curvature of the membrane, C, obtaining a semilinear elliptic model. Next, techniques based on fixed point Theorems provide results of existence, while an approach based on the joint use of Poincaré's inequality and Gronwall's Lemma establish conditions of uniqueness. Finally, some numerical tests complete the work.

Keywords Boundary elliptic problems · Existence and uniqueness for solution · Green function · Membrane MEMS devices · Schauder-Tychonoff theorem

1.1 Introduction to the Problem

In the last decade, MEMSs engineering has acquired an important role in the design of actuators and sensors that often require the use of micro-dimensional devices. This is essentially due to the fact that the MEMSs are a valid link between the physical nature of the problem and the machine language. While, on the one hand, Scientific Research tries to achieve results in the analytical-numerical modeling of such devices, on the other hand, Industry requires the development of models, based on the observed reality [3, 13, 20], with reduced computational load and easily implemented with a clear reduction of costs. Because of the high variety of

L. Fattorusso
DIIES Department, Cittadella Universitaria, "Mediterranea" University, Reggio Calabria, Italy
e-mail: luisa.fattorusso@unirc.it

M. Versaci (✉)
DICEAM Department, Cittadella Universitaria, "Mediterranea" University, Reggio Calabria, Italy
e-mail: mario.versaci@unirc.it

S.-I. Ao et al. (eds.), *Transactions on Engineering Technologies*,
https://doi.org/10.1007/978-981-32-9531-5_1

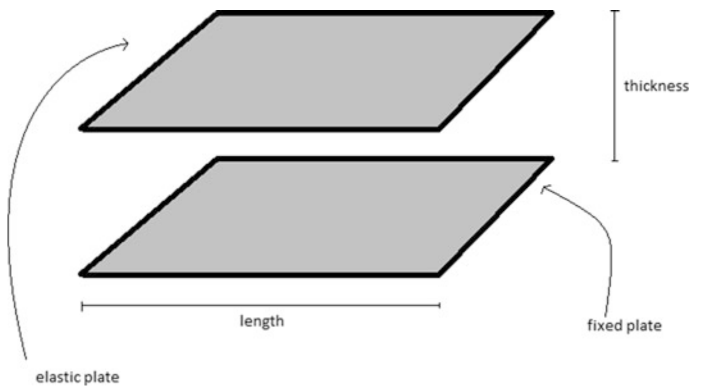

Fig. 1.1 A simplified scheme of the device

application fields of MEMS, the Scientific Community is actively engaged in the study of coupled systems, including magneto-mechanical systems, thermo-elastic systems of biomedical interest [2, 15, 16] and, last but not least, systems related to wave propagation [4, 19, 22].

Although the mathematical models developed have a high degree of sophistication, they do not often allow the obtainment of explicit solutions, so that one is satisfied with numerical solutions [1, 11, 12, 18, 23]. Nevertheless, the risk of obtaining, numerically, ghost solutions is high, so the need to obtain analytical conditions of existence and uniqueness of the solution is evident especially in presence of nonlinear singularities [5–8]. One of the most accredited a dimensionalized models concerns a MEMS device that is composed of two metal plates: one of which is fixed and the other deformable but clumped to its extremes (see, Fig. 1.1). When, a voltage V is applied, the deformable plate deflects towards the fixed one [5, 7, 8] obtaining the following analytical model:

$$\begin{cases} \omega\Delta^2 v = \left(\varrho \int_\Omega |\nabla v|^2 dy + \varsigma\right)\Delta v + \lambda_1 g_1(y)\left((1-v)^\vartheta \left(1 + \alpha \int_\Omega \frac{dy}{(1-v)^{\vartheta-1}}\right)^{-1}\right) \\ v(y) = 0, \quad y \in \partial\Omega \\ 0 < v(y) < 1, \quad y \in \Omega \end{cases} \tag{1.1}$$

where the dielectric properties of the material of the deformable plate are taken into account by the bounded function g_1, while λ_1 is proportional to V. In addition, the positive parameters ω, ϱ, ς, α are related to the electric and mechanic properties of the material and, finally, ϑ could take into account different electrostatic potentials.

Obviously, a useful simplification of model (1.1) can be considered if both the inertial and non-local effects are neglected. In other terms, if $\vartheta = 2$, $\omega = 1$, $\varrho = \varsigma = 0$, and $\alpha = 0$, model (1.1) becomes:

$$\begin{cases} \Delta^2 v(y) = \lambda_1 g_1(y)([1 - v(y)]^2)^{-1} \\ 0 < v(y) < 1 \text{ in } \Omega, \\ v(y) = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

In this chapter, starting from (1.2), we discuss the $1D$ elliptical semi-linear model in which the bottom plate is replaced by a thin membrane attached to the edge whose formulation is:

$$\begin{cases} v'' = -g_2(y)\lambda_1((1 - v(y)^2)^{-1} & \text{in } \Omega = [-A, A] \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where λ_1 is proportional to V and then is also proportional to $|\mathbf{E}|$. In addition, since $\mathbf{E}$, for each point of the membrane, is orthogonal to the tangent of the membrane itself, in this chapter we consider $|\mathbf{E}|$ is considered proportional to the curvature C of the membrane and, in addition, as following specified, the singularity $1 - v(y)$ is not present [10] by introducing a safety distance.

The chapter is organized as follows. Starting from Sect. 1.2 in which some preliminary results about the membrane MEMS are illustrated, Sect. 1.3 describes the approach whose achieved model is expressed in the Dirichlet's form by means of its integral formulation. In addition, if the safety distance τ, that is the gap of the top of the membrane profile from the upper plate, is taken into account, interesting conditions of existence and uniqueness of the solution is presented in Sects. 1.4 and 1.5 respectively. At the end of the chapter, some numerical tests validate the goodness of the proposed approach (Sect. 1.6).

1.2 Electrostatic 1*D* Membrane MEMS Model: Backgrounds

Let us start the treatment by means of a known $1D$ membrane MEMS model in its dimensionless formulation. In particular, in $\mathbb{R}^3$, we consider a system of Cartesian axes $O'y'\eta'\zeta'$ where an electrostatic-elastic device, whose length is $2A$, constituted by two parallel metallic plates (one fixed and the other one elastic but fixed at its edges) lies. The plates are located at a mutual distance h and the axis ζ' is orthogonal to their length. A voltage V is applied on the plates in order that $V = 0$ corresponds to the fixed plates so that V is related to the elastic one. Then, ϕ, the electrostatic potential, satisfies $\Delta\phi = 0$ (Laplace's equation) between the plates such that $\phi = V$ on the elastic plate, and $\phi = 0$ on the fixed one [10]. In such conditions, labeling by w' the deflection of the elastic plate satisfies the following equation [20]:

$$-\vartheta\Delta_\perp w' + D\Delta_\perp^2 w' = -0.5\epsilon_0|\nabla\phi|^2 \tag{1.4}$$

where $\Delta_\perp$ represents the laplacian operator with respect to y' and η'; ϑ and D take into account the mechanical tension and the flexural rigidity of the deformable plate, respectively. As usual, ϵ_o indicates the dielectric permittivity of free space. Since (1.4) represents a macroscopical formulation of the electrostatic problem, in order to apply it to a MEMS device, we need to take into account a set of scaling factors, that is $\Phi = \phi/V$, $u = w'/h$, $y = y'/2A$, $\zeta = \zeta'/h$ and $\eta = \eta'/2A$. Denoting, then, with $\delta = D/((2A)^2\vartheta)$ and $\epsilon = h/(2A)$ the relative importance of tension/rigidity and the aspect ration of the system respectively, Eq. (1.4) is writable as the following system of nonlinear coupled partial differential equations:

$$\begin{cases} \epsilon^2 \Delta_\perp \Phi + \Phi_{\zeta\zeta} = 0 \\ -\Delta_\perp u + \delta \Delta_\perp^2 u = -\lambda^2(\epsilon^2 |\nabla_\perp \Phi|^2 + (\Phi_\zeta)^2) \\ \Phi = 1 \ \ \text{onelasticplate}; \quad \Phi = 0 \ \ \text{onfixedplate} \end{cases} \tag{1.5}$$

in which Φ_ζ and $\Phi_{\zeta\zeta}$ represent the first and second order partial derivative of Φ with respect to ζ respectively. In addition,

$$\lambda_1 = \lambda^2 = \epsilon_0 V^2 (2A)^2 (2h^3 \vartheta)^{-1} = \rho V^2 \tag{1.6}$$

represents the ratio of a reference electrostatic force to a reference elastic force and

$$\varrho = \epsilon_0 (2A)^2 (2h^3 \vartheta)^{-1} \tag{1.7}$$

considers the electro-mechanical properties of the membrane material. However, in dimensionless conditions, laboratory experiences have shown that:

$$\varrho_1 = \epsilon_0 (2\vartheta)^{-1} > 10^{12}. \tag{1.8}$$

In addition, taking into account that the considered formulation of the MEMS device is $1D$ (that is, thickness and width are negligible with respect to its length), system (1.5) is simplified when $\epsilon \to 0$ so that the first equation of (1.5) becomes $\frac{\partial^2 \Phi}{\partial \zeta^2} = 0$ giving the well-know solution $\Phi = \frac{\zeta}{u}$ that, substituted into the second of (1.5), gives us:

$$-\Delta_\perp u + \delta \Delta_\perp^2 u = -\lambda^2 u^{-2} \tag{1.9}$$

which represents a non-linear equation decoupled from the equation containing the potential. Because our interest is focused on membrane MEMS devices, we replace the deformable plate by a deformable membrane fixed along the edge of a crisp and indeformable plate acting as a support deducing that (1.9) is still valid (even if, in this case, the numerical values of the electro-mechanical parameters are different). If then, we consider performant materials with negligible flexural rigidity $D \to 0$ (that is, $\delta = 0$), (1.9) is simplified as follows [9, 10, 20]:

$$\begin{cases} v'' = -\lambda^2(1-v)^{-2} \ \text{ in } \Omega = [-A_1, A_1] \\ v(-A_1) = v(-0.5) = v(A_1) = v(0.5) = 0 \end{cases} \tag{1.10}$$

which represents, a semi-linear model in stationary deflection conditions, in which the orientation of ζ is reversed and the membrane, in the rest condition, is located on the plane $\zeta = 0$. Note that, in (1.10), $u = 1 + v$ and A_1 is dimensionless.

1.3 The Core of the Approach: |E| in Terms of Curvature of the Membrane

Since in (1.10), by the (1.6), λ^2 is proportional to V^2, it makes sense rewrite (1.10) (see, [9]) as follows:

$$\begin{cases} -v'' = \varrho_1 |\mathbf{E}|^2 \ \text{ in } \ \Omega = [-A_1, A_1] \\ v(-A_1) = v(A_1) = 0 \end{cases} \tag{1.11}$$

in which $|\mathbf{E}|^2$ is the square of the electrical field magnitude. As mentioned in the introduction, the main idea of the proposed approach, starting from the fact that che **E**, for each point of the membrane, is always normal to the tangent of the membrane itself, $|\mathbf{E}|$ is expressed proportional to the curvature C of the membrane. So that, we can write $|\mathbf{E}|$ as:

$$|\mathbf{E}(y)| = \mu(y, v(y), \lambda) C(y, v(y)) \tag{1.12}$$

where μ, the coefficient of proportionality, after studies on an hemispherical benchmark [21], takes the following form:

$$\mu(y, u(y), \lambda) = \lambda(1 - u(y) - \tau)^{-1} \tag{1.13}$$

where $\mu(y, u(y), \lambda) \in C^0([-A_1, A_1] \times [0, 1) \times [\overline{\lambda}, \Lambda])$ in which $\overline{\lambda}^2$ is the minimum voltage to apply to the device to win the inertia of the membrane and Λ^2 is the maximum admissible voltage; τ is a critical distance evaluated as $\lambda(\epsilon_t)^{-1}$ (where ϵ_t is the dielectric strength of the material constituting the membrane such that, if $\epsilon_t \to \infty$, model (1.10) is restored). The choice to involve τ in (1.13) derives from the fact that, in the hypothesis of maximum deflection, the membrane must not touch the upper plate of the device. Mathematically, to consider τ translates into avoiding singularity in (1.13). By these premises, (1.11) is rewritten as follows:

$$\begin{cases} -v''(y) = \varrho_1 \mu^2(y, v(y), \lambda) C^2(y, v(y)) = \varrho_1 \lambda^2 C^2(y, v(y))(1 - v(y) - \tau)^{-2} \text{ in } \Omega \\ v(-A_1) = v(A_1) = 0; \quad 0 < v(y) < 1 - \tau, \end{cases} \tag{1.14}$$

and explaining C by its $1D$ formulation [14],

$$C(y, v(y)) = |v''(y)|(1 + |v'(y)|^2)^{-3/2}, \tag{1.15}$$

and taking into account that $v(y) > 0$, from (1.14), we can write:

$$v''(y) + \varrho_1 \mu^2(y, v(y), \lambda)|v''(y)|^2(1 + (v'(y))^2)^{-3} = 0. \tag{1.16}$$

In addition, since $v''(y) = 0$ would provide a linear deflection of the membrane ($v(y) = mx + b$ with m arbitrary constant), impossible when $|\mathbf{E}| \neq 0$, it makes sense write:

$$1 + \varrho_1 \mu^2(y, v(y), \lambda)(v''(y))(1 + (v'(y))^2)^{-3} = 0. \tag{1.17}$$

Definitely (1.14) assumes the following final form:

$$\begin{cases} v''(y) = -(1 + (v'(y))^2)^3(\varrho_1 \mu^2(y, v(y), \lambda))^{-1} \ \text{ in } \ \Omega \\ v(-A_1) = v(A_1) = 0 \\ 0 < v(y) < 1 - \tau \end{cases} \tag{1.18}$$

that can be considered as a particular case of the following problem [10]:

$$\begin{cases} v''(y) + f(x, v(y), v'(y)) = 0 \text{ in } \Omega = [A_1, -A_1] \\ v(-A_1) = v(A_1) = 0 \\ 0 < v(y) < 1 - \tau \quad 1 - \tau \in C^2(\Omega), \end{cases} \tag{1.19}$$

expressed in Dirichlet's form, where $f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R})$, so that

$$f(y, v(y), v'(y)) = (1 + (v'(y))^2)^3(\varrho_1 \mu^2(y, v(y), \lambda))^{-1}, \tag{1.20}$$

and the problem (1.19) assumes the following structure:

$$\begin{cases} v'' = -(1 + (v'(y))^2)^3)(\varrho_1 \mu^2(y, v(y), \lambda)^{-1} = \\ \quad = -(1 + (v'(y))^2)^3(1 - v(y) - \tau)^2(\varrho_1 \lambda^2)^{-1} \text{ in } \Omega \\ v(-A_1) = v(A_1) = 0; \quad 0 < v < 1 - \tau \end{cases} \tag{1.21}$$

in which $v \in C^2(\Omega)$, $\mu = \mu(y, v(y), \lambda) \in C^0(\Omega \times [0, 1], [\overline{\lambda}, \Lambda])$ and $\mu = \lambda(1 - v(y) - \tau)^{-1}$. Finally, the fact that $v \in C^2(\Omega)$ is imperative because, experimentally, both the slopes and curvatures of the membrane vary continually.

1.4 An Interesting Result of Existence

We need to define two functional spaces.

Definition 1 Let us consider $\Omega = [-A_1, A_1]$. In it, we define S and S_1 two functional spaces as follows [10]:

$$S = \{C_0^2(\Omega) : \ 0 < v(y) < 1 - \tau, \ |v'(y)| < M < +\infty\} \tag{1.22}$$

$$S_1 = \{C_0^1(\Omega) : \ 0 < v(y) < 1 - \tau, \ \ |v'(y)| < M < +\infty\} \tag{1.23}$$

Then, (1.19) admits an integral formulation by means of a suitable Green's function $\Xi(y, s)$ [14]. In other words, if

$$v(y) = \int_{-A_1}^{A_1} \Xi(y, s) f(s, v(s), v'(s)) ds, \quad 0 < v < 1 - \tau \tag{1.24}$$

and

$$v'(y) = \int_{-A_1}^{A_1} \Xi_x(y, s) f(s, v(s), v'(s)) ds \tag{1.25}$$

then (1.21) becomes

$$v(y) = \int_{-A_1}^{A_1} \Xi(y, s)(1 + (v'(s))^2)^3)(\varrho_1 \mu^2(s, v(s), \lambda))^{-1} ds. \tag{1.26}$$

To prove the existence of the solution of the problem (1.21), we demonstrate the existence of the solution for the equation

$$\Pi(v) = \omega, \tag{1.27}$$

with $v \in S_1$, exploiting a procedure based on Schauder-Tychonoff fixed point Theorem, from S to S with the operator Π:

$$\Pi(v(y)) = \int_{-A_1}^{A_1} \Xi(y, s)((1 + (v'(s))^2)^3 (\varrho_1 \mu^2(s, v(s), \lambda)^{-1} ds; \tag{1.28}$$

then, we can write

$$\Pi'(v(y)) = \int_{-A_1}^{A_1} \Xi_y(y, s)((1 + (v'(s))^2)^3 (\varrho_1 \mu^2(s, v(s), \lambda))^{-1} ds. \tag{1.29}$$

For our aims, we use the following Green's function [14, 17]:

$$\Xi(y,s)=\begin{cases}(s+A_1)(A_1-y)(2A_1)^{-1} & -A_1\le s\le y\\(A_1-s)(y+A_1)(2A_1)^{-1} & y\le s\le A_1\end{cases}\tag{1.30}$$

from which

$$\Xi_y(y,s)=\begin{cases}-(s+A_1)(2A_1)^{-1} & -A_1\le s\le y\\(A_1-s)(2A_1)^{-1} & y\le s\le A_1.\end{cases}\tag{1.31}$$

By (1.30) and (1.31) we deduce the following properties: (1) $\Xi(y,s)$ is non-negative and continuous; $max(\Xi(y,s))=\Xi(y=s,s=0)=A_1/2$ so that

$$0\le\Xi(y,s)\le 0.5A_1\quad\forall y,s\in\Omega;\tag{1.32}$$

(2) $\int_{-A_1}^{A_1}\Xi(y,s)ds=0.5(A_1-y)(y+A_1)\le 0.5(A_1^2)$ and $\left|\int_{-A_1}^{A_1}\Xi_x(y,s)ds\right|\le$ $\le\int_{-A_1}^{A_1}|\Xi_x(y,s)|ds\le A_1$; (3) finally, $\forall y,s\in(\Omega\times\Omega)$

$$\Xi_x(y,s)\le 0.5;\tag{1.33}$$

useful for the proof of the existence of the solution to the problem (1.21). For this purpose, we premise the following [10]:

Lemma 1 *$\Pi(v)$ defined in (1.28) is an operator from S to S.*

Proof Let us consider:

$$\begin{aligned}||\Pi(v(y))||_{C^2(\Omega)}&=sup_{y\in\Omega}|\Pi(v(y))|+\\+sup_{y\in\Omega}|\Pi'(v(y))|&+sup_{y\in\Omega}|\Pi''(v(y))|<+\infty.\end{aligned}\tag{1.34}$$

Owing the structure of $\Xi(y,s)$, we infer that $\Pi(v)\ge 0$ and $\Pi(v(-A_1))=\Pi(v(A_1))=0$. In addition, from the (1.13) and considering that $|\mathbf{E}|$ for deforming the membrane must win its inertia, it follows that $\mu(y,v(y),\lambda)>1$. Indicating by $\overline{\lambda}>0$ the minimum V necessary to win that inertia, we can write $\overline{\lambda}<\lambda<sup\{\lambda\}<+\infty$ from which $1/\lambda^2<+\infty$. Then, exploiting condition (1.32), it follows:

$$0 \le |\Pi(v(y))| \le sup_{y\in\Omega}|\Pi(v(y))| = sup_{y\in\Omega}\left|\int_\Omega \Xi(y,s)(\varrho_1\mu^2)^{-1}(1+(v'(s))^2)^3)ds\right| \le$$

$$\le (\varrho_1\lambda^2)^{-1}sup_{y\in\Omega}\left|\int_{-A_1}^{y}(2A_1)^{-1}(s+A_1)(A_1-y)(1+(v'(s))^2)^3(1-\tau-v(s))^2ds\right|+$$

$$+(\varrho_1\lambda^2)^{-1}sup_{y\in\Omega}\left|\int_{y}^{A_1}(2A_1)^{-1}(A_1-s)(y+A_1)(1+(v'(s))^2)^3(1-\tau-v(s))^2ds\right| =$$

$$= (1-\tau)(\varrho_1\lambda^2)^{-1}\left\{sup_{y\in\Omega}\left|\int_{-A_1}^{y}(2A_1)^{-1}(s+A_1)(A_1-y)(1+(v'(s))^2)^3ds+\right.\right.$$

$$\left.\left.+\int_{y}^{A_1}(2A_1)^{-1}(A_1-s)(y+A_1)(1+(v'(s))^2)^3ds\right|\right\} \le$$

$$\le 4(1-\tau)(\varrho_1\lambda^2)^{-1}(1+M^6)sup_{y\in\Omega}\left\{\int_{-A_1}^{y}(2A_1)^{-1}(s+A_1)(A_1-y)ds+\right.$$

$$\left.+\int_{y}^{A_1}(2A_1)^{-1}(A_1-s)(y+A_1)ds\right\} \le 4(1-\tau)(\varrho_1\lambda^2)^{-1}(1+M^6)A_1^2 < +\infty. \tag{1.35}$$

In addition:

$$sup_{y\in\Omega}|\Pi'(v(y))| = sup_{y\in\Omega}\left|\int_\Omega \Xi_x(y,s)(\varrho_1\mu^2)^{-1}(1+(v'(s))^2)^3)ds\right| =$$

$$= (\varrho_1\lambda^2)^{-1}sup_{y\in\Omega}\left|\int_{-A_1}^{x} -(2A_1)^{-1}(s+A_1)(1+(v'(s))^2)^3\right.$$

$$\left.(1-\tau-v(s))^2ds + \int_{x}^{A_1} -(2A_1)^{-1}(A_1-s)(1+(v'(s))^2)^3(1-\tau-v(s))^2ds\right| \le$$

$$\le 4(1-\tau)(\varrho_1\lambda^2)^{-1}(1+M^6)sup_{y\in\Omega}\left|\int_{-A_1}^{x} -(2A_1)^{-1}(s+A_1)ds+\right.$$

$$\left.\int_{x}^{A_1}(2A_1)^{-1}(A_1-s)ds\right| \le 4(1-\tau)(\varrho_1\lambda^2)^{-1}(1+M^6)A_1 < +\infty. \tag{1.36}$$

To evaluate $sup_{y\in\Omega}|\Pi''(v(y))|$, taking into account (1.29), (1.31), (1.33) and considering that $|v'| \le M$ and $|1/\mu^2| < 1$, we write:

$$sup_{y\in\Omega}|\Pi''(v(y))| = sup_{y\in\Omega}\left|\frac{d}{dy}\int_{\Omega}\Xi_x(y,s)(\varrho_1\mu^2)^{-1}(1+(v'(s))^2)^3 ds\right| =$$
$$\leq (2\varrho_1\lambda^2)^{-1}\Bigg[sup_{y\in\Omega}\left|\frac{d}{dy}\int_{-A_1}^{y}(1+(v'(s))^2)^3)ds\right| +$$
$$+sup_{y\in\Omega}\left|\frac{d}{dy}\int_{y}^{A_1}(1+(v'(s))^2)^3)ds\right|\Bigg] = (2\varrho_1\lambda^2)^{-1}\Big[2sup_{y\in\Omega}(1+(v'(s)^2)^3\Big] \leq$$
$$\leq (2\varrho_1\lambda^2)^{-1}2(1+M^2)^3 = (\varrho_1\lambda^2)^{-1}(1+M^2)^3 < +\infty; \tag{1.37}$$

from which, substituting (1.35), (1.36) and (1.37) into (1.34), we can write:

$$||\Pi(v(y))||_{C^2(\Omega)} \leq 4(1-\tau)(\varrho_1\lambda^2)^{-1}(1+M^6)A_1^2+ \tag{1.38}$$
$$+4(1-\tau)(\varrho_1\lambda^2)^{-1}(1+M^6)A_1 + (\varrho_1\lambda^2)^{-1}(1+M^2)^3 < +\infty.$$

To verify that $\Pi(u) \in S$, from the (1.35), we must to assume:

$$4(1-\tau)(\varrho_1\lambda^2)^{-1}(1+\mathrm{M}^6)A_1^2 < 1-\tau, \tag{1.39}$$

then

$$1+M^6 < (4A_1^2)^{-1}\varrho_1\overline{\lambda^2} \Rightarrow M < [(4A_1^2)^{-1}\varrho_1\overline{\lambda^2}-1]^{1/6}; \tag{1.40}$$

since both (1.36) and (1.40) are verified, we can consider the following system:

$$\begin{cases} 1+M^6 < M(4(1-\tau)A_1)^{-1}\varrho_1\overline{\lambda^2} \\ 1+M^6 < (4A_1^2)^{-1}\varrho_1\overline{\lambda^2}. \end{cases} \tag{1.41}$$

We observe that, if in (1.41), by absurd, we write:

$$(4A_1^2)^{-1}\varrho_1\overline{\lambda^2} < M(4(1-\tau)A_1)^{-1}\varrho_1\overline{\lambda^2}, \tag{1.42}$$

then, the inequality $M > \frac{1-\tau}{A_1} = 2(1-\tau)$ is verified so that, since $M = \frac{\zeta}{y}$ and $M' = \frac{\zeta'}{y'}$, by means of suitable scaling procedure, it is right to write as follows (see, Sect. 1.2):

$$M = \frac{\zeta}{y} = \frac{\zeta'}{h}\frac{2A}{y'} = H'\frac{2A}{h} > 2(1-\tau'). \tag{1.43}$$

We know that $1-\tau = \frac{1-\tau'}{h}$ then, considering (1.43), $M' > \frac{1-\tau'}{A}$ and, if $A \to 0$, we infer that $\frac{1-\tau'}{A} \to +\infty$ so that $M' = sup|v'| = +\infty$ achieving an absurd. Then, it follows that:

$$(4A_1^2)^{-1}\varrho_1\overline{\lambda}^2 > M(4(1-\tau)A_1)^{-1}\varrho_1\overline{\lambda}^2 \tag{1.44}$$

deducing that system (1.41) is equivalent to the following inequality

$$1+M^6 < (4(1-\tau)A_1)^{-1}M\varrho_1\overline{\lambda}^2 \tag{1.45}$$

to verify that $\Pi(v) : S \to S$.

It is interesting to note that Lemma 1 provides the condition (1.45) that demonstrates the dependence of M on ϱ_1; in other words, M depends on the electromechanical properties of the material constituting the membrane. Finally, applying the previous Lemma 1, we prove the existence of at least a solution for the problem (1.21).

Theorem 1 *Problem* (1.21) *admits at least one solution in the functional space S.*

Proof Always considering $\Omega = [-A_1, A_1]$, taking into account Lemma 1 and, in addition, the verification of both the compact immersions $C_0^2(\Omega) \hookrightarrow C_0^1(\Omega)$ and $S_1 \hookrightarrow S$, by application of the fixed-point Theorem (Schauder-Tychonoff), it follows that, in the functional space S_1, $v = \Pi(s)$ admits at least a fixed point ($v = \Pi(v)$). In other terms, problem (1.21) admits at least a solution.

1.5 On the Uniqueness of the Solution

In this section we prove that problem (1.21) admits unique solution. Additionally, as below specified, the uniqueness does not depend on the material properties of the membrane. In the proof of Theorem 2 this property is justified. In addition, give use important supplementary knowledge.

Theorem 2 $\forall M > 0$, *the uniqueness of the solution v is ensured for problem* (1.21). *Moreover,* $v \in C^\infty(\Omega)$ *and it is symmetric with respect to the origin and*

$$\forall y \in \Omega, \quad |v'(y)| \le |v'(A_1)| = |v'(-A_1)|. \tag{1.46}$$

Proof First, let us prove the inequality (1.46). Considering problem (1.21), we observe that $v"(y) \le 0$ where $y \in \Omega$ (concavity downwards) and therefore, from its equation, we can write:

$$v"(y)([1+(v'(y))^2]^3)^{-1} = -(\varrho_1\overline{\lambda^2})^{-1}[1-\tau-v(y)]^2; \tag{1.47}$$

that, multiplying both member by $v'(y)$, we write:

$$v"(y)\, v'(y)([1+(v'(y))^2]^3)^{-1} = -(\varrho_1\overline{\lambda^2})^{-1}[1-\tau-v(y)]^2\, v'(y) = \\ = -(\varrho_1\overline{\lambda^2})^{-1}(1-\tau)^2 v'(y) + \\ +(\varrho_1\overline{\lambda^2})^{-1}(1-\tau)\frac{d}{dy}[v(y)]^2 - (3\varrho_1\overline{\lambda^2})^{-1}\frac{d}{dy}[v(y)]^3. \tag{1.48}$$

Observing again that

$$v"(y)\, v'(y)([1+(v'(y))^2]^2)^{-1} = -\frac{1}{4}\frac{d}{dy}(1+[v'(y)]^3)^2)^{-1} \tag{1.49}$$

and integrating (1.48), we can write:

$$-\frac{1}{4}(1+[v'(A_1)]^2)^2)^{-1} + \frac{1}{4}(1+[v'(-A_1)]^2)^2)^{-1} = 0, \tag{1.50}$$

from which, the inequality $|v'(-A_1)| = |v'(A_1)|$ holds. Moreover, integrating equality (1.48) from $-A_1$ to t, taking into account that $v(-A_1) = 0$, it is right to write:

$$-\frac{1}{4}(1+[v'(t)]^2)^2)^{-1} + \frac{1}{4}(1+[v'(-A_1)]^2)^2)^{-1} = -(\varrho_1\overline{\lambda}^2)^{-1}(1-\tau)^2 v'(t) + \\ +(\varrho_1\overline{\lambda}^2)^{-1}(1-\tau)\frac{d}{dt}[v(t)]^2 - (3\varrho_1\overline{\lambda}^2)^{-1}\frac{d}{dt}[v(t)]^3. \tag{1.51}$$

Therefore, $\forall t \in [-A_1, A_1]$, we achieve:

$$-(\varrho_1\overline{\lambda}^2)^{-1}(1-\tau)^2 v(t) + (\varrho_1\overline{\lambda}^2)^{-1}(1-\tau)[v(t)]^2 - (3\varrho_1\overline{\lambda}^2)^{-1}[v(t)]^3 = \\ = (\varrho_1\overline{\lambda}^2)^{-1} v(t)\left\{(1-\tau)[v(t)-(1-\tau)] - \frac{1}{3}[v(t)]^2\right\} < 0$$

from which:

$$-\frac{1}{4}(1+[v'(t)]^2)^3)^{-1} + \frac{1}{4}(1+[v'(-A_1)]^2)^3)^{-1} < 0 \tag{1.52}$$

so that, we obtain $\forall t \in \Omega \quad |v'(t)| < |v'(-A_1)|$. With the aim of proving that problem (1.21) admits unique solution, by absurd, we suppose that in S_1 there are two different solutions: v_1 and v_2. By integration, from problem (1.21) and $\forall t \in \Omega$, we can write:

$$v_1'(t) \le M - (\varrho_1\overline{\lambda}^2)^{-1}\int_{-A_1}^{t}[1+(v_1'(y))^2]^3[1-\tau-v_1(y)]^2 dy$$

$$v_2'(t) \leq M - (\varrho_1 \overline{\lambda}^2)^{-1} \int_{-A_1}^{t} [1 + (v_2'(y))^2]^3 [1 - \tau - v_2(y)]^2 dy.$$

After, subtracting on both members, always $\forall t \in \Omega$, we obtain :

$$v_1'(t) - v_2'(t) = (\varrho_1 \overline{\lambda}^2)^{-1} \int_{-A_1}^{t} \{[1 + (v_2'(y)^2]^3 [1 - \tau - v_2(y)]^2 - \\ -[1 + (v_1'(y))^2]^3 [1 - \tau - v_1(y)]^2\} dy. \quad (1.53)$$

To evaluate the term inside the integral, we make the following functions, labeled by F and g, as follows:

$$F(w, v) = [1 + w^2]^3 (1 - \tau - v)^2, \\ g(t) = F(w_t, v_t) = F(tw_1 + (1-t)w_2, \ \ tv_1 + (1-t)v_2) \quad (1.54)$$

so that:

$$g'(t) = F_w(w_t, v_t)(w_1 - w_2) + F_v(w_t, v_t)(v_1 - v_2)$$

and

$$g(1) = F(w_1, v_1), \quad g(0) = F(w_2, v_2) \\ g(1) - g(0) = g'(y), \ y \in (0, 1). \quad (1.55)$$

Simple calculations lead to the following writing:

$$F_w(w_y, v_y) = 6[1 + w_y^2]^2 w_y (1 - \tau - v_y)^2 = \\ = 6\{1 + [yw_1 + (1-y)w_2]^2\}^2 [yw_1 + (1-y)w_2](1 - \tau - v_y)^2 \leq \\ \leq 6\{y[1 + w_1^2]^2 + (1-y)[1 + w_2^2]^2\}[yw_1 + (1-y)w_2](1 - \tau - v_y)^2.$$

Then, taking into account that $w_1 \leq M, \quad w_2 \leq M, \quad v_y \leq 1$, the following inequality holds:

$$\left| F_w(w_y, v_y) \right| \leq 24(1 + M^2)^2 M. \quad (1.56)$$

Therefore, it is easy to get the following inequality:

$$\left| F_v(w_y, v_y) \right| = | - 2[1 + (w_y)^2]^3 (1 - \tau - v_y)| \leq \\ \leq 2|y(1 + w_1^2)^3 + (1-y)(1 + w_2^2)^3| \leq 4(1 + M^2)^3$$

and, applying the Poincaré's inequality and (1.53), it make sense the following chain of inequalities

$$|v_1'(t) - v_2'(t)| \leq 24(\varrho_1 \overline{\lambda}^2)^{-1}(1+M^2)^2 M \int_{-A_1}^{t} |v_1'(y) - v_2'(y)|dy +$$
$$+4(\varrho_1 \overline{\lambda}^2)^{-1}(1+M^2)^3 \int_{-A_1}^{t} |v_1(y) - v_2(y)|dy \leq$$
$$\leq 24(\varrho_1 \overline{\lambda}^2)^{-1}(1+M^2)^2 M \int_{-A_1}^{t} |v_1'(y) - v_2'(y)|dy +$$
$$8A_1(\varrho_1 \overline{\lambda}^2)^{-1}(1+M^2)^3 \int_{-A_1}^{t} |v_1'(y) - v_2'(y)|dy =$$
$$\leq c(M, \overline{\lambda}, A_1, \varrho_1) \int_{-A_1}^{t} |v_1'(y) - v_2'(y)|dy.$$

From which, exploiting the Gronwall's Lemma [14], we infer:

$$\forall t \in \Omega, \quad |v_1'(t) - v_2'(t)| \leq 0 \tag{1.57}$$

and then

$$\forall t \in \Omega, \quad v_1'(t) - v_2'(t) = 0 \tag{1.58}$$

so that $v_1 - v_2 = constant$. In addition, owing

$$v_1(-A_1) = v_2(-A_1) = v_1(A_1) = v_2(A_1) = 0, \tag{1.59}$$

then $v_1 = v_2$ that is the thesis of the Theorem.

To prove the symmetry of v (with respect to the origin), we start to consider a solution v of the problem (1.21). Setting, $\forall t \in \Omega$, $u(t) = v(-t)$ we construct another solution (called u). In fact

$$u'(t) = -v'(-t), \tag{1.60}$$

$$u"(t) = v"(-t), \tag{1.61}$$

substituting both (1.60) and (1.61) in the equation of problem (1.21) we have

$$u"(-t) = -(\varrho_1 \overline{\lambda}^2)^{-1}([1 + (u'(-t))^2]^3)(1 - \tau - u(-t))$$

and exploiting that $u'(-A_1) = -v'(A_1) = v'(-A_1) \leq M$ since the uniqueness of the solution has been proved (then $v(t) = u(t)$), $\forall t \in \Omega$ we deduce that $u(t) = u(-t)$ over Ω.

Finally, for proving that $v \in C^{\infty}(\Omega)$ we observe that $v \in C^2(\Omega)$; then the member on the right of the equation belongs to $C^1(\Omega)$. So, exploiting the induction approach, $v \in C^{\infty}(\Omega)$, as expected.

1.6 Some Numerical Tests

As previously proved, system (1.41) is reducible to the inequality (1.45). To confirm that this reduction is correct also from the numerical point of view, some tests have been implemented exploiting MatLab® (Release 2017a). Particularly, setting in (1.40) $\overline{\lambda} = 1$ and $A_1 = 0.5$ (this is correct from the orders of amplitude point of view), H is smaller than a quantity whose $\mathscr{O}(10^{12})$. Then, it makes sense to write:

$$\begin{cases} 1 + M^6 < (M\varrho_1\overline{\lambda}^{-2})(4(1-\tau)A_1)^{-1} < (10^2 10^{12}\overline{\lambda}^{-2})(2(1-\tau))^{-1} \\ 1 + M^6 < (\varrho_1\overline{\lambda}^{-2})(4A_1^2)^{-1} < 10^{12}\overline{\lambda}^{-2}. \end{cases} \tag{1.62}$$

Moreover, owing $1 - \tau < 1$ and, in addition that

$$(\varrho_1\overline{\lambda}^{-2})(4A_1^2)^{-1} < (M\varrho_1\overline{\lambda}^{-2})(4(1-\tau)A_1)^{-1} \tag{1.63}$$

holds, (1.41) is equivalent to (1.45). Then, (1.62) can be reformulated in the following manner:

$$\begin{cases} f_1(M) = (M 10^{12}\overline{\lambda}^{-2})(2(1-\tau))^{-1} - (M^6 + 1) > 0 \\ f_2(M) = 10^{12}\overline{\lambda}^{-2}(1-\tau) - (M^6 + 1) > 0 \end{cases} \tag{1.64}$$

in which both $f_1(M)$ and $f_2(M)$ have been defined for the aim. Observing that (1.64) must be verified, exploiting the numerical Newton-Raphson's approach with the default tolerance, firstly we found both the zero values. In the follow, they have been considered as sup of the set of the value of M verifying (1.64). The obtained results, shown in both Fig. 1.2a, b, prove, in dimensionless conditions, the agreement of the analytical and numerical results . As an example, for $\overline{\lambda} = 1.02$ and select a suitable range of values of M (in which the numerical procedure is applicable) for both $f_1(M)$ and $f_2(M)$ (considering [220, 240] and [80, 100]) we obtain their zeros (M_1^* and M_2^*, respectively). Then, in order to guarantee the existence of the solution of the problem, we are obliged to choose $sup|M| = min(M_1^*, M_2^*)$ so that $sup|M| = min(M_1^*, M_2^*)_{numerical} = 98.2$. Value 98.2 corresponds, in dimensionless conditions, just a little higher to 87 degrees.

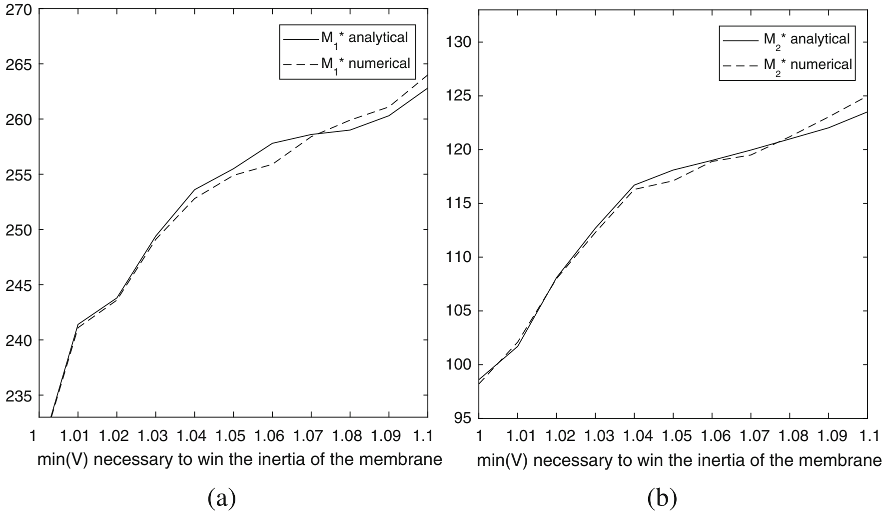

Fig. 1.2 Comparison between numerical and analytical results: (**a**) M_1^* analytical versus M_1^* numerical, (**b**) M_2^* analytical versus M_2^* numerical

References

1. Angiulli, G., Jannelli, A., Morabito, F.C., Versaci, M.: Reconstructing the membrane detection of a 1D electrostatic-driven MEMS device by the shooting method: convergence analysis and ghost solutions identification. Comput. Appl. Math. (2018). https://doi.org/10.1007/s40314-017-0564-4
2. Bhansali, S., Vasudev, A.: MEMS for biomedical applications. In: Woodhead Publishing Series in Biomaterials. Woodhead Publishing Limited. https://doi.org/10.1016/b978-0-85709-129-1.50020-1. ISBN: 978-0-85709-129-1 (2012)
3. Cacciola, M., La Foresta, F., Morabito, F.C., Versaci, M.: Advanced use of soft computing and Eddy current test to evaluate mechanical integrity of metallic plates. NDT E Int. **40**, 357–362 (2007)
4. Cacciola, M., Pellicanó, D., Megali, G., et al.: Aspect about air pollution prediction on urban environment. In: Proceedings of 4th IMEKO TC19 Symposium on Environmental Instrumentation and Measurement: Protection Environment, Climate Changes and Pollution Control
5. Cassani, D., Tarsia, A.: Periodic solutions to nonlocal MEMS equations. Discrete Continuous Dyn. Syst. Ser. S **9**(3), 631–642 (2016)
6. Cassani, D., d'O, M., Ghoussoub, N.: On a fourth order elliptic problem with a singular nonlinearity. Nonlinear Stud. Ser. **9**, 189–209 (2009)
7. Cassani, D., Fattorusso, L., Tarsia, A.: Nonlocal singular problems and application to MEMS. In: Proceedings of WCE 2013
8. Cassani, D., Fattorusso, L., Tarsia, A.: Nonlocal dynamic problems with singular nonlinearities and application to MEMS. Prog. Nonlinear Differ. Equ. Appl. **85**, 185–206 (2014)
9. Di Barba, P., Fattorusso, L., Versaci, M.: Electrostatic field in terms of geometric curvature in membrane MEMS devices. Commun. Appl. Ind. Math. **8**, 165–184 (2017)

10. Fattorusso, L., Versaci, M.: A new one-dimensional electrostatic model for membrane MEMS devices lecture notes. In: Engineering and Computer Science: Proceedings of the World Congress on Engineering 2018, London, vol. 1, pp. 50–55, 4–6 July 2018
11. Fazio, R., Jannelli, A.: Second order numerical operator splitting for 3D advection-diffusion-reaction models. In: Kreiss, G., Lotstedt, P., Malqvist, A., Neytcheva, M. (eds.) Numerical Mathematics and Advanced Applications, pp. 317–324. Springer, Berlin/Heidelberg (2009). https://doi.org/10.1007/978-3-642-11795-433
12. Fazio, R., Jannelli, A.: Finite difference schemes on quasi-uniform grids for BVPs on infinite intervals. J. Comput. Appl. Math. **269**, 14–23 (2014). https://doi.org/10.1016/j.cam.2014.02.036
13. Gad-el-Hak, M.: MEMS: Introduction and Fundamentals. Chapman & Hall/CRC Taylor & Francis Group, Boca Raton/London/New York (2005)
14. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin/Heidelberg/New York (2001)
15. Huja, M., Husak, M.: Thermal microactuators for optical purpose. Coding Comput **15**(1), 137–142 (2001)
16. Khoshnoud, F., Silva, C.W.: Recent advances in MEMS sensor technology – biomedical application. IEEE Instrum. Meas. Mag. **15**(1), 127–138 (1998)
17. Lang, S.: Green's functions on Riemann surfaces. In: Introduction to Arakelov Theory. Springer, New York (1988)
18. Lin, L., Chiao, M.: Electro, thermal and elastic characterization of suspended micro beams. Microelectron. J. **29**(4–5), 104–121 (1998)
19. Morabito, F.C., Versaci, M., Pautasso, G., Tichmann, C.: Fuzzy-Neural approaches to the prediction of disruptions in ASDEX upgrade. Nucl. Fusion **41**, 1715–1723 (2001)
20. Pelesko, J.A., Bernstein, D.H.: Modeling MEMS and NEMS. Chapman & Hall/CRC, Boca Raton/London/New York/Washington, DC (2003)
21. Reson Panat, P.V.: Contributions of maxwell to electromagnetism. Resonance **8**, 17–29 (2003)
22. Rogers, A.A., Samson, S., Kedia, S.: Far-Field evanescent wave propagation using coupled subwavelength gratings for a MEMS sensors. J. Opt. Soc. Am. A. Opt. Image Sci. Vis. **26**(12), 2526–2531 (2009)
23. Versaci, M., Angiulli, G., Fattorusso, L., Jannelli, A.: On the uniqueness of the solution for a semi-linear elliptic boundary value problem of the membrane MEMS device for reconstructing the membrane profile in absence of ghost solutions. Int. J. Non-Linear Mech. **109**, 24–31 (2019)

Chapter 2
Study of Changes of the Individual Parameter of Resources in the Modelling of Renewable Systems

Oleksandr Karelin, Anna Tarasenko, Oleksandr Barabash, Manuel Gonzalez-Hernandez, and Joselito Medina-Marin

Abstract Applying the principles of modelling of renewable resource systems proposed in previous works and the theory of functional operators with shift, we have obtained mathematical models that take into account the reciprocal influences between resources of the systems. If previously special attention has been paid to the study of the density distribution of the group parameters by individual parameters then now we present a study on the dynamics of the individual parameters of resources. A mathematical model for the study of the function of the individual parameter is elaborated. Balance relations are no longer integral equations, but differential equations. Based on these models, possibilities to formulate economic ecological problems that use renewable resource systems are opened.

Keywords Renewable resource systems · Cyclic model · Inverse operator · Degenerate kernel · Shift · Individual parameter

O. Karelin (✉) · M. Gonzalez-Hernandez · J. Medina-Marin
Department of Industrial Engineering, Institute of Basic Sciences and Engineering, Hidalgo State Autonomous University, Pachuca, Hgo, Mexico
e-mail: karelin@uaeh.edu.mx; mghdez@uaeh.edu.mx; jmedina@uaeh.edu.mx

A. Tarasenko
Department of Mathematics, Institute of Basic Sciences and Engineering, Hidalgo State University, Pachuca, Hgo, Mexico
e-mail: anataras@uaeh.edu.mx

O. Barabash
Department of Enterprise Economics, Institute of Business, Economics and Information Technology, Odessa National Politechnic University, Odessa, Ukraine
e-mail: odessaaa48@gmail.com

S.-I. Ao et al. (eds.), *Transactions on Engineering Technologies*,
https://doi.org/10.1007/978-981-32-9531-5_2

2.1 Introduction

In works [1, 2] principles of modelling were proposed and applied to the study of systems with one renewable resource. Cyclic models, where the initial state of the system coincides with the final state, were considered. The balance relation has the form

$$\nu(x) = D(x)v[\beta(x)] + P(x, \nu(x)) + g(x),$$

where $v(x)$ is the initial density of the elements of the system. Here, we take the process of natural mortality and the processes of changes of the individual parameter x into account with the coefficient $D(x)$ and the shift $\beta(x)$. The process of reproduction is represented by the term $P(x, v(x))$, the process of artificial entry of elements into the system and extraction from the system are accounted for by the term $g(x)$. Conditions for the existence and uniqueness of the solution are formulated.

In works [3, 4] we have continued the study of systems whose state depends on time and whose resources are renewable, using functional operators with shift. We made generalizations to systems with two resources. For functional operators with shift, inverse operators in weighted Holder spaces were constructed. In modelling, interactions and reciprocal influence between these two resources were taken into account. We applied our results on invertibility of the operators to the study of balance equations. The balance relation has the form

$$v(x) = D(x)v[\beta(x)] + P_v(x) + R_w(x) + g(x),$$

$$w(x) = C(y)w[\gamma(y)] + P_w(y) + R_v(y) + q(y),$$

where $v(x)$ and $w(y)$ represent initial densities of the distribution of the group parameters by the individual parameters and for the resources λ_1 and λ_2. The terms $P_v(x)$, $P_w(y)$ are responsible for the reproduction and the terms $R_w(x)$, $R_v(y)$ are responsible for the mutual influence and contain integrals with degenerate kernels:

$$P_v(x) = \sum_{i=1}^{n} \mathscr{P}_i p_i(x) \quad \mathscr{P}_1 = \int_{\upsilon_0}^{\upsilon_1} v(x)dx, \quad \mathscr{P}_2 = \int_{\delta_1}^{\upsilon_2} v(x)dx, \ldots, \mathscr{P}_n = \int_{\upsilon_{n-1}}^{\upsilon_n} v(x)dx,$$

$$P_w(y) = \sum_{i=1}^{m} \mathscr{Q}_i q_i(y) \quad \mathscr{Q}_1 = \int_{\mu_0}^{\mu_1} w(y)dy, \quad \mathscr{Q}_2 = \int_{\mu_1}^{\mu_2} w(y)dy, \ldots, \mathscr{Q}_m = \int_{\mu_{m-1}}^{\mu_m} w(y)dy,$$

$$0 = \upsilon_0 < \upsilon_1 < \ldots < \upsilon_n = x_{max}, \quad 0 = \mu_0 < \mu_1 < \ldots < \mu_m = y_{max};$$

$$R_v(x) = \sum_{i=1}^{k} \mathscr{R}_i r_i(x) \quad \mathscr{R}_1 = \int_{\vartheta_0}^{\vartheta_1} v(x)dx, \quad \mathscr{R}_2 = \int_{\vartheta_1}^{\vartheta_2} v(x)dx, \ldots, \mathscr{R}_k = \int_{\vartheta_{k-1}}^{\vartheta_k} v(x)dx,$$

$$R_\omega(y)=\sum_{i=1}^{l}\mathscr{S}_i q_i(y)\quad \mathscr{S}_1=\int_{\xi_0}^{\xi_1} w(y)dy,\quad \mathscr{S}_2=\int_{\xi_1}^{\xi_2} w(y)dy,\ldots,\mathscr{S}_l=\int_{\xi_{l-1}}^{\xi_l} w(y)dy,$$

$$0=\vartheta_0<\vartheta_1<\ldots<\vartheta_k=x_{max},\quad 0=\xi_0<\xi_1<\ldots<\xi_l=y_{max}.$$

For the solution of the balance integral equation with degenerate kernels and inverse operators, a modified Fredholm method [5, 6] for equations of second type is proposed. The equilibrium state of the system is found.

In these works, special attention was paid to the study of the density distribution of the group parameter by individual parameters. In [7], we presented a study on the dynamics of the individual parameters of resources. We carry out a detailed investigation of the graphics of the function of change of the individual parameter (weight) with time (cycles). We show the necessity of introducing the age parameter for elements of the system into consideration. We introduce an age parameter for elements of the system.

This paper is a continuation and expansion of article [7]. In Sect. 2.2 a cyclic model of a system with a set of renewable resources is presented. It is a generalisation from two resources to a finite number of resources. In Sect. 2.3 a mathematical model for study of the function of the changes of the individual parameter is proposed. This significantly expands the ability to analyse individual parameters of resources. Based on these models, possibilities to formulate economic ecological problems that involve renewable resource systems are opened.

A great number of works has been dedicated to systems with renewable resources [8, 9].

The core of the mathematical apparatus used for the study of such systems consists of differential equations in which the sought for function is dependent on time [10, 11].

Our approach presupposes discretization of the processes with respect to time. We move away from tracking the changes in the system continuously to tracking the changes at fixed time points. This discretization and the identification of the individual parameter and the group parameter lead us to functional equations with shift.

2.2 Cyclic Model of a System with a Set of Renewable Resources

Let S be a system with r resources $\lambda^1, \lambda^2, \ldots, \lambda^r$ and let T be a time interval. The choice of T is related to periodic processes taking place in the system and to human interferences.

Let these resources have individual parameters with scales

$x^1_{min}=x^1_1<x^1_2<\ldots<x^1_{n_1}=x^1_{max}$, for the first resource

$x^2_{min} = x^2_1 < x^2_2 < \ldots < x^2_{n_2} = x^2_{max}$, for the second resource

....

$x^r_{min} = x^r_1 < x^r_2 < \ldots < x^r_{n_r} = x^r_{max}$ for the r-th resource.

We introduce the group parameters by functions

$v^1(x^1_i, t),\ v^2(x^2_i, t),\ \ldots,\ v^r(x^r_i, t)$ which express a quantitative estimate of the elements of resources $\lambda^1, \lambda^2, \ldots, \lambda^r$ with the individual parameter $x^1_i, i = 1, 2, \ldots, n_1, x^2_i, i = 1, 2, \ldots, n_2, \quad \ldots, \quad x^r_i, i = 1, 2, \ldots, n_r$ at the time t.

Explain this through an example: the system with two fish resources λ^1 and λ^2. The weight is the individual parameter of the resource λ^1 ;

$$x^1_1 = 100\,\text{gr}, x^1_2 = 200\,\text{gr}, \ldots, x^1_{100} = 10000\,\text{gr}$$

are the values of this individual parameter. The number of fish with a fixed weight x^1_i is a group parameter,

$$v(x_i, t), i = 1, 2, \ldots, 100.$$

The length is the individual parameter of the resource λ^2;

$$x^2_1 = 5\,\text{cm}, x^2_2 = 10\,\text{cm}, \ldots, x^2_{20} = 100\,\text{cm}.$$

The total weight of fish with a fixed length y_i is a group parameter

$$w(x_i, t), i = 1, 2, \ldots, 20.$$

The function $v(x_i, t)$ is the number of fish of the weight x_i at the time t, the function $w(y_i, t)$ is the total weight of fish of the length y_i at the time t.

Passing from discrete description on to a continuous description we obtain the functions $v(x, t),\ w(y, t)$ which are the densities of the objects with the parameters $x,\ y$ at the time t.

Let t_0 be the initial time and S the system under consideration.

As in our previous work [1, 2] on modeling the system, we will hold the following principles:

I. The description of changes that occur on the interval $(t_0, t_0 + T)$ will be substituted by the fixing of the final results at the moment $t_0 + T$;
II. The separation of parameters into individual parameters, group parameters and the study of dependence of group parameters from individual parameters.

The initial state of system S at time t_0 is represented as density functions of a distribution of the group parameter by the individual parameter for each resource

$$v^1(x^1, t_0), x^1_{min} \le x^1 \le x^1_{max},$$

$$v^2(x^2, t_0), x^2_{min} \leq x^2 \leq x^2_{max},$$

....

$$v^r(x^r, t_0), x^r_{min} \leq x^r \leq x^r_{max},$$

which express a quantitative estimate of the elements of resources $\lambda^1, \lambda^2, \ldots, \lambda^r$ with the individual parameter $x^1, x^2, \ldots, x^r$.

If there are no doubts, then we omit t_0 and assume that the initial values of the individual parameters are zero:

$$v^1(x^1, t_0) = v^1(x^1), v^2(x^2, t_0) = v^2(x^2), \ \ldots,$$
$$v^r(x^r, t_0) = v^r(x^r); \quad x^1_{min} = x^2_{min} = \ldots = x^r_{min} = 0.$$

We will now analyse the system's evolution. In the course of time, the elements of the system can change their individual parameter – e.g. fish can change their weight and length.

Modifications in the distributions of the group parameters by the individual parameters are represented by displacements. The state of the system S at the time $t = t_0 + T$ is:

$$v^1(x^1, t_0+T) = \frac{d}{dx^1}\alpha^1(x^1)v^1(\alpha^1(x^1)), \ldots, v^r(x^r, t_0+T) = \frac{d}{dx^r}\alpha^r(x^r)v^r(\alpha^r(x^r)). \tag{2.1}$$

In Sect. 2.3, the appearance of derivatives will be explained.

In the case of two resources λ^1, λ^2 with the individual parameters $x = x^1$, $y = x^1$, the functions of density $v(x) = v^1(x^1)$, $w(x) = v^2(x^2)$ and the displacements $\alpha(x) = \alpha^1(x^1)$, $\beta(x) = \alpha^2(x^2)$, relation (2.1) will look like

$$\left(v(x, t_0 + T) = \frac{d}{dx}\alpha(x) \cdot v(\alpha(x)), \quad w(y, t_0 + T) = \frac{d}{dy}\beta(y) \cdot w(\beta(y)).\right)$$

In the sequel, we will also use parentheses to distinguish a particular case of a system with two resources Further, we will also use parentheses to distinguish between a particular case of a system with two resources.

Over the period $j_0 = [t_0, t_0 + T]$, extractions might be taken from the system as a result of human economic activity; these are represented by summands $\rho^j(x^j)$; if an artificial entrance of elements into the system has taken place, it shall be accounted for by adding terms $\zeta^j(x^j)$; we take natural mortality into account with coefficients $d^j(x^j)$, $1 \leq j \leq r$.

The process of reproduction will be represented by terms

$$\sum_{i=1}^{n_r} P_i^j p_i^j(x^j), \quad 1 \leq j \leq r, \qquad \left(\sum_{i=1}^{n} P_i p_i(x), \quad \sum_{i=1}^{m} Q_i q_i(y),\right)$$