CLASSICS IN MATHEMATICS

Lars Hörmander

The Analysis of Linear Partial Differential Operators IV

Fourier Integral Operators

Springer

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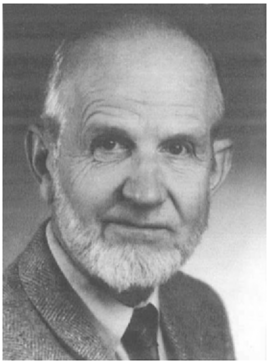

Born on January 24, 1931, on the southern coast of Sweden, Lars Hörmander did his secondary schooling as well as his undergraduate and doctoral studies in Lund. His principal teacher and adviser at the University of Lund was Marcel Riesz until he retired, then Lars Gårding. In 1956 he worked in the USA, at the universities of Chicago, Kansas, Minnesota and New York, before returning to a chair at the University of Stockholm. He remained a frequent visitor to the US, particularly to Stanford and was Professor at the IAS, Princeton from 1964 to 1968. In 1968 he accepted a chair at the University of Lund, Sweden, where, today, he is Emeritus Professor.

Hörmander's lifetime work has been devoted to the study of partial differential equations and its applications in complex analysis. In 1962 he was awarded the Fields Medal for his contributions to the general theory of linear partial differential operators. His book *Linear Partial Differential Operators* published 1963 by Springer in the Grundlehren series was the first major account of this theory. His four volume text *The Analysis of Linear Partial Differential Operators*, published in the same series 20 years later, illustrates the vast expansion of the subject in that period.

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Lars Hörmander

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Fourier Integral Operators

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Preface

to Volumes III and IV

The first two volumes of this monograph can be regarded as an expansion and updating of my book "Linear partial differential operators" published in the Grundlehren series in 1963. However, volumes III and IV are almost entirely new. In fact they are mainly devoted to the theory of linear differential operators as it has developed after 1963. Thus the main topics are pseudo-differential and Fourier integral operators with the underlying symplectic geometry. The contents will be discussed in greater detail in the introduction.

I wish to express here my gratitude to many friends and colleagues who have contributed to this work in various ways. First I wish to mention Richard Melrose. For a while we planned to write these volumes together, and we spent a week in December 1980 discussing what they should contain. Although the plan to write the books jointly was abandoned and the contents have been modified and somewhat contracted, much remains of our discussions then. Shmuel Agmon visited Lund in the fall of 1981 and generously explained to me all the details of his work on long range scattering outlined in the Goulaouic-Schwartz seminars 1978/79. His ideas are crucial in Chapter XXX. When the amount of work involved in writing this book was getting overwhelming Anders Melin lifted my spirits by offering to go through the entire manuscript. His detailed and constructive criticism has been invaluable to me; I as well as the readers of the book owe him a great debt. Bogdan Ziemian's careful proofreading has eliminated numerous typographical flaws. Many others have also helped me in my work, and I thank them all.

Some material intended for this monograph has already been included in various papers of mine. Usually it has been necessary to rewrite these papers completely for the book, but selected passages have been kept from a few of them. I wish to thank the following publishers holding the copyright for granting permission to do so, namely:

Marcel Dekker, Inc. for parts of [41] included in Section 17.2;

Princeton University Press for parts of [38] included in Chapter XXVII;

D. Reidel Publishing Company for parts of [40] included in Section 26.4;

John Wiley & Sons Inc. for parts of [39] included in Chapter XVIII.

(Here [N] refers to Hörmander [N] in the bibliography.)

Finally I wish to thank the Springer-Verlag for all the support I have received during my work on this monograph.

Djursholm in November, 1984 — Lars Hörmander

Contents

Introduction

to Volumes III and IV

A great variety of techniques have been developed during the long history of the theory of linear differential equations with variable coefficients. In this book we shall concentrate on those which have dominated during the latest phase. As a reminder that other earlier techniques are sometimes available and that they may occasionally be preferable, we have devoted the introductory Chapter XVII mainly to such methods in the theory of second order differential equations. Apart from that Volumes III and IV are intended to develop systematically, with typical applications, the three basic tools in the recent theory. These are the theory of pseudo-differential operators (Chapter XVIII), Fourier integral operators and Lagrangian distributions (Chapter XXV), and the underlying symplectic geometry (Chapter XXI). In the choice of applications we have been motivated mainly by the historical development. In addition we have devoted considerable space and effort to questions where these tools have proved their worth by giving fairly complete answers.

Pseudo-differential operators developed from the theory of singular integral operators. In spite of a long tradition these played a very modest role in the theory of differential equations until the appearance of Calderón's uniqueness theorem at the end of the 1950's and the Atiyah-Singer-Bott index theorems in the early 1960's. Thus we have devoted Chapter XXVIII and Chapters XIX, XX to these topics. The early work of Petrowsky on hyperbolic operators might be considered as a precursor of pseudo-differential operator theory. In Chapter XXIII we discuss the Cauchy problem using the improvements of the even older energy integral method given by the calculus of pseudo-differential operators.

The connections between geometrical and wave optics, classical mechanics and quantum mechanics, have a long tradition consisting in part of heuristic arguments. These ideas were developed more systematically by a number of people in the 1960's and early 1970's. Chapter XXV is devoted to the theory of Fourier integral operators which emerged from this. One of its first applications was to the study of asymptotic properties of eigenvalues (eigenfunctions) of higher order elliptic operators. It is therefore discussed in Chapter XXIX here together with a number of later developments which give beautiful proofs of the power of the tool. The study by Lax of the propagation of singularities of solutions to the Cauchy problem was one of the forerunners of the theory. We prove such results using only pseudo-differential operators in Chapter XXIII. In Chapter XXVI the propagation of singularities is discussed at great length for operators of principal type. It is the only known approach to general existence theorems for such operators. The completeness of the results obtained has been the reason for the inclusion of this chapter and the following one on subelliptic operators. In addition to Fourier integral operators one needs a fair amount of symplectic geometry then. This topic, discussed in Chapter XXI, has deep roots in classical mechanics but is now equally indispensible in the theory of linear differential operators. Additional symplectic geometry is provided in the discussion of the mixed problem in Chapter XXIV, which is otherwise based only on pseudo-differential operator theory. The same is true of Chapter XXX which is devoted to long range scattering theory. There too the geometry is a perfect guide to the analytical constructs required.

The most conspicuous omission in these books is perhaps the study of analytic singularities and existence theory for hyperfunction solutions. This would have required another volume - and another author. Very little is also included concerning operators with double characteristics apart from a discussion of hypoellipticity in Chapter XXII. The reason for this is in part shortage of space, in part the fact that few questions concerning such operators have so far obtained complete answers although the total volume of results is large. Finally, we have mainly discussed single operators acting on scalar functions or occasionally determined systems. The extensive work done on for example first order systems of vector fields has not been covered at all.

Chapter XXV. Lagrangian Distributions and Fourier Integral Operators

Summary

In Section 18.2 we introduced the space of conormal distributions associated with a submanifold Y of a manifold X. This is a natural extension of the classical notion of multiple layer on Y. All such distributions have their wave front sets in the normal bundle of Y which is a conic Lagrangian manifold. In Section 25.1 we generalize the notion of conormal distribution by defining the space of Lagrangian distributions associated with an arbitrary conic Lagrangian $\Lambda \subset T^*(X) \setminus 0$. This is the space of distributions u such that there is a fixed bound for the order of $P_1 \dots P_N u$ for any sequence of first order pseudo-differential operators $P_1, \dots, P_N$ with principal symbols vanishing on Λ. This implies that $WF(u) \subset \Lambda$. Symbols can be defined for Lagrangian distributions in much the same way as for conormal distributions. The only essential difference is that the symbols obtained are half densities on the Lagrangian tensored with the Maslov bundle of Section 21.6.

In Section 25.2 we introduce the notion of Fourier integral operator; this is the class of operators having Lagrangian distribution kernels. As in the discussion of wave front sets in Section 8.2 (see also Section 21.2) it is preferable to associate a Fourier integral operator with the canonical relation $\subset (T^*(X) \setminus 0) \times (T^*(Y) \setminus 0)$ obtained by twisting the Lagrangian with reflection in the zero section of $T^*(Y)$. We prove that the adjoint of a Fourier integral operator associated with the canonical relation C is associated with the inverse of C, and that the composition of operators associated with C_1 and C_2 is associated with the composition $C_1 \circ C_2$ when the compositions are defined. Precise results on continuity in the $H_{(s)}$ spaces are proved in Section 25.3 when the canonical relation is the graph of a canonical transformation. We also study in some detail the case where the canonical relation projects into $T^*(X)$ and $T^*(Y)$ with only fold type of singularities.

The real valued C^∞ functions vanishing on a Lagrangian $\subset T^*(X) \setminus 0$ form an ideal with $\dim X$ generators which is closed under Poisson brackets. We define general Lagrangian ideals by taking complex valued functions instead. With suitable local coordinates in X they always have a local system

of generators of the form

$$x_j - \partial H(\xi)/\partial \xi_j, \quad j=1,\dots,n,$$

just as in the real case. The ideal is called positive if $\operatorname{Im} H \leqq 0$. This condition is crucial in the analysis and turns out to have an invariant meaning. Distributions associated with positive Lagrangian ideals are studied in Section 25.4. The corresponding Fourier integral operators are discussed in Section 25.5. The results are completely parallel to those of Sections 25.1, 25.2 and 25.3 apart from the fact that for the sake of brevity we do not extend the notion of principal symbol.

25.1. Lagrangian Distributions

According to Definition 18.2.6 the space $I^m(X, Y; E)$ of conormal distribution sections of the vector bundle E is the largest subspace of ${}^\infty H^{loc}_{(-m-n/4)}(X, E)$, $n = \dim X$, which is left invariant by all first order differential operators tangent to the submanifold Y. It follows from Theorem 18.2.12 that it is even invariant under all first order pseudo-differential operators from E to E with principal symbol vanishing on the conormal bundle of Y. The definition is therefore applicable with no change to any Lagrangian manifold:

Definition 25.1.1. Let X be a C^∞ manifold and $\Lambda \subset T^*(X) \setminus 0$ a C^∞ closed conic Lagrangian submanifold, E a C^∞ vector bundle over X. Then the space $I^m(X, \Lambda; E)$ of Lagrangian distribution sections of E, of order m, is defined as the set of all $u \in \mathscr{D}'(X, E)$ such that

$$L_1 \dots L_N u \in {}^\infty H^{loc}_{(-m-n/4)}(X, E) \tag{25.1.1}$$

for all N and all properly supported $L_j \in \Psi^1(X; E, E)$ with principal symbols L_j^0 vanishing on Λ.

The following lemma allows us to localize the study of $I^m(X, \Lambda; E)$.

Lemma 25.1.2. *If $u \in I^m(X, \Lambda; E)$ then $WF(u) \subset \Lambda$, and $Au \in I^m(X, \Lambda; E)$ if $A \in \Psi^0(X; E, E)$. Conversely, $u \in I^m(X, \Lambda; E)$ if for every $(x_0, \xi_0) \in T^*(X) \setminus 0$ one can find $A \in \Psi^0(X; E, E)$ properly supported and non-characteristic at (x_0, ξ_0) such that $Au \in I^m(X, \Lambda; E)$.*

Proof. If $(x_0, \xi_0) \notin \Lambda$ we can choose $L_1, \dots, L_N$ in (25.1.1) non-characteristic in a conic neighborhood Γ and conclude that $u \in H^{loc}_{(s)}$ in Γ if $s < N - m - n/4$. Hence $WF(u) \cap \Gamma = \emptyset$. To prove the second statement we observe that

$$L_1 \dots L_N A u = L_1 \dots L_{N-1} A L_N u - L_1 \dots L_{N-1} [A, L_N] u.$$

Here $[A, L_N] \in \Psi^0(X; E, E)$ and $L_N u \in I^m(X, \Lambda; E)$ by Definition 25.1.1. By induction with respect to N we conclude that

$$L_1 \dots L_N A u \in {}^\infty H^{\mathrm{loc}}_{(-m-n/4)}(X, E)$$

for all properly supported $A \in \Psi^0(X; E, E)$ and $L_j \in \Psi^1(X; E, E)$ with principal symbols vanishing on Λ. To prove the converse we choose B according to Lemma 18.1.24 so that $(x_0, \xi_0) \notin WF(BA - I)$. Thus $(x_0, \xi_0) \notin WF(BAu - u)$, and since $BAu \in I^m(X, \Lambda; E)$ it follows that

$$L_1 \dots L_N u \in {}^\infty H^{\mathrm{loc}}_{(-m-n/4)} \quad \text{at } (x_0, \xi_0)$$

if $L_1, \dots, L_N$ satisfy the conditions in Definition 25.1.1. Hence (25.1.1) is fulfilled so $u \in I^m(X, \Lambda; E)$.

Remark. So far we have not used that Λ is Lagrangian. However, if (25.1.1) is fulfilled we have $[L_j, L_k]^N u \in {}^\infty H^{\mathrm{loc}}_{(-m-n/4)}(X, E)$ for any N, so $WF(u)$ is contained in the characteristic set of $[L_j, L_k]$ by the first part of the proof. Hence $WF(u)$ cannot contain an arbitrary point in Λ unless Λ is involutive. The hypothesis that Λ is Lagrangian means that Λ is minimal with this property, or alternatively that we have a maximal set of conditions (25.1.1) which do not imply that u is smooth.

Lemma 25.1.2 reduces the study of distributions $u \in I^m(X, \Lambda; E)$ to the case where $WF(u)$ is contained in a small closed conic neighborhood Γ_0 of some point $(x_0, \xi_0) \in \Lambda$, and supp u is close to x_0. In that case Definition 25.1.1 is applicable even if Λ is just defined in an open conic neighborhood Γ_1 of Γ_0, for only the restriction of the principal symbol of L_j to Γ_1 is relevant. More generally, given a conic Lagrangian submanifold Λ of the open cone $\Gamma_1 \subset T^*(X) \setminus 0$ we shall say that $u \in I^m(X, \Lambda; E)$ at $(x_0, \xi_0) \in \Gamma_1$ if there is an open conic neighborhood $\Gamma_0 \subset \Gamma_1$ of (x_0, ξ_0) such that $Au \in I^m(X, \Lambda; E)$ for all properly supported $A \in \Psi^0$ with $WF(A) \subset \Gamma_0$; it suffices to know this for some such A which is non-characteristic at (x_0, ξ_0).

In view of Theorem 21.2.16 we may thus assume now that $X = \mathbb{R}^n$ and that $\Lambda = \{(H'(\xi), \xi); \xi \in \mathbb{R}^n \setminus 0\}$ where H is a real valued function in $C^\infty(\mathbb{R}^n \setminus 0)$ which is homogeneous of degree 1. We may also assume that E is the trivial bundle, which is then omitted from the notation.

Proposition 25.1.3. *If $u \in I^m_{\mathrm{comp}}(\mathbb{R}^n, \Lambda)$, $\Lambda = \{(H'(\xi), \xi);\ \xi \in \mathbb{R}^n \setminus 0\}$, then $\hat{u}(\xi) = e^{-iH(\xi)} v(\xi)$, $|\xi| > 1$, where $v \in S^{m-n/4}(\mathbb{R}^n)$. Conversely, the inverse Fourier transform of $e^{-iH} v$ is in $I^m(\mathbb{R}^n, \Lambda)$ if $v \in S^{m-n/4}(\mathbb{R}^n)$.*

Proof. Choose $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 in a neighborhood of 0 and define h by $\hat{h} = \chi \hat{H}_0$ where $H_0 = (1-\chi)H$. Then $\hat{H}_0 - \hat{h} \in \mathscr{S}$ (see the proof of Theorem 7.1.22), so $H_0 - h \in \mathscr{S}$. Thus $h \in S^1$ has the principal symbol H, so it suffices to prove the result with H replaced by h. Set $h_j(\xi) = \partial h(\xi)/\partial \xi_j$. The operator $h_j(D)$ is convolution with the inverse Fourier transform of h_j so it is properly supported. Hence

(25.1.2) $$D^\beta \prod (x_j - h_j(D))^{\alpha_j} u \in {}^\infty H_{(-m-n/4)} \quad \text{if } |\beta| = |\alpha|$$

for $[x_j - h_j(D), D_k] = i\delta_{jk}$ so commuting the factors D^β we obtain a sum of products of operators of the form $(x_j - h_j(D))D_k$ to which (25.1.1) is applicable. Recalling the definition of ${}^\infty H_{(-m-n/4)}$ we obtain

$$\int_{R/2<|\xi|<2R} |\xi^\beta \prod (-D_j - h_j(\xi))^\alpha \hat{u}(\xi)|^2 d\xi \leqq C_\alpha R^{2(m+n/4)}; \qquad R>1,\ |\beta| = |\alpha|.$$

With the notation $\hat{u}(\xi) = e^{-ih(\xi)} v(\xi)$ this means that

$$\int_{R/2<|\xi|<2R} |\xi|^{2|\alpha|} |D^\alpha v(\xi)|^2 d\xi \leqq C_\alpha R^{2(m+n/4)}.$$

If $v_R(\xi) = v(R\xi)/R^{m-n/4}$ then

$$\int_{\frac{1}{2}<|\xi|<2} |D^\alpha v_R(\xi)|^2 d\xi \leqq C_\alpha$$

which by Lemma 7.6.3 gives uniform bounds for $D^\alpha v_R$ when $|\xi| = 1$, that is, bounds for $|D^\alpha v(\xi)|(1+|\xi|)^{|\alpha|-m+n/4}$. The argument can be reversed to prove the last statement in the proposition, for the passage from the operators $(x_j - h_j(D))D_k$ to the general operators in (25.1.1) can be made by the argument preceding Theorem 18.2.7.

A slight modification of the proof gives precise information about the smoothness of elements in I^m. We state the result directly in a global form.

Theorem 25.1.4. *If $U \in I^m(X, \Lambda)$ and $U \in H_{(s_0)}$ at $(x_0, \xi_0) \in \Lambda$, then $U \in I^\mu(X, \Lambda)$ at (x_0, ξ_0) if $\mu + s_0 + n/4 > 0$.*

Proof. Choose $A \in \Psi^0(X)$ properly supported, non-characteristic at (x_0, ξ_0), so that $AU \in H_{(s_0)}$. By Lemma 25.1.2 we have $AU \in I^m$. We can choose A so that $WF(AU)$ is in a small conic neighborhood of (x_0, ξ_0). Writing $u = AU$ we conclude that it is sufficient to prove that $u \in I^\mu$ if $u \in H_{(s_0)}$ and u satisfies the hypotheses in Proposition 25.1.3. With the notation used there we have

$$\int_{\frac{1}{2}<|\xi|<2} |D^\alpha v_R(\xi)|^2 d\xi \leqq C_\alpha, \qquad \int_{\frac{1}{2}<|\xi|<2} |v_R(\xi)|^2 d\xi \leqq C R^{-2(s_0+m+n/4)}.$$

Let $|\xi| = 1$ and set $V_{R,\xi}(\eta) = v_R(\xi + \eta/R^\delta) R^{-n\delta/2}$ where $\delta > 0$. Then

$$\int_{|\eta|<1} |D^\alpha V_{R,\xi}(\eta)|^2 d\eta \leqq C_\alpha R^{-2|\alpha|\delta}, \qquad \int_{|\eta|<1} |V_{R,\xi}(\eta)|^2 d\eta \leqq C R^{-2(s_0+m+n/4)}.$$

Now we use the Sobolev inequality

$$|D^\beta V(0)|^2 \leqq C_\beta \int_{|\eta|<1} \Big(\sum_{|\alpha|=s} |D^{\alpha+\beta} V(\eta)|^2 + |V(\eta)|^2 \Big) d\eta$$

where $s > n/2$. This is somewhat more general than (7.6.6) but follows from the same proof. Taking s so large that $s\delta > s_0 + m + n/4$ we obtain

$$|D^\beta V_{R,\xi}(0)| \leqq C' R^{-(s_0+m+n/4)},$$

hence

$$|D^\beta v_R(\xi)| \leqq C' R^{\delta(n/2+|\beta|)-(s_0+m+n/4)}, \qquad |\xi|=1,$$
$$|D^\beta v(\xi)| \leqq C' |\xi|^{\delta(n/2+|\beta|)-(s_0+n/2+|\beta|)}, \qquad |\xi|>1.$$

For every β we can choose δ so that the exponent is smaller than $\mu-n/4-|\beta|$, and then we obtain $v\in S^{\mu-n/4}$, hence $u\in I^\mu$.

We shall now prove that elements in $I^m(X,\Lambda)$ can also be represented by means of arbitrary phase functions ϕ parametrizing Λ in the sense of Definition 21.2.15. At first we assume that ϕ is non-degenerate.

Proposition 25.1.5. *Let $\phi(x,\theta)$ be a non-degenerate phase function in an open conic neighborhood of $(x_0,\theta_0)\in\mathbb{R}^n\times(\mathbb{R}^N\setminus 0)$ which parametrizes the Lagrangian manifold Λ in a neighborhood of (x_0,ξ_0); $\xi_0=\phi'_x(x_0,\theta_0)$, $\phi'_\theta(x_0,\theta_0)=0$. If $a\in S^{m+(n-2N)/4}(\mathbb{R}^n\times\mathbb{R}^N)$ has support in the interior of a sufficiently small conic neighborhood Γ of (x_0,θ_0), then the oscillatory integral*

$$u(x)=(2\pi)^{-(n+2N)/4}\int e^{i\phi(x,\theta)}a(x,\theta)\,d\theta \tag{25.1.3}$$

defines a distribution $u\in I^m_{\text{comp}}(\mathbb{R}^n,\Lambda)$. If $\Lambda=\{(H'(\xi),\xi)\}$ as in Proposition 25.1.3 then (for $|\xi|>1$)

$$e^{iH(\xi)}\hat{u}(\xi)-(2\pi)^{n/4}a(x,\theta)\,e^{\pi i/4\,\operatorname{sgn}\Phi}|\det\Phi|^{-\frac{1}{2}}\in S^{m-n/4-1} \tag{25.1.4}$$

where (x,θ) is determined by $\phi'_\theta(x,\theta)=0$, $\phi'_x(x,\theta)=\xi$, and

$$\Phi=\begin{pmatrix}\phi''_{xx} & \phi''_{x\theta}\\ \phi''_{\theta x} & \phi''_{\theta\theta}\end{pmatrix}.$$

Here $a(x,\theta)$ is interpreted as 0 if there is no such point in Γ. $e^{iH(\xi)}\hat{u}(\xi)$ is polyhomogeneous if a is. Conversely, every $u\in I^m(X,\Lambda)$ with $WF(u)$ in a small conic neighborhood of (x_0,ξ_0) can, modulo C^∞, be written in the form (25.1.3).

In the proof we shall need an extension of Lemma 18.1.18.

Lemma 25.1.6. *Let $\Gamma_j\subset\mathbb{R}^{n_j}\times(\mathbb{R}^{N_j}\setminus 0)$, $j=1,2$, be open conic sets and let $\psi:\Gamma_1\to\Gamma_2$ be a C^∞ proper map commuting with multiplication by positive scalars in the second variable. If $a\in S^m(\mathbb{R}^{n_2}\times\mathbb{R}^{N_2})$ has support in the interior of a compactly based cone $\subset\Gamma_2$ then $a\circ\psi\in S^m(\mathbb{R}^{n_1}\times\mathbb{R}^{N_1})$ if the composition is defined as 0 outside Γ_1.*

Proof. The support of $a\circ\psi$ belongs to a compactly based cone $\subset\Gamma_1$ where $\psi(x,\xi)=(y,\eta)$ implies $|\xi|/C<|\eta|<C|\xi|$. The hypothesis on a means that

$$|D^\alpha_{y,\eta}a(y,t\eta)|\leqq C_\alpha t^m, \qquad 1/C<|\eta|<C.$$

Since $a\circ\psi(x,t\xi)=a(.,t.)\circ\psi(x,\xi)$ by the homogeneity of ψ, we obtain

$$|D^\alpha_{x,\xi}(a\circ\psi)(x,t\xi)|\leqq C'_\alpha t^m, \qquad |\xi|=1,$$

by using Leibniz' rule. This proves the lemma.

Proof of Proposition 25.1.5. By hypothesis $\phi'_x(x_0,\theta_0)=\xi_0 \neq 0$, so the oscillatory integral (25.1.3) is well defined. u has compact support if Γ has a compact base. We shall use the method of stationary phase to evaluate

$$e^{iH(\xi)}\hat{u}(\xi)=(2\pi)^{-(n+2N)/4}\iint e^{i(\phi(x,\theta)+H(\xi)-\langle x,\xi\rangle)}a(x,\theta)\,dx\,d\theta. \tag{25.1.5}$$

The exponent has a critical point if

$$\phi'_x(x,\theta)=\xi, \qquad \phi'_\theta=0,$$

which by hypothesis means that $(x,\xi)\in\Lambda$, hence that $x=H'(\xi)$. The critical point is non-degenerate. In fact, the maps

$$C=\{(x,\theta);\ \phi'_\theta=0\}\ni(x,\theta)\mapsto(x,\phi'_x)\in\Lambda \quad\text{and}\quad \Lambda\ni(x,\xi)\mapsto\xi$$

are diffeomorphisms. Hence $C \ni (x,\theta)\mapsto \phi'_x$ is a diffeomorphism, so $d\phi'_x=d\phi'_\theta=0$ implies $dx=d\theta=0$. The matrix Φ is therefore non-singular. If we divide (multiply) the first n (last N) rows (columns) by $|\theta|$ we see that $\det\Phi$ is homogeneous in θ of degree $n-N$. Hence $a(x,\theta)|\det\Phi|^{-\frac{1}{2}}$ is in $S^{m-n/4}$ in a conic neighborhood of C. By Lemma 25.1.6 this remains true for the restriction to C regarded as a function of ξ.

It follows from Theorem 7.7.1 that there is a constant C such that for any N

$$\Big|\int e^{i(\phi(x,\theta)-\langle x,\xi\rangle)}a(x,\theta)\,dx\Big|\leqq C_N(|\xi|+|\theta|)^{-N}, \quad \text{if } |\theta|>C|\xi| \ \text{ or } \ |\xi|>C|\theta|. \tag{25.1.6}$$

In fact, $(\phi(x,\theta)-\langle x,\xi\rangle)/(|\xi|+|\theta|)=f(x)$ is homogeneous in (ξ,θ) of degree 0 and bounded in C^∞. If $(x,\theta)\in\operatorname{supp} a$ we have

$$|f'(x)|\geqq(|\xi|-C_1|\theta|)/(|\xi|+|\theta|)\geqq\tfrac{1}{2} \qquad \text{if } |\theta|/|\xi| \text{ is small,}$$
$$|f'(x)|\geqq(C_2|\theta|-|\xi|)/(|\xi|+|\theta|)>C_2/2 \qquad \text{if } |\xi|/|\theta| \text{ is small.}$$

We can therefore apply Theorem 7.7.1 with $\omega=|\xi|+|\theta|$.

Choose $\chi\in C_0^\infty(\mathbb{R}^N\setminus 0)$ equal to 1 when $1/C<|\theta|<C$. By (25.1.6) the difference between $e^{iH(\xi)}\hat{u}(\xi)$ and

$$U(\xi)=(2\pi)^{-(n+2N)/4}\iint e^{i(\phi(x,\theta)+H(\xi)-\langle x,\xi\rangle)}\chi(\theta/|\xi|)\,a(x,\theta)\,dx\,d\theta$$

is rapidly decreasing as $\xi\to\infty$. Set $|\xi|=t$, $\xi/t=\eta$ and replace θ by $t\theta$. Then

$$U(\xi)=(2\pi)^{-(n+2N)/4}\iint e^{it(\phi(x,\theta)+H(\eta)-\langle x,\eta\rangle)}\chi(\theta)\,a(x,t\theta)t^N\,dx\,d\theta.$$

Here the exponent has only one critical point in the support of the integrand and it is defined by $\phi'_\theta(x,\theta)=0$, $\phi'_x(x,\theta)=\eta$. At that point

$$\phi(x,\theta)=\langle\theta,\phi'_\theta(x,\theta)\rangle=0, \qquad \langle x,\eta\rangle=\langle H'(\eta),\eta\rangle=H(\eta)$$

so the critical value is 0. Using (7.7.13) we obtain an asymptotic expansion of U. Since $\chi=1$ at the critical point, the leading term is

$$(2\pi)^{n/4}\,a(x,t\theta)\,t^{(N-n)/2}\,e^{\pi i/4\,\operatorname{sgn}\Phi}\,|\det\Phi|^{-\frac{1}{2}},$$

that is, the term displayed in (25.1.4) in view of the homogeneity of $\det \Phi$ already pointed out. The k^{th} term will contain another factor t^{-k} and a linear combination of derivatives of $a(x,t\theta)$ with respect to x, θ, so it is in $S^{m-n/4-k}$. In view of Proposition 18.1.4 it follows that we have an asymptotic series in the sense of Proposition 18.1.3, and this completes the proof of the first part of the proposition.

To prove the converse it is by Proposition 25.1.3 sufficient to consider an element $u \in I^m(X,\Lambda)$ with $v=\hat{u}e^{iH} \in S^{m-n/4}$ having support in a small conic neighborhood of ξ_0. Choose a C^∞ map $(x,\theta)\mapsto \psi(x,\theta)\in \mathbb{R}^n\setminus 0$ in a conic neighborhood of (x_0,θ_0) such that ψ is homogeneous of degree 1 and $\psi(x,\theta) = \partial\phi/\partial x$ when $\partial\phi/\partial\theta=0$. Let

$$a_0(x,\theta)=(2\pi)^{-n/4} v\circ\psi(x,\theta)\, e^{-\pi i/4\,\mathrm{sgn}\,\Phi}\,|\det \Phi|^{\frac{1}{2}} \in S^{m+(n-2N)/4}$$

near C, and define u_0 by (25.1.3) with a replaced by a_0. From the first part of the proposition it follows then that $u-u_0 \in I^{m-1}$. Repeating the argument gives a sequence $a_j \in S^{m+(n-2N)/4-j}$ such that $u-u_0-\ldots-u_j \in I^{m-j-1}$ if u_j is defined by (25.1.3) with a replaced by a_j. If a is an asymptotic sum of the series $\sum a_j$ it follows that (25.1.3) is valid modulo C^∞. The proof is complete.

We shall now examine what must be changed in the preceding argument if ϕ is just a clean phase function. We still have (25.1.6) so only $U(\xi)$ is important. However, $\phi(x,\theta)+H(\eta)-\langle x,\eta\rangle$ does not satisfy the hypotheses in Theorem 7.7.6. We do know that (locally)

$$C=\{(x,\theta);\ \partial\phi(x,\theta)/\partial\theta=0\}$$

is a manifold of dimension $e+n$, where e is the excess, and that the composed map $C\to\Lambda\to\mathbb{R}^n$: $(x,\theta)\mapsto(x,\phi'_x)\mapsto\phi'_x$ has surjective differential, hence a fiber C_η of dimension e over η where $x=H'(\eta)$. The critical points of $\phi(x,\theta)+H(\eta)-\langle x,\eta\rangle$ are defined by $\phi'_\theta=0$, $\phi'_x=\eta$, that is, $(x,\theta)\in C_\eta$, and $d\phi'_\theta=0$, $d\phi'_x=0$ precisely along the tangent space of C_η. Note that we have fixed upper and lower bounds for $|\theta|$ on C_η since $|\phi'_x|=1$. We can split the θ variables into two groups θ', θ'' so that the number of θ'' variables is e and the projection $C_\eta \ni (x,\theta)\mapsto\theta''$ has bijective differential. Then $d\phi'_\theta=0$, $d\phi'_x=0$, $d\theta''=0$ implies $dx=d\theta=0$. Thus the Hessian of $\phi(x,\theta)+H(\eta)-\langle x,\eta\rangle$ with respect to (x,θ') is not 0, so the critical point on C_η when θ'' is fixed is non-degenerate. If we change the definition of Φ to

$$\Phi=\begin{pmatrix}\phi''_{xx} & \phi''_{x\theta'}\\ \phi''_{\theta' x} & \phi''_{\theta'\theta'}\end{pmatrix},$$

an application of Theorem 7.7.6 to the integral $U(\xi)$ with respect to the $n+N-e$ variables x, θ' gives, when we integrate with respect to θ'' afterwards,

$$e^{iH(\xi)}\hat{u}(\xi)-(2\pi)^{n/4-e/2}\int_{C_\eta} t^{(N+e-n)/2}\, a(x,t\theta)\, e^{\pi i/4\,\mathrm{sgn}\,\Phi}\,|\det\Phi|^{-\frac{1}{2}}\,d\theta'' \in S^{m+e/2-n/4-1}.$$

Note that the order has increased by $e/2$ since the stationary phase evaluation is applied to e variables less. For the same reason a factor $(2\pi)^{e/2}$ is lost. If we introduce $t\theta$ as a new variable, noting that $\det \Phi$ is homogeneous of degree $n-N+e$ now, we obtain

Proposition 25.1.5'. *Let $\phi(x,\theta)$ be a clean phase function with excess e in an open conic neighborhood of $(x_0,\theta_0)\in\mathbb{R}^n\times(\mathbb{R}^N\setminus 0)$ which parametrizes the Lagrangian manifold Λ in a neighborhood of (x_0,ξ_0); $\xi_0=\phi'_x(x_0,\theta_0)$, $\phi'_\theta(x_0,\theta_0)=0$. If $a\in S^{m+(n-2N-2e)/4}(\mathbb{R}^n\times\mathbb{R}^N)$ has support in the interior of a sufficiently small conic neighborhood Γ of (x_0,θ_0) then the oscillatory integral*

$$u(x)=(2\pi)^{-(n+2N-2e)/4}\int e^{i\phi(x,\theta)}a(x,\theta)\,d\theta \tag{25.1.3'}$$

defines a distribution $u\in I^m_{\text{comp}}(\mathbb{R}^n,\Lambda)$. If $\Lambda=\{(H'(\xi),\xi)\}$ as in Proposition 25.1.3 then

$$e^{iH(\xi)}\hat{u}(\xi)-(2\pi)^{n/4}\int_{C_\xi} a(x,\theta)\,e^{\pi i/4\,\operatorname{sgn}\Phi}\,|\det\Phi|^{-\frac12}\,d\theta''\in S^{m-n/4-1}. \tag{25.1.4'}$$

Here $C_\xi=\{(x,\theta);\ \phi'_\theta(x,\theta)=0,\ \phi'_x(x,\theta)=\xi\}$; $\theta=(\theta',\theta'')$ is a splitting of the θ variables in two groups such that $C_\xi\ni(x,\theta)\mapsto\theta''$ has bijective differential; and

$$\Phi=\begin{pmatrix}\phi''_{xx} & \phi''_{x\theta'}\\ \phi''_{\theta'x} & \phi''_{\theta'\theta'}\end{pmatrix}.$$

Conversely, modulo C^∞ every $u\in I^m(X,\Lambda)$ with $WF(u)$ in a small conic neighborhood of (x_0,ξ_0) can be written in the form (25.1.3)'.

Remark. If $f\in C^\infty(Y)$ has a critical point at $y_0\in Y$ then $|\det f''(y_0)|^{\frac12}$ transforms as a density at y_0. This is why in the standard stationary phase formula the density in the integrand is transformed to a scalar in the asymptotic expansion. If on the other hand f is critical on a submanifold $Z\subset Y$ and is non-degenerate in transversal directions, then the square root of the determinant of the Hessian in transversal planes defines a density in the normal bundle. Dividing a density in Y by it gives a density on Z. This confirms the invariant meaning of the integrand in (25.1.4)'.

There is no difficulty in performing a change of local coordinates x in the representation (25.1.3) of an element in $I^m(X,\Lambda)$, so Proposition 25.1.3 contains all that is needed to define a principal symbol isomorphism for I^m extending Theorem 18.2.11. However, it is instructive to establish first a theorem on limits of elements in I^m which connects the definitions in this section with those given in the linear case in Section 21.6.

Proposition 25.1.7. *Let $u\in I^m_{\text{comp}}(\mathbb{R}^n,\Lambda)$, $\Lambda=\{(H'(\xi),\xi),\ \xi\in\mathbb{R}^n\setminus 0\}$, and set $e^{iH}\hat{u}=(2\pi)^{n/4}v$, $v\in S^{m-n/4}$. If $\psi\in C^\infty(\mathbb{R}^n)$ is real valued, $\psi(x_0)=0$, $\psi'(x_0)=\xi_0\neq 0$, $(x_0,\xi_0)\in\Lambda$, then as $t\to+\infty$*

$$t^{-2m-n/2}(ue^{-it^2\psi})(x_0+x/t)-v(t^2\xi_0)t^{-2m+n/2}u^{\psi}_{x_0,\xi_0}(x)\to 0 \quad \text{in } \mathscr{D}', \tag{25.1.7}$$

where

$$(25.1.8) \qquad u^{\psi}_{x_0,\xi_0}(x)=(2\pi)^{-3n/4}\int \exp iQ^{\psi}_{x_0,\xi_0}(x,\xi)\,d\xi,$$
$$Q^{\psi}_{x_0,\xi_0}(x,\xi)=\langle x,\xi\rangle-\langle\psi''(x_0)x,x\rangle/2-\langle H''(\xi_0)\xi,\xi\rangle/2.$$

Note that the factor $t^{-n/2}$ in the left-hand side of (25.1.7) means that $ue^{-it^2\psi}$ is pulled back as a half density by the map $x\mapsto x_0+x/t$. The other factor t^{-2m} reflects that u is of order m and is examined near the frequency $t^2\xi_0$.

Proof of Proposition 25.1.7. By Fourier's inversion formula

$$u(x)=(2\pi)^{-3n/4}\int e^{i(\langle x,\xi\rangle-H(\xi))}v(\xi)\,d\xi.$$

Replacing ξ by $t^2\xi_0+t\xi$ we obtain if $\chi\in C_0^\infty$

$$(25.1.9) \qquad t^{-2m-n/2}\langle(ue^{-it^2\psi})(x_0+./t),\chi\rangle$$
$$=(2\pi)^{-3n/4}\iint e^{iE_t(x,\xi)}v(t^2\xi_0+t\xi)\,t^{-2m+n/2}\chi(x)\,dx\,d\xi,$$

where

$$E_t(x,\xi)=\langle x_0+x/t,\ t^2\xi_0+t\xi\rangle-t^2\psi(x_0+x/t)-t^2H(\xi_0+\xi/t).$$

Now $H(\xi_0)=\langle H'(\xi_0),\xi_0\rangle=\langle x_0,\xi_0\rangle$, $H'(\xi_0)=x_0$, $\psi(x_0)=0$, $\psi'(x_0)=\xi_0$, so

$$E_t(x,\xi)=Q^{\psi}_{x_0,\xi_0}(x,\xi)+O(1/t)$$

uniformly on compact sets. Hence

$$(25.1.10) \qquad \int e^{iE_t(x,\xi)}\chi(x)\,dx \to \int \exp(iQ^{\psi}_{x_0,\xi_0}(x,\xi))\,\chi(x)\,dx$$

uniformly for ξ in a compact set. If $x\in\operatorname{supp}\chi$ then

$$|\partial E_t(x,\xi)/\partial x|=|t\xi_0+\xi-t\psi'(x_0+x/t)|\geqq|\xi|-C\geqq(|\xi|+1)/2$$

if $|\xi|>2C+1$, so Theorem 7.7.1 shows that the left-hand side of (25.1.10) has a bound $C_N(1+|\xi|)^{-N}$ independent of t, for every N. Thus

$$(2\pi)^{-3n/4}\iint e^{iE_t(x,\xi)}\chi(x)\,dx\,d\xi \to \langle u^{\psi}_{x_0,\xi_0},\chi\rangle,$$

and (25.1.7) follows if we show that for large N

$$\int|v(t^2\xi_0+t\xi)-v(t^2\xi_0)|\,t^{-2m+n/2}(1+|\xi|)^{-N}\,d\xi\to 0, \qquad t\to\infty.$$

The integrand can be estimated by

$$|t\xi|(t^2)^{m-n/4-1}t^{-2m+n/2}(1+|\xi|)^{-N}\leqq t^{-1}|\xi|(1+|\xi|)^{-N},$$

if $|\xi|<t|\xi_0|/2$, so this part of the integral is $O(1/t)$. When $|\xi|>t|\xi_0|/2$ the bound $(1+|\xi|)^{4|m|+n/2-N}$ for the integrand is obvious, which completes the proof.

Remark. If $u\in\mathscr{D}'(\mathbb{R}^n)$, $(x_0,\xi_0)\notin WF(u)$ and $0\neq\xi_0=\psi'(x_0)$, then

$$t^N(ue^{-it^2\psi})(x_0+x/t)\to 0 \quad \text{in } \mathscr{D}'$$

for every N. In fact, replacing u by χu where $\chi\in C_0^\infty$ is 1 in a neighborhood of x_0 and supported in another small neighborhood we can assume that $u\in\mathscr{E}'$ and that $\hat{u}$ is rapidly decreasing in a conic neighborhood of ξ_0. We may also assume that $\psi(x)=\langle x,\xi_0\rangle$ for if ρ vanishes of second order at x_0 then $t^2\rho(x_0+x/t)\to\langle\rho''(x_0)x,x\rangle/2$, $t\to\infty$. The Fourier transform of $t^{N+1}(ue^{-it^2\psi})(x_0+x/t)$ is then $t^{N+n+1}\hat{u}(t^2\xi_0+t\xi)e^{it\langle x_0,\xi\rangle}$ which is bounded when $|\xi|/t$ is small, and uniformly bounded by a power of $(1+|\xi|)$ elsewhere.

If $v\in S_{\text{phg}}^{m-n/4}$ it follows from (25.1.7) that

$$t^{-2m-n/2}(ue^{-it^2\psi})(x_0+x/t)\to v_0(\xi_0)\,u_{x_0,\xi_0}^{\psi}(x)\quad\text{in }\mathscr{D}', \tag{25.1.7$'$}$$

where v_0 is the principal symbol of v. At first sight it might seem that the limit is strongly tied to the specific local coordinates x, but in fact it is not:

Lemma 25.1.8. *Let u_t be distributions in a neighborhood of 0 in $\mathbb{R}^n$ such that $M_t^*u_t\to U$ in $\mathscr{D}'$ as $t\to 0$, where $M_t(x)=tx$. If θ is a local diffeomorphism at 0 with $\theta(0)=0$, it follows then that*

$$M_t^*\theta^*u_t\to\theta_0^*U,\qquad t\to 0,$$

where $\theta_0(x)=\theta'(0)x$ is the differential of θ at 0.

Proof. We can write $M_t^*\theta^*u_t=M_t^*\theta^*M_{1/t}^*M_t^*u_t$. Since

$$M_{1/t}\circ\theta\circ M_t(x)=t^{-1}\theta(tx)\to\theta_0(x)$$

in C^∞ as $t\to 0$, it follows that $M_t^*\theta^*u_t\to\theta_0^*U$.

The existence of the limit U is thus coordinate independent. If we regard u as a distribution on a manifold, the limit is a distribution on the tangent space at 0. If u_t is transformed as a half density distribution under a change of variables, we obtain of course a factor $|\det\theta'(0)|^{\frac{1}{2}}$, so the limit is a half density on the tangent space.

Let us now return to (25.1.7)$'$ where $v\in S_{\text{phg}}^{m-n/4}$ and v_0 is the principal symbol. If u is thought of as a half density $u(x)\,|dx|^{\frac{1}{2}}$ in a manifold X, expressed in the local coordinates x, we conclude that the limit $v_0(\xi_0)\,u_{x_0,\xi_0}^{\psi}$ is a half density on the tangent space $T_{x_0}(X)$. In the tangent space $S=T_{x_0,\xi_0}(T^*(X))$ the tangent planes λ_1 and λ of the graphs of ψ' and of Λ are given by $\xi=\psi''(x_0)x$ and $x=H''(\xi_0)\xi$ in our local coordinates. In S the tangent space of the fiber defined by $x=0$ is a distinguished Lagrangian plane λ_0. If we compare (25.1.8) with (21.6.5) and (21.6.6) it follows that $v(\xi_0)u_{x_0,\xi_0}^{\psi}\in I(\lambda,\lambda_1)$ defines an element in $I(\lambda)$ independent of the choice of ψ, hence an element in the tensor product $M_\lambda\otimes\Omega_\lambda^{\frac{1}{2}}$ where M_λ is the fiber over λ of the Maslov bundle defined on $T^*(X)$ by the tangents of the fibers, and $\Omega_\lambda^{\frac{1}{2}}$ is the fiber of the half density bundle on λ. With the trivialization of the Maslov bundle given by the Lagrangian planes $\xi=0$ in the local coordinates used in Propositions 25.1.5 and 25.1.7, the half density in the tangent space at $(H'(\xi_0), \xi_0) \in \Lambda$ is $v(\xi_0)|d\xi|^{\frac{1}{2}}$ when ξ parametrizes Λ by $x = H'(\xi)$. Thus we obtain an invariant definition of a section of $M_\Lambda \otimes \Omega_\Lambda^{\frac{1}{2}}$ which is homogeneous of degree $m + n/4$. (For the definition of homogeneity see the discussion preceding Definition 18.2.10.) It will be called the *principal symbol* of u.

The preceding discussion motivates our definition of the principal symbol for general $u \in I^m(X, \Lambda)$, but it will actually only depend on Proposition 25.1.5. As a preliminary to the definition we must extend Definition 18.2.10 to symbols on conic manifolds. On any conic manifold V there is defined a multiplication M_t by real numbers $t > 0$, satisfying the conditions in Definition 21.1.8. We define $S^m(V)$ as the set of all $a \in C^\infty(V)$ such that the functions $t^{-m} M_t^* a$ are uniformly bounded in $C^\infty(V)$ when $t \geqq 1$. If for some compact set $K \subset V$ the support of a is contained in $\bigcup_{t \geqq 1} M_t K$, and V is an open subset of $\mathbb{R}^n \times (\mathbb{R}^N \setminus 0)$ with M_t defined as multiplication by t in the second variable, then the proof of Lemma 25.1.6 shows that this definition agrees with our earlier ones. An advantage of the present definition is that it is applicable also if say a is a half density on V. Let a_0 be a fixed positive half density on V which is homogeneous of degree μ, that is, $M_t^* a_0 = t^\mu a_0$. For example, if $V = \mathbb{R}^n \times (\mathbb{R}^N \setminus 0)$ with variables x, θ, then $|dx|^{\frac{1}{2}} |d\theta|^{\frac{1}{2}}$ is a half density which is homogeneous of degree $\mu = N/2$. We can now write every $a \in S^m(V, \Omega_V^{\frac{1}{2}})$ in the form $a = a_0 b$ where $b \in S^{m-\mu}(V)$ is a scalar symbol, and conversely all such products are in $S^m(V, \Omega_V^{\frac{1}{2}})$.

We return now to the definition of the principal symbol of a general $u \in I^m(X, \Lambda)$ where X is a C^∞ manifold and $\Lambda \subset T^*(X) \setminus 0$ is a C^∞ conic Lagrangian manifold. For any $(x_0, \xi_0) \in \Lambda$ we can choose local coordinates x at x_0 such that a conic neighborhood Γ of (x_0, ξ_0) in Λ is defined in the local coordinates by $x = H'(\xi)$ as in Proposition 25.1.3. If $\Gamma_1 \subset \Gamma$ is a compactly generated cone we can use Lemma 25.1.2 to split u into a sum $u_1 + u_2$ where $u_j \in I^m(X, \Lambda)$ and $WF(u_1) \subset \Gamma$, $WF(u_2) \cap \Gamma_1 = \emptyset$. We can take u_1 with compact support in the coordinate patch. For the Fourier transform in the local coordinates we have by Proposition 25.1.3

$$e^{iH(\xi)} \hat{u}_1(\xi) = (2\pi)^{n/4} v(\xi) \in S^{m-n/4}. \tag{25.1.11}$$

If $u = \tilde{u}_1 + \tilde{u}_2$ is another decomposition with the same properties, we have $WF(\tilde{u}_1 - u_1) \cap \Gamma_1 = \emptyset$. Since $WF(\tilde{u}_1 - u_1) \subset \Gamma$ and $\Gamma_1 = \{(H'(\xi), \xi), \xi \in \gamma_1\}$ for some closed cone $\gamma_1 \subset \mathbb{R}^n \setminus 0$ we conclude that the Fourier transform of $\tilde{u}_1 - u_1$ is rapidly decreasing in a conic neighborhood of γ_1. Hence the class of v in $S^{m-n/4}(\gamma_1)/S^{-\infty}(\gamma_1)$ does not depend on the decomposition of u, and we can consider $v|d\xi|^{\frac{1}{2}}$ as an element in $S^{m+n/4}(\Gamma_1, \Omega_{\Gamma_1}^{\frac{1}{2}})/S^{-\infty}(\Gamma_1, \Omega_{\Gamma_1}^{\frac{1}{2}})$ in view of the isomorphism $\gamma_1 \ni \xi \mapsto (H'(\xi), \xi) \in \Gamma_1$. We shall now study to what extent the residue class $\operatorname{mod} S^{m+n/4-1}$ depends on the choice of local coordinates. It is convenient to do so by examining the symbol definition just made when u is defined by (25.1.3) in terms of a non-degenerate phase function but the local coordinates are fixed. Note that (25.1.11) gives

(25.1.11)′ $$u_1(x)=(2\pi)^{-3n/4}\int e^{i(\langle x,\xi\rangle - H(\xi))}v(\xi)\,d\xi$$

which is a special case of (25.1.3) with $\phi(x,\xi)=\langle x,\xi\rangle - H(\xi)$ and $N=n$.

From (25.1.4) it follows that if $u\in I^m_{\text{comp}}(X,\Lambda)$ and $e^{iH}\hat{u}=(2\pi)^{n/4}v$, then

(25.1.12) $$v(\xi)|d\xi|^{\frac12}-a(x,\theta)\,e^{\pi i/4\,\text{sgn}\,\Phi}|\det\Phi|^{-\frac12}|d\xi|^{\frac12}\in S^{m+n/4-1}(\Lambda,\Omega^{\frac12}_\Lambda)$$

where $\phi'_\theta(x,\theta)=0$, $\phi'_x(x,\theta)=\xi$ defines (x,θ) as a function of ξ. Apart from the Maslov factor $\exp(\pi i/4\,\text{sgn}\,\Phi)$ we can interpret (25.1.12) as follows. (Compare (21.6.17)′.) Set as before

$$C=\{(x,\theta);\phi'_\theta(x,\theta)=0\}.$$

The pullback $d_C=\delta(\phi'_\theta)$ of the δ function in $\mathbb{R}^N$ by the map $(x,\theta)\mapsto\phi'_\theta\in\mathbb{R}^N$ is a density on C given by

$$d_C=|d\lambda|\,|D(\lambda,\phi'_\theta)/D(x,\theta)|^{-1}$$

if $\lambda=(\lambda_1,\dots,\lambda_n)$ denote arbitrary local coordinates on C extended to C^∞ functions in a neighborhood and $|d\lambda|$ is the Lebesgue density. This follows from (6.1.1). In particular we can take $\lambda=\phi'_x$ when Λ is parametrized by ξ. Then we obtain $d_C=|d\xi|\,|\det\Phi|^{-1}$, hence

(25.1.13) $$v(\xi)|d\xi|^{\frac12}\equiv a(x,\theta)\,d_C^{\frac12}e^{\pi i/4\,\text{sgn}\,\Phi}\ \text{mod}\ S^{m+n/4-1}(\Lambda,\Omega^{\frac12})$$

where C is identified with Λ by the map $(x,\theta)\mapsto(x,\phi'_x)$.

If we now introduce new coordinates $\tilde{x}$ and transform u as a half density, that is, $\tilde{u}(\tilde{x})=|Dx/D\tilde{x}|^{\frac12}u(x)$, then (25.1.3) gives

$$\tilde{u}(\tilde{x})=(2\pi)^{-(n+2N)/4}\int e^{i\tilde{\phi}(\tilde{x},\theta)}\tilde{a}(\tilde{x},\theta)\,d\theta,$$
$$\tilde{\phi}(\tilde{x},\theta)=\phi(x,\theta),\qquad \tilde{a}(\tilde{x},\theta)=|Dx/D\tilde{x}|^{\frac12}a(x,\theta).$$

With the obvious identification of the manifolds C and $\tilde{C}$ defined by ϕ and by $\tilde{\phi}$, we have $d_C=|Dx/D\tilde{x}|\,d_{\tilde{C}}$ so

(25.1.14) $$\tilde{a}\,d_{\tilde{C}}^{\frac12}=a\,d_C^{\frac12}.$$

The half density $v(\xi)|d\xi|^{\frac12}$ is thus invariant under a change of local coordinates apart from a Maslov factor of absolute value 1. Every non-singular $(n+N)\times(n+N)$ matrix Φ has signature congruent to $n+N$ mod 2 so if $\tilde{\Phi}$ is the matrix replacing Φ in the new coordinate system then the Maslov factor $e^{\pi i/4(\text{sgn}\,\Phi-\text{sgn}\,\tilde{\Phi})}$ which occurs is a power of the imaginary unit i. This means that (25.1.13) gives a principal symbol $\in S^{m+n/4}(\Lambda,\Omega^{\frac12}_\Lambda)$ for the element $u\in I^m$ defined by (25.1.3), which is uniquely determined modulo $S^{m+n/4-1}(\Lambda,\Omega^{\frac12}_\Lambda)$ and is multiplied by a power of i when the local coordinates are changed. For every $u\in I^m(X,\Lambda)$ we therefore get a principal symbol in

$$S^{m+n/4}(\Lambda,M_\Lambda\otimes\Omega^{\frac12}_\Lambda)/S^{m+n/4-1}(\Lambda,M_\Lambda\otimes\Omega^{\frac12}_\Lambda),$$

where M_Λ is a locally constant line bundle. It is defined by a covering $\Lambda=\bigcup\Gamma_j$ of Λ with open cones Γ_j and transition functions which are just powers of i. The discussion of the polyhomogeneous case above or just comparison of (25.1.13) and (21.6.17)′ identifies M_Λ with the Maslov bundle defined more geometrically in Section 21.6. In particular, it follows from (21.6.18) that if we have another local representation

$$u(x)=(2\pi)^{-(n+2\tilde{N})/4}\int e^{i\tilde{\phi}(x,\tilde{\theta})}\tilde{a}(x,\tilde{\theta})\,d\tilde{\theta}$$

in addition to (25.1.3) then

$$\tilde{a}d_{\tilde{C}}^{\frac{1}{2}}-e^{\pi is/4}a d_C^{\frac{1}{2}}\in S^{m+n/4-1}(\Lambda,\Omega_\Lambda^{\frac{1}{2}})$$

in the common domain of definition on Λ if

$$s=(\operatorname{sgn}\phi''_{\theta\theta}(x,\theta)-\operatorname{sgn}\tilde{\phi}''_{\tilde{\theta}\tilde{\theta}}(x,\tilde{\theta}))$$

where $\phi'(x,\theta)=\tilde{\phi}'(x,\tilde{\theta})=0$ and $\phi'_x(x,\theta)=\tilde{\phi}'_x(x,\tilde{\theta})=\xi$. Here the integer s is locally constant, and the x coordinates are now arbitrary. This connects with the definition of the Maslov bundle indicated after (21.6.17).

Summing up, we have now proved the following extension of Theorem 18.2.11 where we allow again the presence of a general vector bundle:

Theorem 25.1.9. *Let X be a C^∞ manifold, $\Lambda\subset T^*(X)\setminus 0$ a C^∞ conic Lagrangian submanifold, and E a C^∞ complex vector bundle over X. Then we have an isomorphism*

$$I^m(X,\Lambda;\Omega_X^{\frac{1}{2}}\otimes E)/I^{m-1}(X,\Lambda;\Omega_X^{\frac{1}{2}}\otimes E)$$
$$\to S^{m+n/4}(\Lambda,M_\Lambda\otimes\Omega_\Lambda^{\frac{1}{2}}\otimes\hat{E})/S^{m+n/4-1}(\Lambda,M_\Lambda\otimes\Omega_\Lambda^{\frac{1}{2}}\otimes\hat{E}).$$

Here $\hat{E}$ is the lifting of the bundle E to Λ. The image under the map is called the principal symbol.

Proof. By Lemma 25.1.2 this only has to be verified locally. For suitable fixed local coordinates the statement follows from Proposition 25.1.3. The Maslov bundle has been defined so that it is independent of the local coordinates chosen. – We shall often write E instead of $\hat{E}$ when no confusion seems possible.

Under the hypotheses in Proposition 25.1.5′ the principal symbol of u expressed in terms of the local coordinates there is equal to

$$|d\xi|^{\frac{1}{2}}\int_{C_\xi} a(x,\theta)e^{\pi i/4\,\operatorname{sgn}\Phi}|\det\Phi|^{-\frac{1}{2}}\,d\theta''.$$

This follows from (25.1.4)′, and C_ξ,Φ have been defined in Proposition 25.1.5′. We want to compare this with the definitions in Section 21.6. To every $(x_1,\theta_1)\in C_\xi$ the Hessian Q of $\phi/2$ at (x_1,θ_1) defines

$$U=a(x_1,\theta_1)(2\pi)^{-(n+2N-2e)/4}\int e^{iQ(x,\theta)}\,d\theta\in I(\lambda,\lambda_1)\otimes\Omega(R)$$

where R is the radical of Q_{x_1,θ_1} in the θ direction, λ is the tangent plane of Λ at $(H'(\xi),\xi)$ and λ_1 is the horizontal Lagrangian plane defined by $d\xi=0$ there. By hypothesis $R\ni\theta\mapsto\theta''$ is bijective, and the symbol of U as defined in Section 21.6 with the local coordinates x, ξ is

$$|d\xi|^{\frac{1}{2}} a(x_1, \theta_1) e^{\pi i/4 \operatorname{sgn} \Phi} |\det \Phi|^{-\frac{1}{2}} |d\theta''|.$$

Since $C_\xi \ni (x,\theta) \mapsto \theta''$ is bijective, $|d\theta''|$ is a positive density on C_ξ. Thus the symbol of u is the integral over C_ξ of the density on C_ξ with values in $(M_\Lambda \otimes \Omega_\Lambda^{\frac{1}{2}})_{(H'(\xi),\xi)}$ defined according to Section 21.6.

The phase function $-\phi$ defines the Lagrangian $\check{\Lambda} = i\Lambda$ where i: $T^*(X) \to T^*(X)$ is defined by $i(x,\xi) = (x, -\xi)$. From (25.1.4) it follows that the principal symbol of $\bar{u}$ is defined on $\check{\Lambda}$ by

$$\overline{a(x,\theta)}\, e^{-\pi i/4 \operatorname{sgn} \Phi} |\det \Phi|^{-\frac{1}{2}}, \tag{25.1.15}$$

that is, we obtain the pullback by i of the complex conjugate of the principal symbol of u. Now the complex conjugate of a section of $M_\Lambda \otimes \Omega_\Lambda^{\frac{1}{2}}$ is a section of $M_\Lambda^{-1} \otimes \Omega_\Lambda^{\frac{1}{2}}$, and $i^* M_\Lambda^{-1} = M_{\check{\Lambda}}$ by (21.6.5) since $i^*\sigma = -\sigma$. (This is just another way of expressing the complex conjugation of the Maslov factor in (25.1.15).) The pullback of a section of $M_\Lambda^{-1} \otimes \Omega_\Lambda^{\frac{1}{2}}$ can thus be identified with a section of $M_{\check{\Lambda}} \otimes \Omega_{\check{\Lambda}}^{\frac{1}{2}}$. Summing up, we have

Theorem 25.1.10. *Let the hypotheses of Theorem 25.1.9 be fulfilled and let j: $E \to F$ be an antilinear bundle map. Then $u \in I^m(X, \Lambda; \Omega_X^{\frac{1}{2}} \otimes E)$ implies $ju \in I^m(X, \check{\Lambda}; \Omega_X^{\frac{1}{2}} \otimes F)$ if $\check{\Lambda} = i\Lambda$, $i(x,\xi) = (x, -\xi)$; and $i^* ja \in S^{m+n/4}(X, \check{\Lambda};\ M_{\check{\Lambda}} \otimes \Omega_{\check{\Lambda}}^{\frac{1}{2}} \otimes \hat{F})$ is a principal symbol of ju if $a \in S^{m+n/4}(X, \Lambda;\ M_\Lambda \otimes \Omega_\Lambda^{\frac{1}{2}} \otimes \hat{E})$ is one for u.*

As in Section 18.1 we could have used considerably more general symbols in the preceding discussion. Lemma 25.1.6 remains valid for the symbol spaces $S_\rho^m = S_{\rho,1-\rho}^m$. More generally, we can define $S_\rho^m(V)$ if V is a conic manifold as the set of $a \in C^\infty(V)$ such that when $t \geq 1$

$$t^{-m-(1-\rho)k} M_t^* a$$

is uniformly bounded in $C^k(V)$ for every $k \geq 0$. We abandon now the intrinsic Definition 25.1.1 and define I_ρ^m when $\rho > \frac{1}{2}$ as the set of distributions which are microlocally of the form (25.1.11)$'$ with $v \in S_\rho^{m-n/4}$. An analogue of Proposition 25.1.5 follows with no essential change of the proof, and it leads to a principal symbol isomorphism

$$I_\rho^m(X, \Lambda; \Omega_X^{\frac{1}{2}} \otimes E)/I_\rho^{m+1-2\rho}(X, \Lambda; \Omega_X^{\frac{1}{2}} \otimes E) \to$$
$$S_\rho^{m+n/2}(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{\frac{1}{2}} \otimes \hat{E})/S_\rho^{m+n/2+1-2\rho}(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{\frac{1}{2}} \otimes \hat{E}).$$

It would also have been possible to define I_ρ^m by the condition (25.1.1) in Definition 25.1.1 for all $L_j \in \Psi_\rho^{2\rho-1}(X; E, E)$ with principal symbol vanishing on Λ. However, the proof of the analogue of Proposition 25.1.3 becomes somewhat longer since, we must consider on one hand operators with symbols of the form $|\xi|^\rho \chi((x-h'(\xi))|\xi|^{1-\rho})(x_j - h_j(\xi))$ and on the other hand operators with symbols of the form $|\xi|^{2\rho-1}(1-\chi((x-h'(\xi))|\xi|^{1-\rho}))$. Here $\chi \in C_0^\infty$ is equal to 1 in a neighborhood of the origin. We leave the details for the interested and energetic reader.

25.2. The Calculus of Fourier Integral Operators

Let X and Y be two C^∞ manifolds and E, F two complex vector bundles on X, Y. Then every $A\in\mathscr{D}'(X\times Y, \Omega^{\frac{1}{2}}_{X\times Y}\otimes \operatorname{Hom}(F,E))$ defines a continuous map

$$\mathscr{A}:\ C_0^\infty(Y,\Omega_Y^{\frac{1}{2}}\otimes F)\to\mathscr{D}'(X,\Omega_X^{\frac{1}{2}}\otimes E)$$

and conversely. (See Section 5.2 and, for the role of the half densities, also Section 18.1.) Here the fiber of the vector bundle $\operatorname{Hom}(F,E)$ at (x,y) consists of the linear maps $F_y\to E_x$. In particular, if Λ is a closed conic Lagrangian submanifold of $T^*(X\times Y)\setminus 0$ we can identify $I^m(X\times Y,\Lambda;\Omega^{\frac{1}{2}}_{X\times Y}\otimes\operatorname{Hom}(F,E))$ with a space of such maps. If we have

$$\Lambda\subset(T^*(X)\setminus 0)\times(T^*(Y)\setminus 0) \tag{25.2.1}$$

then it follows from Theorem 8.2.13 and Lemma 25.1.2 that $\mathscr{A}$ is even a continuous map from $C_0^\infty(Y)$ to $C^\infty(X)$ which can be extended to a continuous map from $\mathscr{E}'(Y)$ to $\mathscr{D}'(X)$ with

$$WF(\mathscr{A}u)\subset C(WF(u)),\qquad u\in\mathscr{E}'(Y,\Omega_Y^{\frac{1}{2}}\otimes F), \tag{25.2.2}$$

where

$$C=\Lambda'=\{(x,\xi,y,-\eta)\in(T^*(X)\setminus 0)\times(T^*(Y)\setminus 0);(x,\xi,y,\eta)\in\Lambda\}$$

is a canonical relation from $T^*(Y)\setminus 0$ to $T^*(X)\setminus 0$. (See Definition 21.2.12.) As in Section 21.2 we call $\Lambda=C'$ the twisted canonical relation. The Maslov bundle M_Λ can be regarded as a bundle M_C on C defined by C and the product symplectic form $\sigma_X-\sigma_Y$.

Definition 25.2.1. Let C be a homogeneous canonical relation from $T^*(Y)\setminus 0$ to $T^*(X)\setminus 0$ which is closed in $T^*(X\times Y)\setminus 0$, and let E, F be vector bundles on X, Y. Then the operators with kernel belonging to $I^m(X\times Y,\ C';\ \Omega^{\frac{1}{2}}_{X\times Y}\otimes\operatorname{Hom}(F,E))$ are called Fourier integral operators of order m from sections of F to sections of E, associated with the canonical relation C.

Let E^* be the vector bundle with fiber E_x^* at $x\in X$ antidual to the fiber E_x of E. Then we have a pairing

$$u,v\mapsto\int(u,v)(x);\quad u\in C_0^\infty(X,\Omega_X^{\frac{1}{2}}\otimes E),\quad v\in\mathscr{D}'(X,\Omega_X^{\frac{1}{2}}\otimes E^*),$$

and a similar one for Y and F. If $A\in\mathscr{D}'(X\times Y,\ \Omega^{\frac{1}{2}}_{X\times Y}\otimes\operatorname{Hom}(F,E))$ then the adjoint of the map $C_0^\infty(Y,\Omega_Y^{\frac{1}{2}}\otimes F)\to\mathscr{D}'(X,\Omega_X^{\frac{1}{2}}\otimes E)$ defined by A is defined by $A^*\in\mathscr{D}'(Y\times X,\Omega^{\frac{1}{2}}_{Y\times X}\otimes\operatorname{Hom}(E^*,F^*))$. If s is the map $Y\times X\to X\times Y$ interchanging the two factors then A^* is obtained by composing s^*A with the antilinear bundle map $\operatorname{Hom}(F,E)\to\operatorname{Hom}(E^*,F^*)$ given by taking adjoints. If

$$A\in I^m(X\times Y,C';\Omega^{\frac{1}{2}}_{X\times Y}\otimes\operatorname{Hom}(F,E))$$

then

$$s^*A\in I^m(Y\times X,s^*C';\Omega^{\frac{1}{2}}_{Y\times X}\otimes\operatorname{Hom}(F,E));$$

if a is the principal symbol of A then s^*a is the principal symbol of s^*A. This is obvious by the invariance of our constructions. With the same notation i for the reflection in the cotangent bundle as in Theorem 25.1.10 we have $i^*s^*C'=(C^{-1})'$ where C^{-1} is the inverse canonical relation obtained by interchanging $T^*(X)$ and $T^*(Y)$. Thus we obtain in view of Theorem 25.1.10:

Theorem 25.2.2. *Let C be a homogeneous canonical relation from $T^*(Y)\setminus 0$ to $T^*(X)\setminus 0$ which is closed in $T^*(X\times Y)\setminus 0$, and let E, F be vector bundles on X, Y. If $A\in I^m(X\times Y, C'; \Omega^{\frac12}_{X\times Y}\otimes \operatorname{Hom}(F,E))$, identified with the corresponding linear operator, then $A^*\in I^m(Y\times X, (C^{-1})'; \Omega^{\frac12}_{Y\times X}\otimes\operatorname{Hom}(E^*,F^*))$. If $a\in S^{m+n/4}(C; M_C\otimes\Omega^{\frac12}_C\otimes\operatorname{Hom}(F,E))$ is a principal symbol for A, where $n=\dim(X\times Y)$, then $s^*a^*\in S^{m+n/4}(C^{-1}, M_{C^{-1}}\otimes\Omega^{\frac12}_{C^{-1}}\otimes\operatorname{Hom}(E^*,F^*))$ is a principal symbol for A^*. Here s is the interchanging map $Y\times X\to X\times Y$.*

Note that we have here chosen to regard the principal symbol as defined on C rather than on C'. This is usually more convenient in connection with Fourier integral operators and should cause no confusion.

We shall now discuss products, so let C_1 be a homogeneous canonical relation from $T^*(Y)\setminus 0$ to $T^*(X)\setminus 0$ and C_2 another from $T^*(Z)\setminus 0$ to $T^*(Y)\setminus 0$ where X, Y, Z are three manifolds, with vector bundles, E, F, G. Let

$$A_1\in I^{m_1}(X\times Y, C_1'; \Omega^{\frac12}_{X\times Y}\otimes\operatorname{Hom}(F,E)),$$
$$A_2\in I^{m_2}(Y\times Z, C_2'; \Omega^{\frac12}_{Y\times Z}\otimes\operatorname{Hom}(G,F))$$

and assume that both are properly supported so that the composition A_1A_2 of the corresponding operators is defined. We want to show that it is associated with the composition C of the canonical relations C_1 and C_2 provided that the composition is clean, proper and connected in a sense which we shall now define. Already after the statement of Theorem 21.2.14 we defined the composition to be *clean* if $C_1\times C_2$ intersects $T^*(X)\times\Delta(T^*(Y))\times T^*(Z)$ cleanly, that is, in a manifold $\hat{C}$ with tangent plane everywhere equal to the intersection of the tangent planes of the intersecting manifolds. We shall say that the composition is *proper* if the map

$$\hat{C}\to T^*(X\times Z)\setminus 0$$

is proper. (When Y is compact this is automatically true since C_1 and C_2 are closed in $T^*(X\times Y)\setminus 0$ and $T^*(Y\times Z)\setminus 0$ respectively but contained in $(T^*(X)\setminus 0)\times(T^*(Y)\setminus 0)$ and $(T^*(Y)\setminus 0)\times(T^*(Z)\setminus 0)$.) Then the range C is a closed subset of $T^*(X\times Z)\setminus 0$ contained in $(T^*(X)\setminus 0)\times(T^*(Z)\setminus 0)$. The inverse image C_γ in $\hat{C}$ of $\gamma\in C$ is a compact manifold of dimension equal to the excess e of the clean intersection. To avoid self-intersections of C we assume that the composition is *connected* in the sense that C_γ is connected for every $\gamma\in C$. Then it follows from Theorem 21.2.14 that C is also a canonical relation. We shall prove that

(25.2.3) $$A_1 A_2 \in I^{m_1+m_2+e/2}(X\times Z, C; \Omega^{\frac{1}{2}}_{X\times Z}\otimes \operatorname{Hom}(G,E))$$

and compute the principal symbol. Note that when the composition is *transversal*, that is, the excess $e=0$, then $A_1A_2\in I^{m_1+m_2}$. The normalizations introduced in Section 25.1 were to a large extent motivated by our wish to maintain this natural property of the order of differential and pseudo-differential operators.

By a partition of unity we can reduce the proof of (25.2.3) to the local case where $X\subset\mathbb{R}^{n_X}$, $Y\subset\mathbb{R}^{n_Y}$, $Z\subset\mathbb{R}^{n_Z}$, the bundles E, F, G are trivial and

$$A_1(x,y)=(2\pi)^{-(n_X+n_Y+2N_1)/4}\int e^{i\phi(x,y,\theta)}a_1(x,y,\theta)\,d\theta,$$
$$A_2(y,z)=(2\pi)^{-(n_Y+n_Z+2N_2)/4}\int e^{i\psi(y,z,\tau)}a_2(y,z,\tau)\,d\tau.$$

Here ϕ is a non-degenerate phase function in a conic neighborhood of $(x_0,y_0,\theta_0)\in X\times Y\times(\mathbb{R}^{N_1}\setminus 0)$ parametrizing C_1 in a conic neighborhood of (x_0,ξ_0,y_0,η_0), thus

$$\phi'_\theta=0,\quad \phi'_x=\xi_0,\quad \phi'_y=-\eta_0 \quad \text{at } (x_0,y_0,\theta_0).$$

Similarly ψ is a non-degenerate phase function in a conic neighborhood of $(y_0,z_0,\tau_0)\in Y\times Z\times(\mathbb{R}^{N_2}\setminus 0)$ parametrizing C_2 in a conic neighborhood of (y_0,η_0,z_0,ζ_0), thus

$$\psi'_\tau=0,\quad \psi'_y=\eta_0,\quad \psi'_z=-\zeta_0 \quad \text{at } (y_0,z_0,\tau_0).$$

The amplitudes a_1, a_2 have supports in small conic neighborhoods of (x_0,y_0,θ_0) and (y_0,z_0,τ_0) respectively, and

(25.2.4) $$a_1\in S^{m_1+(n_X+n_Y-2N_1)/4},\qquad a_2\in S^{m_2+(n_Y+n_Z-2N_2)/4}.$$

If $a_j\in S^{-\infty}$ then $A=A_1A_2$ is given by

(25.2.5) $$A(x,z)=\int A_1(x,y)A_2(y,z)\,dy$$
$$=(2\pi)^{-(n_X+n_Z+2(n_Y+N_1+N_2))/4}\iiint e^{i\Phi(x,z,y,\theta,\tau)}a(x,z,y,\theta,\tau)\,dy\,d\theta\,d\tau$$

where

$$\Phi(x,z,y,\theta,\tau)=\phi(x,y,\theta)+\psi(y,z,\tau);$$
$$a(x,z,y,\theta,\tau)=a_1(x,y,\theta)\,a_2(y,z,\tau).$$

From Proposition 21.2.19 we know that Φ is a clean phase function defining C, in a conic neighborhood of $(x_0,z_0,y_0,\theta_0,\tau_0)$. This will lead to a proof of (25.2.3) when we have proved that the integration can be restricted to a set where $|\theta|$ and $|\tau|$ have the same order of magnitude so that a is a well behaved symbol. This is not the case for the function a as it stands since, for example, differentiation with respect to θ only improves the magnitude by a factor $1/(1+|\theta|)$ and not by $1/(1+|\theta|+|\tau|)$.

We assume that a_1 and a_2 have supports in compactly generated cones Γ_1 and Γ_2 where $\partial\phi(x,y,\theta)/\partial y$ and $\partial\psi(y,z,\tau)/\partial y$ never vanish. Then we can