BestMasters

Paulo Guilherme Santos

Diagonalization in Formal Mathematics

Springer Spektrum

BestMasters

Mit **„BestMasters“** zeichnet Springer die besten Masterarbeiten aus, die an renommierten Hochschulen in Deutschland, Österreich und der Schweiz entstanden sind. Die mit Höchstnote ausgezeichneten Arbeiten wurden durch Gutachter zur Veröffentlichung empfohlen und behandeln aktuelle Themen aus unterschiedlichen Fachgebieten der Naturwissenschaften, Psychologie, Technik und Wirtschaftswissenschaften. Die Reihe wendet sich an Praktiker und Wissenschaftler gleichermaßen und soll insbesondere auch Nachwuchswissenschaftlern Orientierung geben.

Springer awards **"BestMasters"** to the best master's theses which have been completed at renowned Universities in Germany, Austria, and Switzerland. The studies received highest marks and were recommended for publication by supervisors. They address current issues from various fields of research in natural sciences, psychology, technology, and economics. The series addresses practitioners as well as scientists and, in particular, offers guidance for early stage researchers.

More information about this series at http://www.springer.com/series/13198

Paulo Guilherme Santos

Diagonalization in Formal Mathematics

Springer Spektrum

Paulo Guilherme Santos
Universidade Nova de Lisboa
Lisboa, Portugal

The work of this thesis was presented at FCT-UNL for the Master's Degree in Pure Mathematics. It was funded by the FCT project: PTDC/MHC-FIL/2583/2014.

ISSN 2625-3577 ISSN 2625-3615 (electronic)
BestMasters
ISBN 978-3-658-29110-5 ISBN 978-3-658-29111-2 (eBook)
https://doi.org/10.1007/978-3-658-29111-2

This Springer Spektrum imprint is published by the registered company Springer Fachmedien Wiesbaden GmbH part of Springer Nature.
The registered company address is: Abraham-Lincoln-Str. 46, 65189 Wiesbaden, Germany

To all that were an active part in shaping me in what I am, that aided me to dream higher, and that taught me happiness.
To my parents: José Paulo and Isabel.
To my grandparents: Adriana, Guilherme, Carminda, and António.
To Pi.

Acknowledgment

I am very thankful to my Advisers—Professor Isabel Oitavem and Professor Reinhard Kahle—for their help in the work developed in the current thesis. I am deeply thankful to Professor Reinhard Kahle for helping me fulfilling my professional dreams. In our meetings—a time where I have learned to be creative and to always be curious—, I have had the opportunity to contact with a great variety of subjects and new ideas that have been essential to the investigation that I have developed under his tuition. I am profoundly grateful to my parents—José Paulo and Isabel—, to my grandparents—Adriana, Guilherme, Carminda, and António—, and to my girlfriend Pilar. I also thank to my friends and family for their love.

The present work started under the Gulbenkian scholarship "Novos Talentos em Matemática" where I had Professor Kahle as my supervisor. The thesis was developed with the collaboration of Tübingen University via Professor Kahle and presented at FCT-NOVA. The work was developed with support of the project *Hilbert's* 24^{th} *Problem*, funded by the Portuguese Science Foundation FCT (PTDC/MHC-FIL/2583/2014).

Contents

1 Preliminaries

As the main subject of the current thesis is Mathematical Logic, we will assume that the main definitions and results of Logic are known (the definition of formula, the connectives, first-order theories, etc); for more information in introductory notions of Logic see: [Sho18], [Bar93], [Rau06], and [EFT96]. We will also assume the main definitions and results of Category Theory, a domain where we will use the right-to-left notation (in the rest we will use the usual function notation): af will denote, in the context of categories, the composition of f with a (see [Lan13] for more informations).

We continue the preliminaries by remembering the main notions needed to study theories of Arithmetic. Following the notation of [Rau06], let $\mathbf{F}_n$ denote the set of all n-ary functions with arguments and values in $\mathbb{N}$ and let $\mathbf{F} := \bigcup_{n\in\mathbb{N}} \mathbf{F}_n$. For $f \in \mathbf{F}_m$ and $g_1,\ldots,g_m \in \mathbf{F}_n$, we call $h : \vec{a} \mapsto f(g_1(\vec{a}),\ldots,g_m(\vec{a}))$ the (generalised) *composition* of f and g_i and write $h = f[g_1,\ldots,g_m]$. The set of *primitive recursive* functions is the minimal set of function on $\mathbb{N}$ such that:

Initial The constant function equal to 0, the successor function S, and the projection functions $I^n_\nu : \vec{a} \mapsto a_\nu$ $(1 \le \nu \le n,\ n \in \mathbb{N})$ are primitive recursive;

Oc If $h \in \mathbf{F}_m$ and $g_1,\ldots,g_m \in \mathbf{F}_n$ are primitive recursive, then $f = h[g_1,\ldots,g_m]$ is primitive recursive;

Op If $g \in \mathbf{F}_n$ and $h \in \mathbf{F}_{n+2}$ are primitive recursive, then so is $f \in \mathbf{F}_{n+1}$ uniquely determined by the equations

$$f(\vec{a},0) = g(\vec{a}); \qquad f(\vec{a},\mathrm{S}(b)) = h(\vec{a},b,f(\vec{a},b)).$$

The set of *recursive* functions is the minimal set of function on $\mathbb{N}$ that includes the primitive recursive functions and obeys:

P. G. Santos, *Diagonalization in Formal Mathematics*, BestMasters,
https://doi.org/10.1007/978-3-658-29111-2_1