Malo Tarpin

Non-perturbative Renormalization Group Approach to Some Out-of-Equilibrium Systems

Diffusive Epidemic Process and Fully Developed Turbulence

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Diffusive Epidemic Process and Fully Developed Turbulence

Springer

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À Ambre

Supervisor's Foreword

The scope of this thesis is the study of scale invariance in non-equilibrium systems in statistical physics. Out of equilibrium, systems exhibit a great wealth of scaling behaviours. Not only new intrinsically non-equilibrium universality classes have been discovered, but also new behaviours, which have no counter-parts in equilibrium systems. A typical example is self-organised criticality, in which the dynamics itself drives the system to a critical state, without fine-tuning any external parameters, contrary to a standard phase transition. Another intriguing phenomenon, which has been unveiled in active matter systems, is the spontaneous breaking of a continuous symmetry in dimension less than two, which is not permitted at equilibrium since it violates the Mermin-Wagner theorem.

To investigate systems out of equilibrium, one cannot rely on the standard tools of statistical mechanics available for equilibrium systems. New theoretical approaches have to be developed. A very versatile and powerful one to study scaling phenomena is the renormalisation group (RG). In particular, in recent decades, a modern formulation of the RG has emerged, which is both functional and non-perturbative (NPRG), and has allowed one to address genuinely strong-coupling problems.

We have initiated the application of these techniques to classical non-equilibrium systems in the 2000s. We have first focused on single-species reaction-diffusion processes, which are simple models describing particles that diffuse randomly on a lattice and interact when they encounter. These systems exhibit absorbing phase transitions—transitions between an active and a non-fluctuating state—belonging to non-equilibrium universality classes. Another important application we have considered concerns stochastic interface growth and kinetic roughening, as described by the celebrated Kardar-Parisi-Zhang equation. A randomly growing interface always becomes rough as it grows, and this rough phase is scale invariant. This is an example of self-organised criticality. The rough phase corresponds in dimensions greater than one to a strong-coupling fixed point, unaccessible at any order of perturbation theory, and we have developed a suitable framework within the NPRG which enables one to describe it.

Malo's thesis represents an important contribution in this field, by pushing further the applications of the NPRG method in two respects: he addresses for the first time a two species reaction-diffusion process, and he develops a new scheme within NPRG to address the long-standing problem of turbulence. Let me be more specific on these two aspects.

In the first part of this thesis, Malo studies a two-species reaction-diffusion system which is called Diffusive Epidemic Process. This model exhibits an absorbing phase transition, in which universality class depends on the relative diffusion rates of the two species. One of these cases is very controversial, since it has been argued that the transition could be first order, or be continuous but with debated universal properties. To tackle this problem, Malo develops the first implementation of NPRG methods for a two-species reaction-diffusion process. He unveils subtle issues concerning the very definition of the model and the ensuing symmetries which in fact do not coincide between the different definitions. This clarifies some of the disagreements present in the literature. He then analyses the model within the Local Potential Approximation, which is a standard approximation scheme within NPRG. Although the outcome of this analysis does not bring a definite answer to the problem—which would require going to a higher-order approximation—all the framework to address multi-species reaction-diffusion processes is set up and the main technical issues are discussed. This part constitutes a useful basis for a reader interested in applying NPRG techniques to other similar problems.

In my opinion, the most beautiful breakthrough of Malo's work concerns fully developed turbulence, in homogeneous, isotropic and stationary conditions. We had started to work on this subject a few years before Malo started his Ph.D. Our aim was more or less to transpose our experience with the KPZ equation, which maps to the Burgers equation and hence can be viewed as a simplified model for turbulence, to the full Navier–Stokes problem. To our surprise, we found a new time-gauged symmetry of the Navier–Stokes field theory, related to a shift in the response field sector. We realised that this symmetry was crucial since it enabled us to close exactly the flow equation for the two-point correlation function in the limit of large wave-number. Malo's thesis reveals the full power of this approach, since he obtains an analytical expression for any multi-point correlation functions of turbulence, which is exact in the limit of large wave-numbers.

The crux of this derivation is to combine an existing approximation scheme within NPRG, called the Blaizot-Mendez-Wschebor scheme, with the time-gauged symmetries of the Navier–Stokes field theory, in a new scheme which can be called large-momentum expansion. In this expansion, Malo derives a proof showing that the flow equation of any n-point correlation function can be closed exactly at leading order in this expansion, and bears a simple expression. He then obtains the solution of these flow equations at the fixed point, in both regimes of small and large time delays. These results constitute a milestone for NPRG methods, since they show that within this framework, the whole hierarchy of flow equations for the correlation functions can be treated in a systematic and fully analytical way. They hence pave the way to new types of approaches. Malo's results are also an

important contribution in the domain of turbulence, where controlled results derived directly from the Navier–Stokes equation, without phenomenological inputs, are—to say the least—scarce. I am sure that the interested reader will find inspiring materials in this thesis.

Grenoble, France Prof. Léonie Canet Isère
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My work has also benefited from the hospitality of the LPMMC team and I would like to thank them warmly. Similarly, I would like to thank the IFFI team and its director Daniel Ariosa, who welcomed me with open arms during my stay in Montevideo. It is also an opportunity to thank all those with whom I have had the opportunity to work during these three years and who have had the patience to support me: Steven Mathey, Magali Le Goff, Carlo Pagani, Vivien Lecomte, Davide Squizzato, each time it has been a source of exciting and enriching discussions.

These three years of thesis would have been very empty without the presence of all those who shared these Grenoble moments: the long coffee breaks with the colleagues, the Thursday evening meals, the climbing and ski touring trips, and more. There are too many to name them, but they will recognize themselves.

Finally, my thoughts go to all those from whom I received and learned and those with whom I shared the path that led me to here. My parents and my family, the childhood friends of École Michaël, the teachers who knew how to awaken my interest for science since primary school, the whole team of the 130 bar of ESPCI, with whom I spent formative years, my master professors, Leticia Cugliandolo, Jean-Baptiste Fournier, Julien Serreau and Michel Bauer who really got me into the scientific world, Marcela, Gonzalo, and the others who welcomed me warmly in Montevideo. Finally, Jules, Thibault and Géraldine who accompanied me throughout the thesis and with whom I remade the world a hundred times. Thank you Eugenia, for the interesting times.

Contents

Chapter 1
General Introduction

This manuscript presents the study of two physical systems belonging to the field of out-of-equilibrium statistical physics: the diffusive epidemic process, and homogeneous isotropic fully developed turbulence. The former is a simplified model for the diffusion of an epidemic in a population. More specifically, we focus on the continuous phase transition it undergoes when the population density is varied. The second system is a fluid in a turbulent stationary state, as described by the Navier–Stokes equation subjected to a random forcing. Both systems, in addition to share the property of being intrinsically out-of-equilibrium, are examples of critical phenomena. In this work, the study of each system is conducted using the tools coming from the framework known as the non-perturbative (or functional) renormalization group. Before delving into the particular physics of each system, let us present in this introduction the more general context of universal and critical phenomena in statistical physics, with an emphasis on the case of out-of-equilibrium systems, as well as the field theoretical methods developed to study them.

Statistical physics is the study of systems containing a large number of degrees of freedom. Its aim is to give a description of the global macroscopic phenomena of such system as emerging from the fluctuations of its microscopic elementary, possibly interacting, constituents. In order to reduce the complexity of the description, one aims at building a minimal microscopic model, in the sense that it should reproduce all the known macroscopic features of the statistical system under study in the simplest way and with the least possible amount of ingredients needed. The rationale behind such approach lies in the fact that macroscopic observables are built up by the contributions of a large number of microscopic degrees of freedom. Thus it is reasonable to hope that for well-chosen macroscopic observables, some form of self-averaging takes place and these quantities are not sensitive to some details of the microscopic description. The resolution of the model can in turn lead to new predictions and suggests new experiments. In this back-and-forth process, one hopes to find unifying pictures or mechanisms which shed light on universal phenomena in physics.

A prominent example of such universality is given by critical phenomena, such as continuous phase transitions. Indeed, in a critical phenomenon, the degrees of freedom become correlated over all the range of scales spanned by the system. As a consequence, the long distance behavior of the system loses memory of most of the physics at the microscopic scales. This is reflected for example in the appearance of scaling laws, with universal exponents, at the approach to a continuous phase transition. Unfortunately, these scaling laws signal the existence of singularities, which hinder the approaches traditionally applied to derive the macroscopic behavior from a microscopic model. Critical phenomena in statistical physics were identified to be closely related to the problem of renormalization in quantum field theory. Thus, it was tempting to apply the methods developed in this framework (Bogolyubov and Shirkov 1959; Dyson 1949; Stueckelberg and Petermann 1953), to study the critical properties of such systems. In the case of equilibrium physics, this bridge was made by Wilson and Kogut (1974), Fisher (1974), building on earlier work by Kadanoff (1966). They interpreted the early renormalization schemes developed for quantum field theory in a new framework, the Renormalisation Group (RG). The general idea of this method is to construct an effective theory for the macroscopic observable not by trying to calculate the contributions coming from the degrees of freedom living at all scales at once, but to do so progressively. One starts with the fluctuations having as typical scale the scale at which is defined the microscopic physics, named the ultraviolet (UV) cutoff of the system, and ends at the scale of the macroscopic observables, the infrared (IR) cutoff. If the system is at a critical point, the integration of all degrees of freedom from the UV to the IR generates singularities. To do the integration infinitesimally allows one to understand how these singularities appear. This operation can be formulated as a differential equation giving the evolution of the system under a change of the RG scale. An exact equation for the RG flow was given by Polchinski (1984). In the following decade, this exact RG flow was reformulated in terms of the effective action by Wetterich (1993), Morris (1994), Ellwanger (1994). This approach is now given the standard name of Non-Perturbative (also named functional) Renormalisation Group (NPRG).

Now, let us emphasize somes specificities of out-of-equilibrium systems in statistical physics. The most successful framework to take into account microscopic fluctuations is the theory of systems at equilibrium with a thermal bath. For a system at equilibrium, the logarithm of the probability of a given microscopic configuration is assumed to be proportional to the energy associated to the configuration (Gibbs 1902). However, a large part of phenomena in statistical physics do not fit in the framework of equilibrium or perturbation to equilibrium. Indeed in these latter cases, the statistical correlations exhibited by the system and its statistical response to a perturbation are not independent: they are found to satisfy what is known as the fluctuation-dissipation theorem. This fact pertaining to systems at equilibrium can be traced back to a property named detailed balance. Because the dynamics of out-of-equilibrium systems are not constrained to satisfy detailed balance, they describe a richer physics than the one accessible to systems at or dissipating to equilibrium. For example, out-of-equilibrium systems can exhibit continuous phase transitions between fluctuating and non-fluctuating steady states (Hinrichsen 2000), which is impossible at

equilibrium. Moreover, numerous out-of-equilibrium systems are found to be generically critical. For systems at equilibrium, critical behaviors are generally associated to continuous phase transitions. In these cases the critical behavior emerges from a fine-tuning of some parameters of the theory. However, for many out-of-equilibrium systems, this critical behavior emerges without any fine-tuning of the parameters. This phenomenon, sometimes termed self-organized criticality (Bak et al. 1987), has maybe as most famous example interface growth, modeled by the celebrated Kardar–Parisi–Zhang equation (Kardar et al. 1986). These peculiarities of classical out-of-equilibrium systems make it an exciting playground to study the physics of critical phenomena using the tools of the RG, as pioneered in Janssen (1979). See Täuber (2014) for a review sticking to perturbative RG and Canet et al. (2004), Canet et al. (2010) for two modern examples using NPRG.

Unfortunately, contrary to equilibrium systems, for out-of-equilibrium systems there does not exist an a priori probability distribution for the microscopic configurations. Thus one has to model explicitly the fluctuations within the dynamics of the microscopic degrees of freedom. There are two traditional ways used by physicists to generate such stochastic dynamics for classical systems. The first one is to pertub the deterministic dynamic followed by the microscopic degrees of freedom with a noise term, which is generally assumed to be Gaussian. This facilitating hypothesis is justified by viewing the noise as emerging from the sum of many unknown small independent effects. When the number of degrees of freedom is countable, one can write a set of Langevin equations. However, it is often more convenient to represent the degrees of freedom as fields, whose evolution is then given by stochastic partial differential equation (SPDE). The second way to build a dynamics is to assume that the process can be described by a time-continuous Markov chain and in this case, the prescription of the dynamics is done by giving the master equation of the process. However, both of these formulations are not straightforwardly amenable to the treatment by renormalization methods. This gap led to many developments in the '60, notably for applications to turbulence (Kraichnan 1961; Wyld 1961), which were synthetized and popularized by Martin et al. (1973). It was realized that the dynamics of a statistical system given by a SPDE could be written in formal closeness with quantum field theory at the price of introducing an extra field for each degree of freedom of the theory. This breakthrough, named the response field formalism, enabled the use of the tools of quantum field theory to tackle critical phenomena in out-of-equilibrium statistical systems. Later it was shown by Janssen (1976), De Dominicis (1976) that the field theory for the observables and the response fields could be formulated as a partition function, summing over configurations in space-time weighted by the exponential of an action. This formulation opens the way to powerful approximations relying on saddle-point methods, to a systematic way to account for the symmetries of a model and put out-of-equilibrium field theories on the same footing as equilibrium ones for a RG treatment *à la* Wilson. The mapping from a SPDE to a partition function for a field theory is known collectively as the Martin–Siggia–Rose–Janssen–de Dominicis (MSRJD) formalism. Although it is not the focus of this work, let us note that in the case of quantum systems there also exist a formalism to write a partition function when the system is not at equilibrium. It

is known as the Schwinger–Keldysh formalism (Schwinger 1960, 1961; Keldysh 1964) and its semi-classical limit gives back the action of MSRJD (Kamenev 2011). In the same decade, starting directly from the evolution of the probability distribution of the observables, Doi devised another method to map statistical systems into field theory akin to the "second-quantization" in quantum systems (Doi 1976a, b). This method leads to a different field theory from the one obtained by the MSRJD formalism and the link between the two is still subject to some discussions. The method was presented and refined in the case of reaction-diffusion processes (also named birth-death processes) on a lattice in Peliti (1985) and since then bears the name of Doi-Peliti formalism. In the present days, out-of-equilibrium statistical physics is often presented as subdivided in two categories. On the one hand, processes which are defined in the continuum through their SPDE and casted to a field theory using the MSRJD formalism. On the other hand, jump processes with countable state space such as reaction-diffusion processes on a lattice, defined by their master equation and whose continuum limit is taken at the level of the Lagrangian, after using Doi-Peliti formalism. This separation is represented in this work, as the forced Navier–Stokes equation is a SPDE, and the diffusive epidemic process is a originally formulated as a reaction-diffusion process on a lattice.

In Chap. 2 we will present the physics of both systems and the open problems which motivated our study. In Chap. 3, we will make a short presentation of the framework of the NPRG, with an emphasis on its application to out-of-equilibrium field theories. Finally, in Chaps. 4 and 5, we will present respectively our take on the characterization of the phase transition of the diffusive epidemic process, and on the breaking of scale invariance in homogeneous isotropic fully developed turbulence.

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Wyld H (1961) Formulation of the theory of turbulence in an incompressible fluid. Ann Phys 14:143–165. https://doi.org/10.1016/0003-4916(61)90056-2

Chapter 2
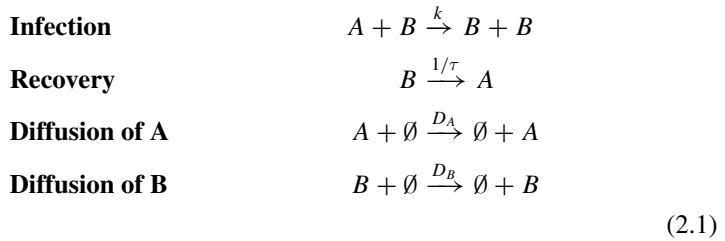# Universal Behaviors in the Diffusive Epidemic Process and in Fully Developed Turbulence

In this chapter, the phenomenology and challenges of the two systems studied as part of the thesis work are presented. Firstly, in Sect. 2.1 we give a short account on the physics of the diffusive epidemic process and in particular of the phase transition between a fluctuating state and an absorbing state that this system undergoes. We take the time to present the existing literature on the subject and to uncover some remaining issues in the established description of this system. Secondly, in Sect. 2.2 after giving the general phenomenology and challenges of fully developed turbulence in fluids, we focus on the two subjects studied here: the time-dependence of correlation functions in both two- and three-dimensional turbulence, and the existence of intermittency in two-dimensional turbulence.

2.1 The Absorbing Phase Transition in the Diffusive Epidemic Process

The first part of the manuscript is devoted to studying the absorbing phase transition occurring in the diffusive epidemic process. Absorbing phase transitions are phase transitions to a state from which the system cannot escape, a phenomenon exclusive to out-of-equilibrium physics. The diffusive epidemic process (DEP), proposed in van Wijland (1998), is a stochastic process which serves as a streamlined model to describe the propagation of an epidemy in a population without immunization. The absorbing phase in this case is the state without any individuals infected such that the epidemy has disappeared. As announced in the introduction, DEP belongs to the class of reaction-diffusion processes. Let us first give a brief general introduction to these models.

The denomination of reaction-diffusion process covers in this work Markov processes continuous in time and with countable state space. The process is understood as describing a set of particles species undergoing random and independent events, for example the annihilation of two particles when they cohabit on the same site.

These processes are very simple to formulate in the form of a master equation but they can encompass an extremely rich set of phenomena. The hoppings and reactions rates are most often chosen local, in the sense that each individual particle performs a random walks, with or without exclusion, and the reactions happen between particles on the same site. Because each reaction can be chosen individually to model a particular process, reaction-diffusion processes are a rich playground to formulate and test minimal models. To fix the idea, let us already describe the set of reactions and diffusions defining DEP. DEP is a process with two species of particles, healthy and infected individuals, noted respectively A and B. A and B particles can hop to neighbouring sites, without exclusion effects, at rate D_A and D_B respectively. Furthermore, when a A and a B individuals are on the same site, B can infect A at the rate k. Finally, B individuals recover and transform into A at rate $1/\tau$. This set of reaction can be summarized symbolically as

$$
\begin{array}{lc}
\textbf{Infection} & A + B \xrightarrow{k} B + B \\[2mm]
\textbf{Recovery} & B \xrightarrow{1/\tau} A \\[2mm]
\textbf{Diffusion of A} & A + \emptyset \xrightarrow{D_A} \emptyset + A \\[2mm]
\textbf{Diffusion of B} & B + \emptyset \xrightarrow{D_B} \emptyset + B
\end{array}
$$

$$(2.1)$$

Let us review quickly how the physics of reaction-diffusion processes can be investigated theoretically. The simplest approximation consists in assuming well-mixing, meaning that the densities of A and B are homogeneous, and to neglect all correlations, which is called the (homogeneous) mean-field approximation. This gives simply the generalization of the law of mass action to out-of-equilibrium for the process considered. Still neglecting correlations but taking into account non-homogeneity, one obtains partial differential equations which are a subject of study in themselves (Kolmogorov et al. 1991; Turing 1952). However, the mean-field approximation may fail to describe the physics at hand. Notably it is known to not be applicable at a continuous phase transition. In order to theoretically study such systems further than their mean-field description, we will see later that one generically has to solve an infinite hierarchy of coupled temporal evolutions obeyed by the moments of the observables of the system.

A subset of the reaction-diffusion processes are said to be integrable. Formally, these are systems which possess enough conserved quantities such that one can decouple all the degrees of freedom of the system. For these systems, one can hope to find closed analytical expressions for any averaged observables. Integrable stochastic processes are closely linked to quantum integrable models. In particular, stochastic processes in one spatial dimension and with exclusion can often be mapped to a quantum spin chain problem. The methods developed in this field are thus closely related to their quantum mechanical counterparts (Babelon et al. 2003).
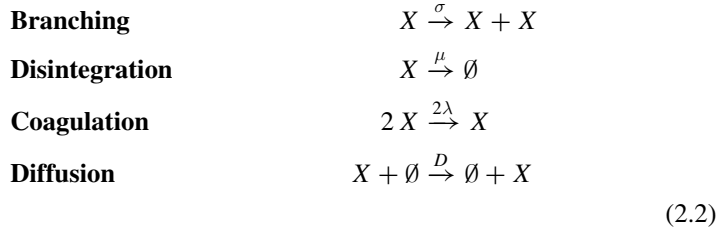
However, the largest part of physically relevant reaction-diffusion processes do not satisfy the conditions of integrability. One way to tackle this difficulty is to turn to numerical studies. Reaction-diffusion processes are conceptually simple to simulate numerically and can be implemented with cost-effective methods. In general, these methods are based on the Monte-Carlo algorithm to explore the phase space of the system through jumps between states (Marro and Dickman 1999). If one is not satisfied with numerical simulations, approximations has to be devised in order to go further. Many approaches to get an approximate picture from the exact hierarchy of equations have been attempted over the years. Among these and without exhaustivity, let us cite three which have been successful in describing critical phenomena in reaction-diffusion systems. The first kind of approaches consist in modifying the reaction rates such as to inhibit the propagation of the correlations. In this family one finds for example the cluster mean-field method (Gutowitz et al. 1987). Another type of approach aims at devising mesoscopic Langevin equations for the coarse-grained observables. These Langevin equations are often justified on phenomenological grounds (Janssen 1981; Wiese 2016), but they can sometimes be derived rigorously (Gardiner et al. 1976; Kampen 2007; Kurtz 1978). Finally, reaction-diffusion processes can be mapped to a field theory in order to use the tools of RG and NPRG. As announced in the introduction, this is our choice in this work.

After this general survey on the different approaches to study reaction-diffusion processes, let us turn to their phenomenology. In order to do so, before tackling DEP, we present in the next section the directed percolation process, which is a simpler and well-studied one-species model.

2.1.1 Directed Percolation

In section, we give a brief summary of the directed percolation process. This will turn useful because directed percolation (DP) is the most paradigmatic model for transitions to an absorbing state and DEP can be seen as a extension of it. Furthermore, it can serve as a pedagogical introduction to the framework of reaction-diffusion processes for unfamiliar readers.

The DP process is given by the evolution of a population of particles, noted X, distributed on the sites of a lattice (most generally a d-dimensional hypercubic one). Each particle can hop to neighbouring sites with diffusion rate D. Moreover each particle can replicate itself with rate σ and disintegrate with rate μ, and two particles can merge with rate 2λ upon encountering. These rules are symbolically summarized in the following table:

Branching	$X \xrightarrow{\sigma} X + X$
Disintegration	$X \xrightarrow{\mu} \emptyset$
Coagulation	$2\,X \xrightarrow{2\lambda} X$
Diffusion	$X + \emptyset \xrightarrow{D} \emptyset + X$

$$(2.2)$$

The names branching, coagulation, and directed percolation come from an equivalent formulation of this model in term of a percolation problem with a preferred direction. We refer to Hinrichsen (2000) for an extensive review of the subject of DP. The set of rules above can lead the system to a state with zero particles but do not allow to escape from it. The zero particle state is the simplest example of what is called an absorbing state. The natural question at this point is whether and for which initial conditions and values of the parameters the system will fall into this absorbing state.

Let us give a first rule of thumb answer. In order to do so, we assume well-mixing and make the mean-field approximation. Namely, we assume that the diffusion rate is much faster than the branching, disintegration and coagulation rates, such that the system can be considered as homogeneous and can be described by only one observable: the total number of particles, N. The branching and disintegration rates are then proportional to N. Furthermore, for N large, the correlations between particles should be negligible and the coagulation rate should be proportional to N^2, as it is proportional to the number of pairs. In this approximation, the variation of N is thus given by

$$\partial_t N = (\sigma - \mu)N - \lambda N^2 \,. \tag{2.3}$$

A quick stability analysis of this equation tells us that for $\sigma - \mu < 0$, the only stationary state is $N = 0$, which turns out to be stable, while for $\sigma - \mu > 0$ the only stable stationary state is $N = (\sigma - \mu)/\lambda \equiv N^*$. The system thus undergoes a phase transition from a fluctuating occupied state ($N = N^*$) to the absorbing extinct state ($N = 0$). Furthermore, it is readily calculated that for $\sigma - \mu \neq 0$, (2.3) gives an exponential approach to the stationary state, with a typical time $\tau = (\sigma - \mu)^{-1}$ while it acquires an algebraic behavior, decaying as t^{-1}, at $\sigma - \mu = 0$. This critical slowing down of the dynamics is typical of second-order phase transitions. Of course, this first approximation has the obvious drawback that it describes a transition to a state with zero particles by making the assumption of a large number of particles in presence and by neglecting the pair correlations.

These findings prompt to give a more precise description of the model. The rules (2.2) has to be interpreted in terms of a Markov process, whose master equation is given in Appendix A. In that appendix, we explain how the time evolution of the averaged observables can be derived using the generating function. In particular, the exact equation for the mean occupation number at the site k reads