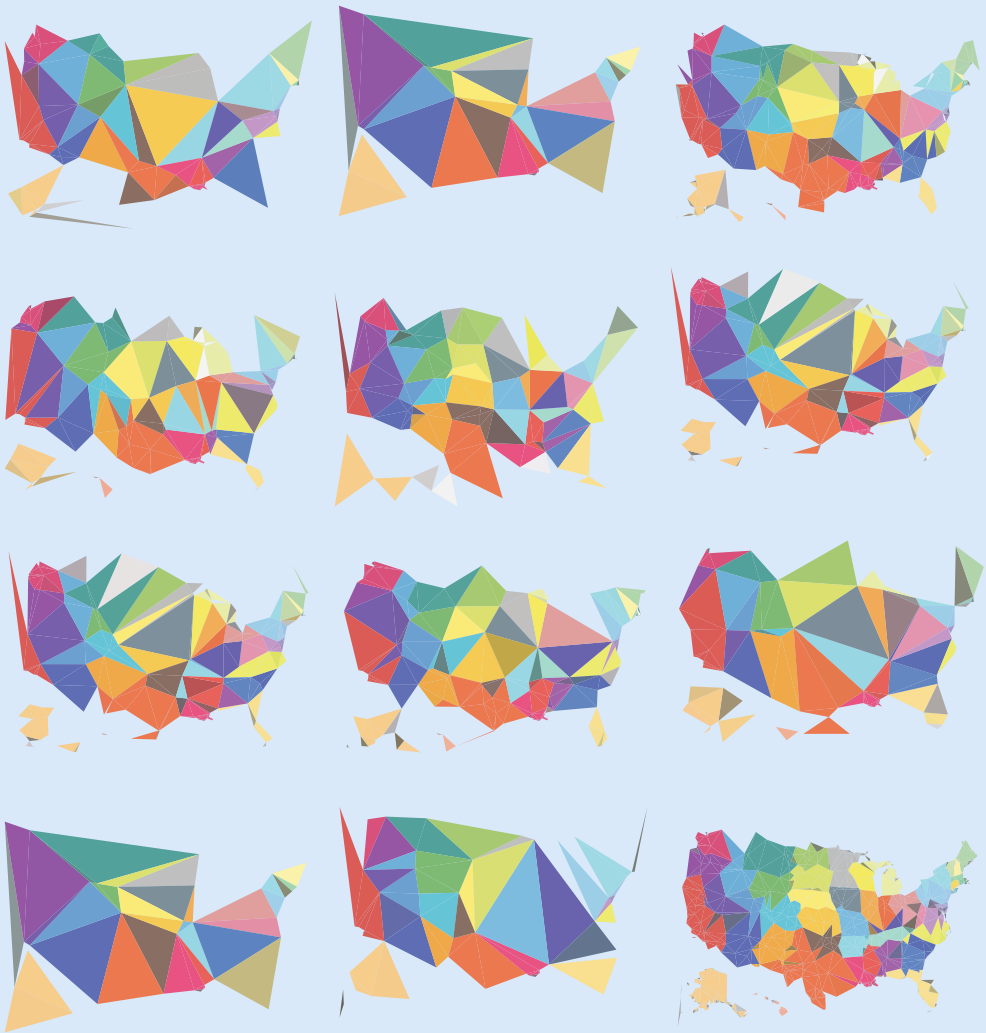


Moon Duchin, Olivia Walch Editors



Political Geometry

Rethinking Redistricting in the US
with Math, Law, and Everything
In Between

 Birkhäuser

Moon Duchin • Olivia Walch
Editors

Political Geometry

Rethinking Redistricting in the US with
Math, Law, and Everything In Between

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Preface

This unusual book was originally conceived as the proceedings of an unusual conference. In 2016-2017, a kitchen-table collective of mathematicians formed the Metric Geometry and Gerrymandering Group, and in July 2017 we planned a Geometry of Redistricting Workshop at Tufts University. Speakers came from many fields and vocations—scholars, technologists, organizers, and litigators—and the conference drew so much interest that we had to hold it at a theater off campus. More than four years later, the collective is now a Lab, and the Lab is engaged in the decennial redistricting in some way in over a dozen states. Meanwhile, the conference proceedings evolved into the ambitious boundary-flouting volume you see here.

What is “political geometry”? For starters, it’s a riff on “political geography,” an established academic discipline that looks at the spatial dimensions of elections and governance. Where there’s space, there’s shape, and mathematicians might have something useful to say. And it’s not just math, geography, and political science: there’s software, graph algorithms, policy, civil rights, history, political philosophy, and of course law in the mix. This book was designed to serve up a multi-disciplinary buffet, with both traditional fare and fresh fusions.

WHAT’S IN THIS BOOK

We’ve divided this book into five broad parts, coarsely chunked by domain but with lots of overlaps and cross-talk: Political Thought, Law, Geography, Math and Computer Science, and “On the Ground.” Each chapter is written by a different author, or set of authors, drawing from a huge variety of backgrounds and perspectives. We’ve added smaller “Explainers” in a few key places that treat important topics in a stand-alone fashion. (They are marked with colored strips on the page corners so they’re easy to flip to.) We’ve also sprinkled interviews with a range of practitioners and theorists throughout.

OK, BUT REALLY, WHAT’S IN THIS BOOK

We start the book off with an introduction and overview from *Moon Duchin* (Chapter 0). She’ll cover some of the basics: Can you use shape to define a gerrymander? What about judging from results themselves, where representation is out of whack

with the vote balance? If not either of those, what can you do? This chapter surveys the lay of the land in data-driven redistricting.

Chapter 1, “Explainer: Compactness by the numbers,” builds on the Introduction, defining scores commonly used to judge the shape of a plan and identifying some of their basic shortcomings.

Then we’re off to the Parts:

POLITICAL THOUGHT

Traditionally, the identification of gerrymandering has been the province of political science.

In Chapter 2, “Measuring partisan fairness,” *Mira Bernstein* and *Olivia Walch* provide a mathematical view on some metrics of fairness from the political science literature.

But what do we even mean by fairness? In Chapter 3, “Concepts of Representation,” we interview four political thinkers—philosophers *Elizabeth Anderson*, *Ryan Muldoon*, and *Brian Kogelmann* and political scientist *Claudine Gay*—on what fairness in representation looks like.

In Chapter 4, “Redistricting: Math, systems, or people?,” *Keith Gaddie* gives a high-level take on “the redistricting problem” from his perspective as a political scientist and veteran redistricting expert.

Chapter 5, “Political geography and representation,” goes in-depth on a particular conundrum raised throughout this Part: We know that geography matters in elections, but how? Here, *Jonathan Rodden*, a political scientist who specializes in political economics and geography, teams up with data scientist *Thomas Weighill* to take a look at how political geography is reflected in districting outcomes, particularly addressing questions of size and scale.

LAW

Then we turn to the law.

In Chapter 6, “Explainer: A brief introduction to the VRA,” *Arusha Gordon* and *Doug Spencer* provide the reader with key background on the Voting Rights Act and its relevance to redistricting.

Leading voting rights scholar *Ellen Katz* jumps off from there in Chapter 7, “Race and Redistricting,” where she surveys the law of racial gerrymandering with a detailed look at the intertwining roles of Congress and the Courts.

Chapter 8, “Law, computing and redistricting in the 1960s,” brings a historian’s perspective to the book. In it, *Alma Steingart* looks back to the moment that the U.S. Supreme Court introduced its One Person, One Vote standard, which made computing a permanent part of the redistricting scene.

Finally, in Chapter 9, “The law of gerrymandering” *Guy-Uriel Charles*, an expert in race and constitutional law, and *Doug Spencer*, who teaches both law and policy, examine the parallels between racial and partisan gerrymandering law.

GEOGRAPHY

As a discipline, geography spans from philosophy of place to technologies of space.

Chris Fowler is a geographer who studies cities, planning, and neighborhood change. In Chapter 10, “Race, space, and the geography of representation,” he reminds us not to take demographic distributions for granted, but to put them in historical and social context.

In Chapter 11, “The elusive geographies of communities,” *Garrett Dash Nelson*, a historical geographer and the curator of maps at the Boston Public Library, takes a close look at community and regionalization.

Chapter 12, “Explainer: Communities of interest,” *Heather Rosenfeld* teams up with Moon for a practical primer on what constitutes a “COI” where redistricting is concerned.

In Chapter 13, “Geography as data,” geographer and data scientist *Ruth Buck* comes together with *Lee Hachadoorian*, whose work uses geospatial technology in urban and demographic analysis, to give us a close look at geo-electoral data and the software that wrangles it.

MATH AND COMPUTER SCIENCE

To a mathematician, the redistricting problem can feel like a playground where any idea from math might find fruitful application.

In Chapter 14, “Three applications of entropy,” *Larry Guth*, *Ari Nieh*, and *Thomas Weighill* test this out by taking the math/physics idea of entropy and seeing where it fits. They end up describing three use cases: 1) how different are two plans? 2) how much does a map split counties? 3) how segregated is a city?

Of course, entropy isn’t the only hammer in the toolkit. In Chapter 15, “Explainer: Measuring clustering and segregation,” *Moon Duchin* and *James Murphy* tackle metrics of spatial patterning in a completely different way, examining a construct that geographers call “Moran’s I.”

In Chapter 16, “Redistricting algorithms,” computer scientists *Amariah Becker* and *Justin Solomon* give a big, big picture overview of how computing can bear on the redistricting story. It’s notoriously hard to compare algorithmic strategies against each other because implementation can be very finicky, and the goals of different researchers don’t line up perfectly in the first place. But they go for it anyway, and it makes for some very illuminating comparisons.

Next comes Chapter 17, “Random walks,” by mathematicians *Daryl DeFord* and *Moon Duchin*. This chapter looks at one of those algorithmic strategies, Markov chain sampling, in closer detail.

ON THE GROUND

We close with the voices of practitioners.

Megan Gall, *Karin Mac Donald*, and *Fred McBride* all have political science training but now work hands-on with maps and data in the field. In Chapter 18, “Making maps,” they talk about their experiences drawing in the real world.

Nate Persily is a law professor who has frequently been appointed by courts to draw the lines when the primary parties can’t agree. In Chapter 19, we interview him about his experiences.

The longer we spend in the redistricting world, the more our focus moves from flagging the rulebreakers to upgrading the rules. Chapter 20, “Explainer: Ranked choice voting” acts as a primer on one policy change that may be able to do just that.

Next is a redistricting story very close to MGGG’s home in Boston—in Chapter 21, *Iván Espinoza-Madrugal* and *Oren Sellstrom* tell us about the voting rights suit that they filed on behalf of plaintiffs in Lowell, Massachusetts—the first to base a voting rights claim on a coalition between Asian and Latino voters. As we write, the lines are being drawn (by Nate Persily!) for a brand new city council structure.

After that is Chapter 22, “Explainer: Race vs. Party,” a brief look at how race and party preference intertwine in voting patterns, and how this has played out in several recent cases.

We close with Chapter 23, “The state of play in voting rights,” from *Kristen Clarke*, who has held many civil rights law positions across government and nonprofit organizations, and *Arusha Gordon*, an attorney who is her former colleague at the Lawyers’ Committee for Civil Rights Under Law (LCCR). They leave us with a wide-angle view of where voting rights stand today, and point to possible locations of battlefields to come.

Who’s the audience for this book? Great question.

We have tried to frame the book to be simultaneously engaging to community organizers, math-curious high school students, philosophers, programmers, and election lawyers. We hope it’s suitable to assign in a political science class and to ground a data science curriculum. Not every chapter is written at the same level, and they’re certainly not written in the same voice. We think this keeps it true to its interdisciplinary conference roots: a symposium of sorts, bringing a lot of different people together to share tools and ideas. There are illustrations (many by Olivia) throughout the book, which provides one kind of throughline. There are also sidebars, often written in the voice of the editors, that can be skipped without cost to the exposition but should provide more depth or color in strategic places.

For readers, we hope there is something for every taste. You will get the most out of this buffet by giving all the dishes a try. The authors have made a real effort to make their flavors accessible but complex.

Code-heavy chapters have a corresponding GitHub repo (github.com/political-geometry). We hope that both the code and the book overall can be public resources in the long and wild mapping wars that are already underway.

EDITORS' ACKNOWLEDGMENTS

MD: I wandered into my current obsession with redistricting through teaching a class on abstract voting theory (Mathematics of Social Choice), primed by two happy coincidences. One, I had attended a session of the American Political Science Association a few years earlier and I chanced on a talk that introduced me to the idea of “compactness.” Two, one of my closest and oldest friends, Kristen Clarke, is a prominent civil rights attorney, who was just telling me that voting rights litigation was in need of a new generation of experts. So I had election systems, district shape, and civil rights on my mind. I teamed up with two multi-talented, multi-curious mathematician friends, Ari Nieh and Mira Bernstein, and the *Metric Geometry and Gerrymandering Group* was born. We recruited the brilliant and demonically productive computer scientist Justin Solomon (who’d read about us on Reddit!) not long after. I thank Kristen, Mira, Ari, and Justin first for the life-changing inspiration and collaboration.

Five years later, this is my main research focus, and I run a Lab that is deeply engaged in public mapping, plan evaluation, and scholarship in data science for civil rights. And while this book was in press, Kristen became the brand new Assistant Attorney General of the United States—the top civil rights attorney in the land!



Kristen and Moon in 2017 at the first MGGG workshop

MD thanks: Daryl DeFord, Jordan Ellenberg, Aidan Kestigian, Thomas Weighill, Sam Gutekunst, John Powers, Sarah Cannon, Max Hully, Ruth Buck, Parker Rule, JN

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Alfonso Gracia-Saz is a rock-star educator, crack square-dancer, and hilarious and loving human who helped develop the Educator materials for the original MGGG workshop cycle. He tragically died in the COVID epidemic earlier this year. I love and miss you, Alfonso.

OW: My happy coincidence is that I only met Moon at MGGG's first conference in 2017, when I went to draw a comic about it, and not a few years earlier, when we were both at the University of Michigan and independently going to occasional poker games at Dick Canary's house. If I had met her then, I would definitely have lost a lot of money to her.

OW thanks the authors, Mira Bernstein, Thomas Weighill, Heather Rosenfeld, and (of course) Moon Duchin. She also thanks David Renardy and Matt Jacobs for their love and support.

We thank The Norman B. Leventhal Map & Education Center for their kind permission to use the historical maps included in this volume, as well as Mapbox for permission to use screenshots from the Districtr tool.

Chapter 0

Introduction

MOON DUCHIN

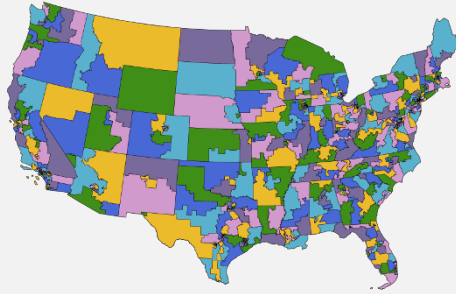
You've probably been hearing about gerrymandering lately. As I write in 2021, the U.S. Supreme Court has heard cases from Wisconsin, Maryland, Texas, Virginia, and North Carolina in its last two terms, if only to give back equivocal answers. Meanwhile, in Pennsylvania, plaintiffs pressed a partisan gerrymandering suit and prevailed in state court in 2018, followed by a frenzied few weeks with new proposed maps flying around, and finally a brand-new set of congressional districts, with sweeping effects for the Congressional delegation.¹ In North Carolina, multiple state-level cases fared better than the federal one, and the state put new congressional and legislative districts in place by the 2020 election, one tick before they would have had to be redrawn anyway in the new Census cycle.

What was at stake in these cases? Apparently quite a few seats, for one thing. Pennsylvania's new map coincided with a major shift in its congressional delegation, from 13–5 Republican control to a 9–9 split. Was that responsive to shifts in the vote, or a mere function of the carefully re-drawn lines?

Gerrymandering, or agenda-driven line-drawing, is a practice (and an anxiety) as old as the Republic. In a country that vests power in elected representatives, there will always be skirmishes and scrapping for control of the process, and in a system like our House of Representatives where winner-takes-all within a geographical district, the delineation itself is a natural battleground.

¹Full disclosure: I got a front-seat view of Pennsylvania's districting reboot as a consulting expert for Governor Tom Wolf.

0.1 U.S. ELECTORAL DISTRICTS 101



- The U.S. House of Representatives has 435 members, ever since 1911. They must be elected one per district by common practice that was made official in law in the 1960s.^a The picture shows the 432 in the continental U.S. circa 2019.
- State legislatures currently have 7308 members, all elected from districts. (Over 1000 of them are elected from multi-member districts.)
- Many thousands more elected representatives sit on city councils that are elected from districts—New York and Chicago have the two largest city councils, with 51 and 50 districts respectively.
- And then there are school districts, county commissions, ambulance districts, water boards, executive commissions, and more. The U.S. Census Bureau conducted a count of local governments in 2017, enumerating around 90,000 across the country.^b
- “One person, one vote” jurisprudence from the Supreme Court (from the 1960s onward) tells us that electoral districts should be population-balanced within their jurisdiction—so zones to elect a school board must have nearly the same population, even if zones defining school attendance need not.
- We usually use *plurality* or “first past the post” voting in districts—i.e., the single candidate with the most votes wins. There are exceptions, like the multi-member legislative districts mentioned above, and the many local elections that use “at-large” schemes to fill several seats at once.
- We have two major parties, but the parties have shifted significantly over American history and certainly might continue to do so. For elections where candidates run with a party ID, there is often a primary several months in advance to pick the nominee in each party before the inter-party competition in the general election.
- Incumbency advantage is enormous. U.S. House races happen every two years, and for instance in 2016 only five incumbents ran for re-election but lost in the primary; another eight lost in the general election; and 380 were re-elected, for an overall success rate in the neighborhood of 97%.^c

^aApportionment Act of 1911, Pub.L. 62–5; 2 U.S.C. §2c “no district to elect more than one Representative”

^b<https://www.census.gov/programs-surveys/cog.html>

^cBrookings Institute, *Vital Statistics on Congress*.

I am a mathematician with a background in geometry (shape) and dynamics (systems in motion). I have a long-standing investment in civil rights work and social movements. I'm also invested in social studies of science, and I like to think about how scientific argument circulates in politics, policy, and law—how technical expertise acts in the social sphere. This comes with a healthy skepticism of scores and metrics that promise to take a *complicated* thing and make it *simple*. So gerrymandering is an irresistible problem for me. It's all about peeling back layers of intuition about shapes, numbers, and power.

Today the primary image of gerrymandering centers on party politics, but the long history of manipulative redistricting has been driven by many other agendas, like back-room deals to make safe seats for incumbents or to dice up a district to stick it to a hated rival. And it's impossible to understand the current context or the bulk of the jurisprudence without contending with the history of schemes to suppress the political power of racial minorities, especially Black and Latino voters—not a practice of the past, but one that's even arguably on the rise in places where new demographic formations are visible. All of these flavors of gerrymandering have in common their basic structure: draw the lines to arrange pluralities for one set of voters and dilute the influence of the other voters.

This is stubbornly difficult to identify. People think they know gerrymandering by two hallmarks: bizarre shapes and disproportionate outcomes. But neither one is reliable.

1 HOW (NOT) TO SPOT A GERRYMANDER

1.1 BIZARRE SHAPES

We think crazy shapes tip us off to moustache-twirling gerrymanderers for a few reasons. The simplest is that we can easily imagine that the district line had to veer around wildly to include *this* pocket of people, but not *that* one. This seems especially likely if a district has been made to narrowly favor one party's voters in election after election. Another reason—if we expect that different kinds of people with shared community interests tend to clump together—is that jagged lines may indicate that an unspoken agenda has dominated over the contours of neighborhoods and communities. Finally and possibly most persuasively, we may worry that those who draw the lines just have too much detailed control over outcomes. Wildly winding boundaries flaunt the power of the pen.

The 1812 episode that gave us the word “gerrymander” sprang from this same pile of intuitions. The name is derived from Elbridge Gerry, governor of Massachusetts at the time. Gerry has quite a Founding Father pedigree—member of Congress, James Madison's vice president, a major player at the U.S. Constitutional Convention—so it's remarkable that he's mainly remembered in connection with nefarious redistricting. The “Gerry-mander,” or Gerry's salamander, was the curvy state Senate district in Boston's North Shore that was allegedly drawn to favor one party, Gerry's Democratic-Republicans, over the rival Federalists (see Figure 1). A woodcut political cartoon ran in the Boston Gazette in 1812 with wings and claws and fangs

suggestively added to the contours of the district to heighten its appearance of reptilian contortions—Figure 2 shows a Salem Gazette adaptation the next year.

So the idea that eccentric shapes are red flags for wrongdoing is old. And just as old is the idea that close-knit districts promote democratic ideals. Even before the notorious Gerry-mander, James Madison had written in the Federalist Papers (1787) that “the natural limit of a democracy is that distance from the central point which will just permit the most remote citizens to assemble as often as their public functions demand”—in other words, districts should be *transitable* to promote the possibility of deliberation. The new federalist model would knit these together: the United States was to be a republic built from these districts, serving as its constituent democracies. Forming districts of manageable size would ensure that the representatives have knowledge of “peculiar local interests” to be conveyed to the legislature (Fed. 14, 37, 56). So here, shape is in the mix but only as a correlate of function. In 1901, a federal apportionment act marked the first appearance in U.S. law of the vague desideratum that districts should be composed of “compact territory.” That word *compact* then proliferated throughout the legal landscape of redistricting as a districting criterion, but almost always without a definition.²



Figure 1: Democratic-Republican Thomas Jefferson (left) and Federalist Alexander Hamilton (right) disagreeing over the compactness of this district. (Reenactment.)

Going back to the 1810s, the language from the Original Gerrymander broadsides is instructive. In the Salem Gazette, the *democratic* sins of the district are that it “cut up and severed this Commonwealth” much like “the arbitrary deeds of Napoleon when he partitioned the territories of innocent nations to suit his sovereign will.” The *geographic* sins are those of its “peculiarities” of shape: three towns too far north, a town from a separate county “tacked on,” and so on.

²Apportionment Act of 1901, 31 Statute 733. For a precise definition, see Chapter 1.



SALEM GAZETTE.

VOLUME XXVII.]

SALEM, FRIDAY, APRIL 2, 1813.

[No. 27.]

THE GERRY-MANDER, OR ESSEX SOUTH DISTRICT FORMED INTO A MONSTER!



FEDERALISTS! FOLLOWERS OF WASHINGTON!

AGAIN behold and shudder at the exhibition of this terrific Dragon, brought forth to swallow and devour your Liberties and equal Rights. Unholy party spirit and inordinate love of power gave it birth;—your patriotism and hatred of tyranny must by one vigorous struggle strangle it in its infancy. The iniquitous Law, which cut up and severed this Commonwealth into Districts, is kindred to the arbitrary deeds of NAPOLEON when he partitioned the territories of innocent nations to suit his sovereign will. This Law inflicted a grievous wound on the Constitution,—it in fact subverts and changes our Form of Government, which ceases to be Republican as long as an Aristocratic House of Lords under the form of a Senate tyrannizes over the People, and silences and stifles the voice of the Majority.

When Tyranny and arbitrary Power thus make inroads upon the Rights of the People, what becomes the duty of the citizen? Shall he submit quietly and ignominiously to the decrees of Usurpers? Are the citizens of this Republic less jealous of their rights than their ancestors? Will you, then, permit a Party to disfranchise the People,—to convert the Senate Chamber into a Fortress in which ambitious office-seekers may entrench themselves and set at defiance the frowns of the People? No,—this usurping Faction must be dislodged from its strong-hold.

Arise, then, injured Citizens! Turn out! turn out! let Monday Next be the day of your Emancipation—by one manful Struggle reclaim your usurped Rights—and from into obscurity those audacious men who unblushingly boasted—“*We have secured the Senate for ten years, and should have been Fools if we had not done it.*” Prove on election day that the Polly of their men is equal to their want of honesty and contempt of the People. Elect patriots who will be loyal to the Constitution, and faithful to the interests of the State,—men who will harmonize with our illustrious STRONG,—and who will not sell our Drafted Soldiery, like Hessians, to any Foreign Power.

requilition their utmost cunning to devise means expressly to accommodate their own views and to promote their own interests, should still be unwilling to adhere to the policy prescribed by themselves.

Such a man would have to learn, that violence and subordination must be employed to accomplish what cunning and trick had failed to secure. It is in fact the fatted maxim of forms, who, if the testimony of their own professions is to be credited, have much love for the people and a sacred regard for their rights, to condemn all legal provisions, which impede their ambitious designs, how ever just and necessary to secure the peace and order of society; to execute by force whatever they are prohibited to do by law, and to aggrandize themselves without respect to justice or decency. Upon no other principle can we account for the conduct and activity of those, who, at a late election in this town, scrupled not to bring to the poll, not only alien residents, who were known never to have been naturalized, nor only a certain clergyman (the learned Conventor of Politics for this vicinity) whom they had, in common with the rest of his brethren, disqualified to vote in the choice of town officers, and thus to involve him in the turpitude of their transgressions, and to multiply the causes of repentance, (unless he for far believes, “that the end would justify the means,” that there would be *non culpa peccata*); but even to induce citizens from other towns, to invell the minority with a power, which they would be furs to abuse. He, who promises freedom to the captive, when he increases the number and weight of his chains, acts as confidently, as he does, who declaims much and loud about the liberties of the people and the rights of his country, when all his actions evince, that he regards neither the one nor the other, any farther, than he discovers a direct advancement of his own interest. It is greatly to be desired, that honest men would sometimes flatter themselves to be influenced as well by objects of vision, as hearing; and compare actions with profssion; and then let judgment decide upon the worth of character, and the safety of reposing confidence.

Mr. PEISTER.—I am a native of *Gesellschaft*, and some years ago was situated at *Guttenburg*; but for several months I have been engaged in trade in this State. I have seen very pleasing notices of the Method for alien enemies to report themselves forthwith while I remain here I will to comply with the laws and ordinances of the country, and feel no enmity towards the government or people. A peace or change of times would destroy the profits of my present speculations, and in that case I should probably soon depart. Moreover, I do not find that the U. S. of America are at war with my adopted country. None I will thank any of your correspondents to inform me what is proper or expedient from one in my circumstances; as these are many in this country, and more expected, finally situated.

“**WOMAN A SINNER.**”
The correct, and was once the universally received phrase. But a spirit of innovation and hypercriticism directed by gross misconception has engendered and brought into use general use the awkward and absurd term, *vice a signatur.* To show how mischievous tolerated abuses are, we would refer to the following from the Boston *Free Press*:
“*Mrs. Russell & Colver*—I have read a very simple, but small tract *Concern your manhood, gentlemen* (read) *over the above title.*”

Now, the tract, *con*” This title is wholly all praise for its correctness and candor, and may be read without offence and certainly with advantage, by all parties. It is previously calculated for distribution;—it repels no reader, whatever his politics or character.

Given over my hand and seal, March, 1813.
THE GOOD-OLD WAY.

DARKNESS VISIBLE.
Messrs. GARDNER has directed the Lights near the Chesapeake and Nantuxet to be extinguished, so that our friends may be left in the dark, and of course it will be perfectly easy to place Torpedoes under the Feet and blow us out of our waters.—How perfectly in character is it for our present rulers to extinguish instead of illumining, and to scatter darkness where their predecessors have left light. It is the sport of these Preliminary-men, the Mad Tom.
To play at bowls with sun and moon,
And shuffle them in couples.

It is expected that every man who does not wish to be drafted as a conscript and sent out of the State to fight the battles of the Southern Nobles—every Farmer who wishes to prevent his farm and stock from being forced and sold to pay the enormous taxes which are thereby to be laid on them, will turn out on the first Monday of April next, and support the Peace Treaty. You will find him in a perfect example; let us follow it. You all know you were proficients and happy under Washington; you are now wretched and miserable under Madison.—You cannot, you will not give your suffrages for men who are in favor of the present war, which you are obliged to your duty and unfaithful to your country.—*Angels Herald.*

Salem, FRIDAY, APRIL 2, 1813. SOUTH with a Vengeance!

The Senatorial District made and called “ESSEX SOUTH District,” by a Gerry-mander Legislature, has the following peculiarities:

1. It contains three Towns (Salisbury, Amesbury and Haverhill) more territory than any other town in the County.
2. An East and West line drawn through the centre of the County leaves Salisbury, Amesbury, Haverhill, Methuen, a large part of Andover and a small part of Middlesex [sic] on the Northern side.
3. The only towns wholly in the Southern section of the county that the district contains are Danvers, Lynnfield, Salem, Middlesex and Lynn [sic].
4. The northernmost town of another county is tacked on it.
5. There neighbor, take a piece of chalk and see if it does not come out fo.

Measures for Defence have been adopted in all the ports threatened by the British Squadron At Wilmington a Committee of Safety has been appointed. The Honorable Messrs. BATES and HOSMER (U. S. Senators from Delaware) and other zealous Federalists constitute this Committee. These are the Patriots on whom the country must rely when the hour of extreme peril approaches. Our Democratic rulers have sufficient talents to plunge the nation in misery and calamity, but none but Federalists can raise the nation and save its “drowning honors.” The deceased Heroes WASHINGTON, LINCOLN, KNOX, WAYNE, MONROE and HAMILTON,—the Soldiers of Liberty who achieved our Independence,—were zealous and decided *Federalists* to the last hour of their lives,—and even WASHINGTON was culminated and traduced as a Death Tory by the Democratic party,—which party he solemnly declared in a letter to a friend was “A CURSE TO THIS COUNTRY.” This same party is now scourging the country with War, and the People feel the truth of the words of WASHINGTON.

The Right of Suffrage.
ALL men will agree in the importance of having the right of suffrage properly defined by law; and noble men, when unimpeached by passion, will acknowledge the necessity of enforcing the law with the utmost firmness, and can correctly calculate the immense benefits which would result from such a course.—It is the natural dictate of the understanding and of conscience. But man in theory and in action is not the same being. In the dispassionate moments of speculation, he listens to the suggestions of his reason, and approves of the admissions of his understanding; but in practice he acknowledges no law but convenience, and feels no satisfaction for his exertions except in triumph and victory.—You a person, whose attainments had not been too often excited, will be had become incapable of new surpris; by the moral phenomena which are continually occurring in the political world in these days of strange delusion and of sinistries it would appear incredible, that a party of men, after they had had the opportunity of putting in

Figure 2: The O.G. (Original Gerrymander)

There's no question that the outline of a crooked reptile is doing a lot of work on your intuition. If this feels like an "eyeball test," that's exactly what it is, and it's a major driver in redistricting to this day. Thirty-seven states have some sort of shape regulation on the books, and in almost every case (at least until the map goes to court!) the eyeball test is king.

But the problem is that the eyeballed outline of a district tells you a very partial, and often very misleading, story. Take Alabama's 1st district (Figure 3), bounded to the south by the jagged Gulf coast and to the north by a pair of rivers.



Figure 3: Alabama's 1st district.

The parts of its boundary that are not dictated by physical geography tend to follow county lines fairly faithfully. And county lines may be tortuous themselves, but you wouldn't want to punish a district for following them! (In fact, many states have rules telling you that district lines should follow county lines "to the extent practicable.") This spotlights a balancing act that is both real and often scapegoated: districters quite often will claim that other districting rules and principles forced horrible shapes on them. The plausibility of this claim.... varies, to say the least.

For instance, Figure 4 shows a pair of notorious districts, one from North Carolina and one from Maryland. Party politicians on both sides of the aisle claimed (and may have believed!) that the shape of NC-12 was forced on them by the Voting Rights Act. And at least one leading political figure asserted (but surely did not believe) that MD-3 had to look this tortured to hit a precise population number.³

Certainly there can be benign reasons for ugly shapes. Even more importantly, districts that are plump and squat and symmetrical to the eye offer no real seal of quality. For example, in the 2018 redistricting scrum in Pennsylvania, the state Supreme Court invalidated the 2011 Congressional plan and asked for a new one. Interestingly, the court order named a number of metrics that had to be calculated for any new plan, including five somewhat different scores of shape to be reported for each district, without specifying what role all those numbers would have in their decision. First crack at a new plan goes to the legislature, which had the opportunity to commission a new plan and to pass it as a bill. They didn't end up voting on it, but the Republican leaders Mike Turzai (House) and Joe Scarnati

³John T. Willis, the Democratic party stalwart who chaired the redistricting subcommittee, said: "It's a very complex situation, and population is the No. 1 driving characteristic. ...All of our congressional districts don't deviate by more than one person." See E. Batista, "Politics Makes Strange Bedfellows, Even Stranger Congressional Boundaries," perma.cc/P6Z4-S2NL.

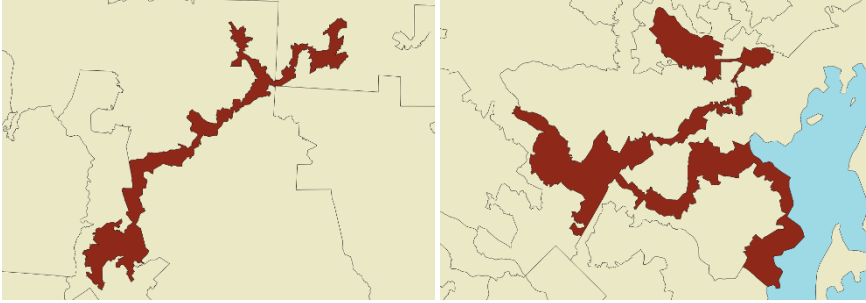


Figure 4: Examples of scapegoating: North Carolina's 12th district and Maryland's 3rd, circa 2013.

(Senate) floated an alternative plan on Twitter before filing it with the court—so I'll call it "the Twitter plan." The Twitter plan achieved glowing compactness scores, under all five formulas specified by the court, relative to the much-mocked 2011 enacted plan that it was aimed at replacing. But the court found that despite its more pleasing forms, the plan locked in the same extreme partisan skew as its predecessor. So in Pennsylvania, you can get extreme performance with innocent shapes (see Figure 5).

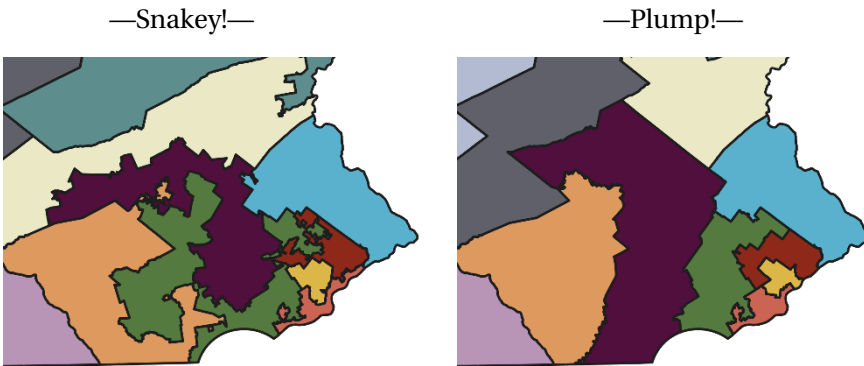


Figure 5: Philadelphia-area inset of the 2011 enacted Congressional plan (left) and the replacement map proposed in 2018 by Republican legislative leaders (the "Twitter plan," right). The replacement looks great, but both plans only have 4 out of 18 Democratic-majority districts when laid over the 2016 Senate vote pattern, which was nearly equal between the two major parties. (Compare Figure 12.)

In fact, some of the reason why shapes were often so flagrantly ugly in the past is not that horrible contours were strictly needed for more extreme partisan performance, but that the right kinds of pressure were not yet in force to rein them in.⁴ What's more, the Twitter plan is not the exception—even strong shape imperatives may fail to constrain. Under scrutiny, line-drawers can often lock in all the advantage afforded by an ugly plan while keeping the shapes nice and plump.

⁴The era of shape-based legal invalidation really began in the 1990s with the so-called Shaw Line of Supreme Court cases (see Chapter 7), when the court grumbled about—but still began to engage in—"endless beauty contests" about district appearance.

0.2 THE RULES OF REDISTRICTING

Redistricting is made extremely complicated by a patchwork of rules that are typically unranked and often fuzzy to boot. Here's a quick primer on the "traditional redistricting principles" across the country, plus some that are less traditional but still make appearances.



Equal population—*Districts within a polity should all have very close to the same population.* The standard way to count is to use the Decennial Census numbers, which is one of the reasons why the Census is so important. This rule applies to the whole nation, and these days any two Congressional districts within a state will most often have a zero- or one-person difference in their Census count!^a

Contiguous—*Each district should be a single connected component.* You may be surprised to hear that only around 30 states require this property by law. This rule is mostly straightforward except when you're building from units that are themselves disconnected, or where there are water crossings to consider.

Compact—*The districts should be reasonably shaped.* ...Whatever that means! Language varies on this one, but for the most part it's a matter of the eyeball test. At least 37 states reference this principle.

Voting Rights Act—*The districts must not undercut the opportunity for minority communities to elect candidates of choice.* This has been a federal law on the books since 1965 and has a formidable (and formidably complicated) legal history and practice.



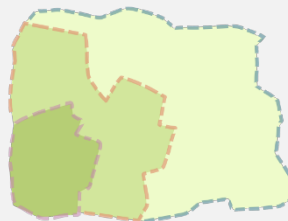
Communities of interest—*Groups with significant shared interests should be strategically placed in order to boost their voice in government.* While it's one of the most conceptually important, this principle is especially open-ended. Shared interests could be about industry, environment, or culture, and groups are sometimes better served by being kept together and sometimes by forming a significant part of multiple districts. More states will take concrete steps toward COI consideration in the 2021 redistricting cycle than in any previous cycle.



Political boundaries—*Counties, cities, and other relevant jurisdictions should not be split among multiple districts when there is a way to keep them whole.* In some states, this is phrased as a preference that district lines should follow political boundary lines.

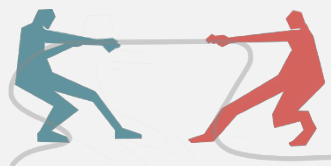
Units—Some states prescribe which building-block pieces plans should be assembled from. For instance, Louisiana and New Mexico mandate whole precincts in legislative plans, and Iowa requires that counties be kept whole in congressional districts.

Nesting—Eight states currently require the state House districts to nest inside the state Senate districts two-to-one, and two additional states require three-to-one nesting.



Incumbency—In some states, there is a rule on the books that implies that *new maps should avoid pairing incumbents to run against each other*. (Pairing incumbents also goes by the colorful name “double-bunking”!) In other states, the rules forbid having the redistricters consider incumbency at all.

Partisan properties—A handful of states have rules indicating that there is a priority on the creation of *competitive* districts or districts that react *responsively* to changes in voter opinion, and numerous states have considered adopting language of that kind. Several other states forbid considering partisan data in the redistricting process.



^ancesl.org/research/redistricting/2010-ncesl-redistricting-deviation-table.aspx

1.2 DISPROPORTIONATE OUTCOMES

So district shape will not do the trick on its own. How about if we cut out the middleman and get right to the bottom line, studying the extent to which the representatives match the electorate. Many people hold the strong intuition that disproportions give *prima facie* evidence of abuse. That is, a group with 30% of the votes *would have gotten* 30% of the seats, if the lines had not been rigged.

But not so fast. Let’s zoom in on a particular case to understand some of the root causes of disproportionate outcomes. We’ll look at a subgroup that reliably has over 1/3 of the votes but is locked out of even 1/9 of the Congressional representation: Republicans in my home state of Massachusetts.⁵ This is a situation where even

⁵Let me flag at the outset that it’s hard to directly measure people’s party preferences for Congress

if your heart expects or desires a proportional outcome, the structure gods are cruel—it can't be done.

If you consider the elections for President and U.S. Senate held in MA since the year 2000, the Republican share of the statewide vote is most often between 30 and 40%, averaging over 36%. Since that's well over a third of the vote and we have nine seats to fill, you might expect a fair map to send three Republicans to the House in each cycle; meanwhile, the last time a Republican won *any* MA Congressional district was in 1994.⁶ That is thirteen straight election cycles of total Republican lockout. So we must be looking at a vicious gerrymander that denies Republicans their rightful opportunity districts, right?

Except the mathematics here exonerates the Bay State line-drawers. The Bush-Gore election in 2000 is a great example. There is literally no way to put together a subset of the state's 351 towns making up enough population for a district—no matter now disconnected and scattered—that preferred Bush. That sounds like a paradox, but it's easily explained. Though Bush won 35.2% of the statewide vote, only 32 towns preferred Bush outright, making up under 3% of the state population. Preferences were very flat around the average, and there just aren't enough Bush-majority towns to anchor a district, no matter how cleverly you group them.⁷

The state started reporting more granular precinct-level results just after that, giving us an opportunity to see that the pattern held up in Massachusetts all through that Census cycle. Kenneth Chase, the Republican challenger to Ted Kennedy in 2006, cracked 30% of the statewide vote. But once again the districting numbers don't shake out for Chase voters. It is mathematically impossible to create a single district-sized grouping of *precincts* that preferred Chase; this is a realistic redistricting setting because precincts are typically preserved whole in Massachusetts legislative plans and rarely broken up in congressional plans. Chase voters simply were not clustered enough for a district to give them access to representation.

The problem is that even though Republican voters are nearly a third of the state, they are also about a third of every town and a third of every precinct—and a third of every household, as far as I know!—so no combination of units can combine to form a Republican majority, even if you throw niceties like compactness and contiguity to the winds. And this phenomenon carries over to any group in the numerical minority. You need a certain level of nonuniformity in the distribution for districting to offer even a theoretical opportunity to elect. The takeaway is that districts are ineffective if a minority is dispersed.

from Massachusetts, because the races are so often uncontested, as five out of nine seats were in 2016. Also, like many states, Massachusetts votes Democratic for national office but loves its Republican governors. But really I just want to make a point about the consequences of certain distributions of *votes*, so we can look at statewide elections for federal office—Senate and President—to understand that.

⁶If there are nine districts, each has about 11% of the state population—since you just need a plurality to prevail, you should be able to control a district with just 6% of the statewide vote. (The apportionment for Massachusetts dropped from 10 to 9 during this timeframe, but I will stick with 9 districts to simplify the discussion.) So Republicans routinely get more than *six times* the vote support needed to control a district.

⁷Exercise for the enterprising reader: collecting units in order of Republican vote share is a less effective greedy strategy than going by Republican margin per capita. You can find an appropriate sorting lemma in Duchin et al. [1].

So as a system of representation, districting doesn't start out looking like it will provide strong guarantees for minority groups (which includes Republicans in Massachusetts). And that's only looking at the population shares in the units and not at their spatial arrangement, which often compounds the difficulty. Even the mildest constraints on shape, like requiring that each district be one connected piece, make it harder to convert scattered votes for a minority-preferred candidate into representation. In a fairly ironic turn, this means that minority groups with the most strongly segregated geographic patterns—like racial groups historically targeted by discriminatory housing policy—may be in the *best* position to leverage the system of districts to secure representation. Spatially dispersed groups have no hope. Suddenly it looks unreasonable to expect that representative democracy can make good use of winner-take-all districts.

We have identified a problem with the system: districts beget disproportion. Let's look to mathematics to measure the extent of this problem, and to try to understand some of the mechanisms that cause it.

2 THE UNIVERSE OF POSSIBILITIES

We want to understand how districts might be able to provide some minority representation within a majoritarian paradigm. More broadly, we just want to understand what they can and can't do. It would be enormously useful to be able to survey all of the possible districting plans that satisfy some basic constraints, and then reason from there.

Mathematicians like to ease into a hard problem by first abstracting to a “toy problem”—an ultra-simplified model that helps illuminate structural issues. So instead of directly tackling the question at hand (what are all the ways to divide the geographical units in a state into nearly equipopulous districts?) we'll start much simpler: redistricting a small square grid with homogeneous units. This is already hard, as it turns out.

2.1 NUMBER

Maybe I've made some progress in convincing you that neither weird shape nor glaring disproportion gives a sure stand-alone indicator of gerrymandering. If you want to evaluate whether an election result should make you distrust the districts, you should really be comparing the plan to *other possible ways of districting the same jurisdiction*. The catch is that studying the universe of possible plans becomes an intractable problem.

When you think about “big data,” you probably think of space exploration and medical imaging. It may come as a surprise that the humble math problem of how many ways to cut up a square pie belongs in the same conversation.

Think of a simple 4×4 grid, and suppose you want to divide it into four “districts” of equal size, 4 units each. The only requirement is that the districts should be contiguous. If we imagine the regions on a chessboard and we interpret contiguity

to mean that a rook (traveling vertically and horizontally) should be able to visit the whole district, then there are exactly 117 ways to do it, summarized in Figure 6.⁸ I'll denote rook-contiguous partitions of the $n \times n$ grid into k equal districts as the $n \times n \rightarrow k$ problem, for short. A cleverly programmed laptop can generate the $4 \times 4 \rightarrow 4$ solutions instantly.

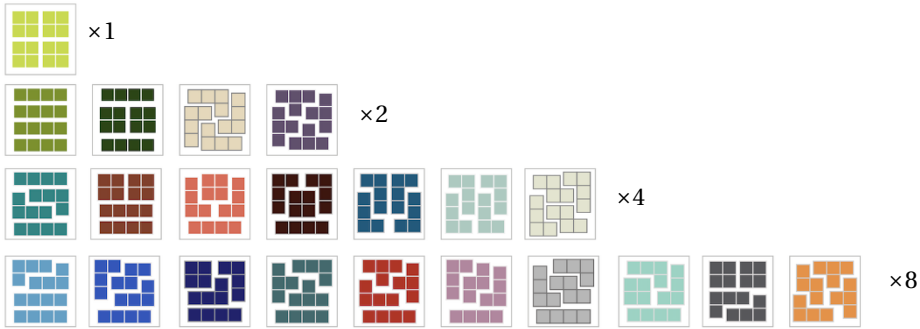


Figure 6: The 4×4 grid has 117 four-district plans—start with the 22 types shown here and apply rotations and reflections to get the full list. Try for yourself! No matter how you rotate or reflect the first plan, it looks the same (so it only contributes $\times 1$ to the ultimate list), but each plan on the next row is one of a pair of variants (so they contribute $\times 2$).

But to my surprise—forgive me, I'm trained as a theorist—I've learned that it's not obvious how to get even a high-performance machine running the best known algorithms to count all the possible configurations in a reasonable amount of time. At the time of writing, our best methods can handle $7 \times 7 \rightarrow 7$ in seconds and $8 \times 8 \rightarrow 8$ in minutes, but the 9×9 is a much more formidable computing task and the 10×10 is out of reach. Now try 18 districts built from Pennsylvania precincts!—it's not only a far bigger problem (9059 units) but has a more complicated connection topology of the units, with no symmetry to exploit. Forget about getting an answer during a 10-year census cycle; this complete enumeration calculation almost certainly can't be done before the heat death of the universe.

One reason for that is that the contiguity and balance constraints are stubbornly nonlocal, meaning that if you just look in one small neighborhood you can't be sure that a district is globally connected or that it's the right size. And these requirements have a lot of bite: unconstrained, there are roughly $4^{16} / 4! = 179$ million ways to label 16 grid squares as belonging to district 1, 2, 3, or 4. Balance (insisting that each district is of equal size) cuts it down to $\binom{16}{4} \binom{12}{4} \binom{8}{4} / 4! = 2.6$ million. Contiguity without balance cuts it down to 62,741. And both together leave you with just 117. So "brute force" algorithms that have to check all possible labelings just don't scale. This seems to call for a clever idea and not just the determination to search exhaustively.

Unfortunately, the problem doesn't reduce in a nice way: knowing the full answer for smaller grids gets us nowhere at all with the $n \times n$. (In math-speak, the problem lacks *recursive* structure.) So to find the very large number of valid partitions, you're searching blindly in an exorbitantly larger ambient space.

⁸If corner adjacency is permitted—so-called "queen contiguity"—the number jumps to 2620.

2.2 CLUSTERING

Since we can't simply build out all the plans, we will need to start understanding what *features* of the problem have important consequences for the measurements we care about. If we are trying to divide a population of two types into districts, it really matters how that population is laid out over the area we are dividing. Let's call this the *political geography*. We've already seen that political geography doomed the hapless Republican voters of Massachusetts—they were too uniformly distributed across the units (towns or precincts) to secure representation. They were not clustered enough.

On the other hand, conventional wisdom in redistricting carries the strong view that 21st century Democrats are disadvantaged by *excessive* clustering. "Democrats pack themselves!" as the slogan goes—because the votes are densely arranged in dense cities, even party-blind redistricting tends to create wastefully high Democratic percentages in urban districts, causing inefficient *packing* (shares far higher than needed) in parts of the map and *cracking* (shares just below the winning threshold) in others. But the math is actually subtle.

If too little clustering is bad and too much clustering is bad, is there is a sweet spot? Let's explore in a toy grid. Below I've represented four configurations in a 6 × 6 grid, each with one-third of units marked with a club suit (Figure 7).

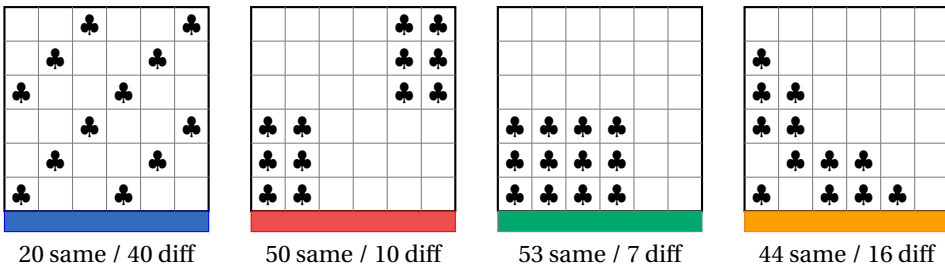


Figure 7: Spatiality matters! For each of these different ways of arranging 12 clubs voters, I've used a "same/different" count to measure clustering, by noting how many pairs of neighbors have matching or different symbols; for instance, 44 pairs of neighbors in the orange grid have the same marking (club-club or blank-blank) while 16 neighbor pairs are different (club-blank). In network science, this kind of same/different statistic is called *assortativity*. This captures something (but not everything) about the geometry of the configuration.

I can try partitioning these into six same-sized "districts" and see how much the layout matters, even while the vote share stays constant. Try it for yourself—some of these symbol layouts give you greater control of the outcome than others. Some spatial arrangements make it possible to lock out the clubs voters from representation entirely; in other arrangements, it's possible to overshoot proportional representation. For instance, I can shut out the blue grid's clubs voters by drawing vertical-stripes districts. The best I can do if I'm trying to *maximize* their representation, on the other hand, is to draw a plan that gives them two districts out of six, and that's not so easy to find. This world of possibility is almost disjoint from the one afforded by the political geography in the orange grid!

How the distribution of clubs votes relates to district outcomes is surprisingly

subtle. But in this small example, it’s the *most* clustered arrangement (green) that is in line with proportional representation (1/3 of the votes tending to earn 1/3 of the seats), and this is way better than the outcomes I should expect of a typical layout. We can see histograms summarizing *all* the possible ways of districting these grids in Figure 8. The very best layout possible for 12 clubs voters is the one shown in orange—the expectation is actually slightly super-proportional!—but others with a similar clustering score are not as advantageous. The spatial effects are stubbornly multidimensional; political geography is not captured in a single clustering score.⁹

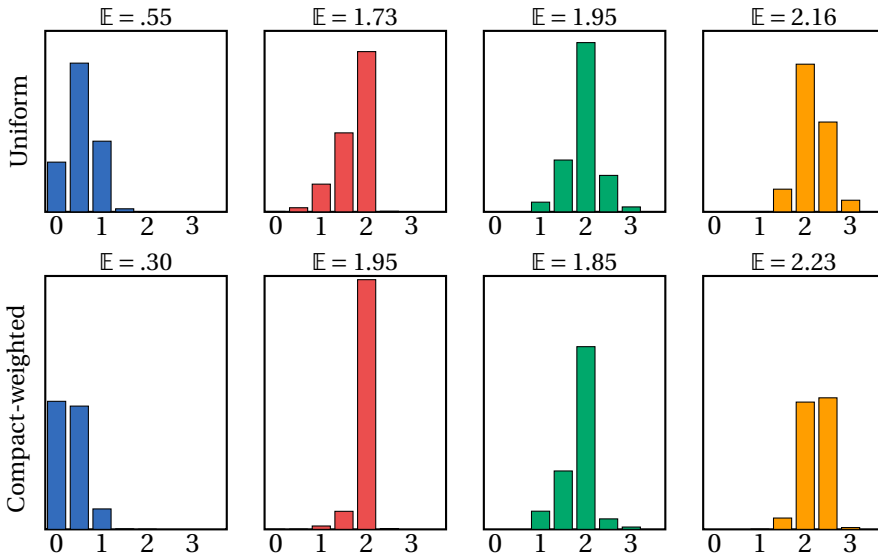


Figure 8: How much can I gerrymander? This plot shows how many seats would be won by the clubs party for every single way of districting the grid—there are 451,206 contiguous plans in all. (I gave clubs credit for .5 of a district if it got three out of six votes.) The top row is uniform: all plans are weighted equally, so for instance the most common outcome on the red grid is that two out of six seats are held by the clubs party. The bottom row shows the exact same set of possibilities, but where plans are weighted according to compactness—plump plans get heavier weight and snakey plans contribute more lightly to their histogram bars. (To be precise, this uses *spanning tree weighting*, which will be explained a bit further below.) So if there’s a preference for choosing compact plans, the two-seats outcome becomes overwhelmingly likely on the red grid.

We were able to unearth considerable complexity in the problem by completely enumerating the plans for the $6 \times 6 \rightarrow 6$ districting problem. Now consider that I’m not able to construct all the plans even for a 9×9 grid, and I can’t even *count* all the possibilities for a 10×10 grid or reasonably *estimate* the possibilities for Pennsylvania’s precincts. How can I assess the consequences of the “political geography” to disentangle gerrymandering from the neutral consequences of districting?

⁹Or at least not *this* clustering score (meaning assortativity, or the “same/diff” count shown above), or any that I have seen or tried—see Chapter 15 for more discussion of spatial statistics. It would take a lot of space to provide enough examples to make this point fully, but you can play with spatial effects yourself at maggg.org/metagraph.

2.3 SAMPLING

This sounds like a hopeless state of affairs. We're trying to evaluate one way of cutting up a state, but without any measure of the size of the universe of alternatives, let alone a catalog of its properties to compare against. This sounds like groping around in a dark, infinite wilderness.

The good news is that even universes that can't be definitively mapped can often be effectively explored with random sampling. You don't need to talk to every American to conduct a good poll; you can use statistics from a representative sample to understand the wider universe. To do this well, you'll need to think about weighting and sample size. (We'll return to this below.)

There is a bevy of sampling techniques you might use for redistricting (Chapter 16, Chapter 17). Instead of profiling those, let's stay broad. What it is to be a representative sample in any context?

Building a Sample

Step 1. Come up with relevant categories or types;

Step 2. Construct a raw sample that encounters all relevant types;

Step 3. Re-weight the raw sample to reflect the population you want to represent.

Let's stick with polling to illustrate some of the issues in play. There are a lot of ways to fail as a pollster! Suppose my ultimate goal is to get a sample of intended voters that is representative of the electorate. If my whole poll is conducted by cell phone calling, then I will entirely miss some kinds of people—those who don't have cell phones, or those who don't pick up from an unknown number. If a lot of people hang up on me when they hear my first question, I'll have too few responses from a certain type of voter. In order to counteract the over-representation and under-representation in my raw sample (relative to the electorate), I need to do work to come up with relevant categories, such as "Angry White Guy Who Thinks Coronavirus Is A Hoax" (AWGWTCIAH). I will then need a sense of how much of the electorate is made up of AWGWTCIAH so that I can counteract the skew in my sample relative to the universe I want to represent. That lets me re-weight my raw sample so that AWGWTCIAH voting preferences are in balance.

If you are thinking "Well, I don't know any AWGWTCIAH!"—yes, that is kind of my point. A uniform distribution on your friends and family, or even a uniform distribution on the voting-age public, is not going to give you a sample that represents the electorate. It's easy to miss that a lot of fundamental conceptual work happens in Step 1 and Step 3. It's also easy to forget that if Step 2 fails, so that you *never* encounter certain types, it can't be corrected by re-weighting.

These reweighting ideas are crucially important in redistricting, because there is a type of silly and unreasonable districting plan that wants to dominate your sample if you let it! Namely, there is an over-supply of plans that are so wild and snakey and flagrantly noncompact that they look like fractals and put the original gerrymander to shame (Figure 9).

0.3 BUILDING RANDOM SAMPLES OF PLANS

Markov chain Monte Carlo, or MCMC, is an industry standard across scientific domains for impossible search tasks such as ours. It's a tool capable of decoding ciphertext, probing the properties and phase transitions of liquids, finding provably accurate fast approximations for hard computational problems, and more. (Persi Diaconis's classic 2009 survey [3] estimates that 10–15% of statistical work in science and engineering is driven by MCMC, and the number has probably only gone up since then.)

Essentially, the strategy of MCMC for sampling a collection of objects is just to take a *random walk* in the universe of objects and see what you see. In our case, you can start at any districting plan and make a random transformation to obtain another, then iterate as many times as you like. Then you can compare a proposed plan to the *ensemble* that you encountered on your random walk. It turns out that for many problems where solutions are hard to construct exhaustively, you can still sample quite efficiently if you have a well-designed engine for making those iterative alterations. You're building out what you need from a starting point, using chains of elementary moves.

So you take a million, a billion, or a trillion steps and look at the aggregate statistics. There's mathematics in the background (ergodic theory, to be precise) guaranteeing that if you random-walk in a suitable space for long enough, you'll hit a probabilistic steady-state. This allows you to collect a sample whose properties are representative of the overall universe, typically far before you've encountered every alternative.

I've been involved in developing a family of samplers called "ReCom" (or recombination) that are powered by large moves in the space of plans, and for which we have a good approximate description of how their ensembles ultimately distribute. Heads up: recombination-style samplers do not weight all districting plans equally, *and that's a good thing!* Weighting all plans equally would tend to put far more weight on noncompact districting plans than on plausibly compact ones, just because of how many more ways there are to be snakey than plump.

ReCom works by fusing two whole districts at every step, choosing a district skeleton called a *spanning tree*, then finding a place to snip the tree that leaves two population-balanced pieces behind. Once they converge, ReCom samplers draw from (approximately) the distribution that weights plans according to the number of spanning trees of their districts. (A different elementary move would target a different distribution; you can think of this as the *distributional design* of the random walk.) This distribution is precisely the "compact-weighted" one that produced the club-suit statistics in the bottom row of Figure 8.^a The spanning tree distribution nicely blends visual compactness, in that it favors fat over spindly districts, and functional compactness, because it favors plans that have more connections *within* relative to connections *between* the districts.

Today the cutting-edge scientific questions concern better theoretical guarantees and convergence diagnostics, as well as efficient implementations. And together with all this, we must keep building persuasive ways of talking about it!

^aThis distribution gives the most compact partitions of the grid over 11 million times the weight of the least compact ones. Chapter 17 and particularly Sidebar 17.8 provide the ingredients to compute this for yourself!