SISSA Springer Series 2

Nicola Gigli
Enrico Pasqualetto

Lectures on Nonsmooth Differential Geometry

Springer

SISSA Springer Series

Volume 2

The **SISSA Springer Series** publishes research monographs, contributed volumes, conference proceedings and lectures notes in English language resulting from workshops, conferences, courses, schools, seminars, and research activities carried out by SISSA: https://www.sissa.it/.

The books in the series will discuss recent results and analyze new trends focusing on the following areas: geometry, mathematical analysis, mathematical modelling, mathematical physics, numerical analysis and scientific computing, showing a fruitful collaboration of scientists with researchers from other fields.

The series is aimed at providing useful reference material to students, academic and researchers at an international level.

More information about this series at http://www.springer.com/series/16109

Nicola Gigli • Enrico Pasqualetto

Lectures on Nonsmooth Differential Geometry

Springer

Nicola Gigli
Department of Mathematics
International School for Advanced Studies (SISSA)
Trieste, Italy

Enrico Pasqualetto
Department of Mathematics and Statistics
University of Jyväskylä
Jyväskylä, Finland

ISSN 2524-857X ISSN 2524-8588 (electronic)
SISSA Springer Series
ISBN 978-3-030-38612-2 ISBN 978-3-030-38613-9 (eBook)
https://doi.org/10.1007/978-3-030-38613-9

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

These are the lecture notes of the Ph.D. level course 'Nonsmooth Differential Geometry' given by the first author at SISSA (Trieste, Italy) from October 2017 to March 2018. The material discussed in the classroom has been collected and reorganised by the second author.

The course was intended for students with no prior exposure to non-smooth calculus and aimed at giving a rather complete picture of first-order Sobolev calculus on general metric measure spaces and a glimpse at second order calculus on RCD spaces. For this reason the first lectures covered basic material like the concept of absolutely continuous curve or Bochner integration. This material is collected in Chap. 1.

A great deal of time has been spent at introducing the by-now classical concept of real valued Sobolev function on a metric measure space. Out of the several equivalent definitions, the approach chosen in the course has been the one based on the concept of 'test plan' introduced in [4] as it better fits what comes next. The original approach by relaxation due to Cheeger [13] and the one by Shanmugalingam [28] based on the concept of 'modulus of a family of curves' are presented, but for time constraint the equivalence of these notions with the one related to test plans has not been proved. These topics are covered in Chap. 2.

The definition of Sobolev map on a metric measure space does not come with a notion of differential, as it happens in the Euclidean setting, but rather with an object, called minimal weak upper gradient, which plays the role of 'modulus of the distributional differential'. One of the recent achievements of the theory, obtained in [17], has been to show that actually a well-defined notion of differential exists also in this setting: its introduction is based on the concept of L^∞/L^0-normed module. Chapter 3 investigates these structures from a rather abstract perspective without insisting on their use in non-smooth analysis.

The core of the course is then covered in Chap. 4, where first-order calculus is studied in great detail and the key notions of tangent/cotangent modules are introduced. Beside the notion of differential of a Sobolev map, other topics discussed are the dual concept of divergence of a vector field and how these behave under transformation of the metric measure structures. For simplicity, some of the constructions, like the one of speed of a test plan, are presented only in the technically convenient case of infinitesimally Hilbertian metric measure spaces, i.e. those for which the corresponding Sobolev space $W^{1,2}$ is Hilbert.

A basic need in most branches of mathematical analysis is that of a regularisation procedure. In working on a non-smooth environment this is true more than ever and classical tools like covering arguments are typically unavailable if one does not assume at least a doubling property at the metric level. Instead the key, and often only, tool one has at disposal is that of regularisation via the heat flow (which behaves particularly well under a lower Ricci curvature bound, a situation which the theory presented here aims to cover). Such flow can be introduced in a purely variational way as gradient flow of the 'Dirichlet energy' (in this setting called Cheeger energy) in the Hilbert space L^2, and thus can be defined in general metric measure spaces. In Chap. 5 we present a quick overview of the general theory of gradient flows in Hilbert spaces and then we discuss its application to the study of the heat flow in the 'linear' case of infinitesimally Hilbertian spaces.

Finally, the last lessons aimed at a quick guided tour in the world of RCD spaces and second order calculus on them. This material is collected in Chap. 6, where:

- We define $\mathsf{RCD}(K, \infty)$ spaces.
- Prove some better estimates for the heat flow on them.
- Introduce the algebra of 'test functions' on RCD spaces, which is the 'largest algebra of smooth functions' that we have at disposal in this environment, in a sense.
- Quickly develop the second-order differential calculus on RCD spaces, by building on top of the first-order one. Meaningful and 'operative' definitions (among others) of Hessian, covariant derivative, exterior derivative and Hodge Laplacian are discussed.

These lecture notes are mostly self-contained and should be accessible to any Ph.D. student with a standard background in analysis and geometry: having basic notions of measure theory, functional analysis and Riemannian geometry suffices to navigate this text. Hopefully, this should provide a hands-on guide to recent mathematical theories accessible to the widest possible audience.

The most recent research-level material contained here comes, to a big extent, from the paper [17], see also the survey [19]. With respect to these presentations, the current text offers a gentler introduction to all the topics, paying little in terms of generality: as such it is the most suitable source for the young researcher who is willing to learn about this fast growing research direction. The presentation is also complemented by a collection of exercises scattered through the text; since these are at times essential for the results presented, their solutions are reported (or, sometimes, just sketched) in Appendix B.

We wish to thank Emanuele Caputo, Francesco Nobili, Francesco Sapio and Ivan Yuri Violo for their careful reading of a preliminary version of this manuscript.

Trieste, Italy
Jyväskylä, Finland
January 2019

Nicola Gigli
Enrico Pasqualetto

Contents

About the Authors

Nicola Gigli studied Mathematics at the Scuola Normale Superiore of Pisa and is Professor of Mathematical Analysis at SISSA, Trieste. He is interested in calculus of variations, optimal transport, and geometric and nonsmooth analysis, with particular focus on properties of spaces with curvature bounded from below.

Enrico Pasqualetto earned his Ph.D. degree in Mathematics at SISSA (Trieste) in 2018 and is currently a postdoctoral researcher at the University of Jyväskylä. His fields of interest consist in functional analysis and geometric measure theory on non-smooth metric structures, mainly in the presence of synthetic curvature bounds. Within this framework, his research topics include rectifiability properties, Sobolev spaces, and sets of finite perimeter.

Chapter 1
Preliminaries

In this chapter we introduce several classic notions that will be needed in the sequel. Namely, in Sect. 1.1 we review the basics of measure theory, with a particular accent on the space $L^0(\mathfrak{m})$ of Borel functions considered up to $\mathfrak{m}$-almost everywhere equality (see Sect. 1.1.2); in Sect. 1.2 we discuss about continuous, absolutely continuous and geodesic curves on metric spaces; in Sect. 1.3 we collect the most important results about Bochner integration. Some functional analytic tools will be treated in Appendix A.

1.1 General Measure Theory

1.1.1 Borel Probability Measures

Given a complete and separable metric space (X, d), let us denote

$$\begin{aligned} \mathscr{P}(\mathrm{X}) &:= \big\{\text{Borel probability measures on } (\mathrm{X}, \mathsf{d})\big\}, \\ C_b(\mathrm{X}) &:= \big\{\text{bounded continuous functions } f : \mathrm{X} \to \mathbb{R}\big\}. \end{aligned} \tag{1.1}$$

We can define a topology on $\mathscr{P}(\mathrm{X})$, called *weak topology*, as follows:

Definition 1.1.1 (Weak Topology) The *weak topology* on $\mathscr{P}(\mathrm{X})$ is defined as the coarsest topology on $\mathscr{P}(\mathrm{X})$ such that:

$$\text{the function } \mathscr{P}(\mathrm{X}) \ni \mu \longmapsto \int f \,\mathrm{d}\mu \text{ is continuous,} \qquad \text{for every } f \in C_b(\mathrm{X}). \tag{1.2}$$

N. Gigli, E. Pasqualetto, *Lectures on Nonsmooth Differential Geometry*,
SISSA Springer Series 2, https://doi.org/10.1007/978-3-030-38613-9_1

Remark 1.1.2 If a sequence of measures $(\mu_n)_n$ weakly converges to a limit measure μ, then

$$\mu(\Omega) \leq \varliminf_{n\to\infty} \mu_n(\Omega) \qquad \text{for every } \Omega \subseteq \mathrm{X} \text{ open.} \tag{1.3}$$

Indeed, let $f_k := k\,\mathsf{d}(\cdot, \mathrm{X}\setminus\Omega) \wedge 1 \in C_b(\mathrm{X})$ for $k \in \mathbb{N}$. Hence $f_k(x) \nearrow \chi_\Omega(x)$ for all $x \in \mathrm{X}$, so that $\mu(\Omega) = \sup_k \int f_k \,\mathrm{d}\mu$ by monotone convergence theorem. Since $\nu \mapsto \int f_k \,\mathrm{d}\nu$ is continuous for any k, we deduce that the function $\nu \mapsto \nu(\Omega)$ is lower semicontinuous as supremum of continuous functions, thus yielding (1.3).

In particular, if a sequence $(\mu_n)_n \subseteq \mathscr{P}(\mathrm{X})$ weakly converges to some $\mu \in \mathscr{P}(\mathrm{X})$, then

$$\mu(C) \geq \varlimsup_{n\to\infty} \mu_n(C) \qquad \text{for every } C \subseteq \mathrm{X} \text{ closed.} \tag{1.4}$$

To prove it, just apply (1.3) to $\Omega := \mathrm{X}\setminus C$. ■

Remark 1.1.3 We claim that if $\int f \,\mathrm{d}\mu = \int f \,\mathrm{d}\nu$ for every $f \in C_b(\mathrm{X})$, then $\mu = \nu$. Indeed, $\mu(C) = \nu(C)$ for any $C \subseteq \mathrm{X}$ closed as a consequence of (1.4), whence $\mu = \nu$ by the monotone class theorem. ■

Remark 1.1.4 Given any Banach space V, we denote by V' its dual Banach space. Then

$$\mathscr{P}(\mathrm{X}) \ \text{ is continuously embedded into } \ C_b(X)'. \tag{1.5}$$

Such embedding is given by the operator sending $\mu \in \mathscr{P}(\mathrm{X})$ to the map $C_b(\mathrm{X}) \ni f \mapsto \int f \,\mathrm{d}\mu$, which is injective by Remark 1.1.3 and linear by definition. Finally, continuity stems from the inequality $\left|\int f \,\mathrm{d}\mu\right| \leq \|f\|_{C_b(\mathrm{X})}$, which holds for any $f \in C_b(\mathrm{X})$. ■

Fix a countable dense subset $(x_n)_n$ of X. Let us define

$$\begin{aligned} \mathcal{A} &:= \Big\{\big(a - b\,\mathsf{d}(\cdot, x_n)\big) \vee c \ : \ a, b, c \in \mathbb{Q},\ n \in \mathbb{N}\Big\}, \\ \widetilde{\mathcal{A}} &:= \big\{f_1 \vee \ldots \vee f_n \ : \ n \in \mathbb{N},\ f_1, \ldots, f_n \in \mathcal{A}\big\}. \end{aligned} \tag{1.6}$$

Observe that $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are countable subsets of $C_b(\mathrm{X})$. We claim that:

$$f(x) = \sup\big\{g(x) \ : \ g \in \mathcal{A},\ g \leq f\big\} \qquad \text{for every } f \in C_b(\mathrm{X}) \text{ and } x \in \mathrm{X}. \tag{1.7}$$

Indeed, the inequality $\geq$ is trivial, while to prove $\leq$ fix $x \in \mathrm{X}$ and $\varepsilon > 0$. The function f being continuous, there is a neighbourhood U of x such that $f(y) \geq f(x) - \varepsilon$ for all $y \in U$. Then we can easily build a function $g \in \mathcal{A}$ such that $g \leq f$ and $g(x) \geq f(x) - 2\,\varepsilon$. By arbitrariness of $x \in \mathrm{X}$ and $\varepsilon > 0$, we thus proved the validity of (1.7).

Exercise 1.1.5 Suppose that X is compact. Prove that if a sequence $(f_n)_n \subseteq C(\mathrm{X})$ satisfies $f_n(x) \searrow 0$ for every $x \in \mathrm{X}$, then $f_n \to 0$ uniformly on X. ■

Corollary 1.1.6 *Suppose that* X *is compact. Then* $\widetilde{\mathcal{A}}$ *is dense in* $C(\mathrm{X}) = C_b(\mathrm{X})$. *In particular, the space* $C(\mathrm{X})$ *is separable.*

Proof Fix $f \in C(\mathrm{X})$. Enumerate $\{g \in \mathcal{A} : g \leq f\}$ as $(g_n)_n$. Call $h_n := g_1 \vee \ldots \vee g_n \in \widetilde{\mathcal{A}}$ for each $n \in \mathbb{N}$, thus $h_n(x) \nearrow f(x)$ for all $x \in \mathrm{X}$ by (1.7). Hence $(f - h_n)(x) \searrow 0$ for all $x \in \mathrm{X}$ and accordingly $f - h_n \to 0$ in $C(\mathrm{X})$ by Exercise 1.1.5, proving the statement. □

The converse implication holds true as well:

Exercise 1.1.7 Let (X, d) be a complete and separable metric space. Prove that if $C_b(\mathrm{X})$ is separable, then the space X is compact. ■

Corollary 1.1.8 *It holds that*

$$\int f \, \mathrm{d}\mu = \sup\left\{ \int g \, \mathrm{d}\mu \;\middle|\; g \in \widetilde{\mathcal{A}},\ g \leq f \right\} \quad \textit{for every } \mu \in \mathscr{P}(\mathrm{X}) \textit{ and } f \in C_b(\mathrm{X}). \tag{1.8}$$

Proof Call $(g_n)_n = \big\{g \in \mathcal{A} : g \leq f\big\}$ and put $h_n := g_1 \vee \ldots \vee g_n \in \widetilde{\mathcal{A}}$, thus $h_n(x) \nearrow f(x)$ for all $x \in \mathrm{X}$ and accordingly $\int f \, \mathrm{d}\mu = \lim_n \int h_n \, \mathrm{d}\mu$, proving (1.8). □

We endow $\mathscr{P}(\mathrm{X})$ with a distance δ. Enumerate $\big\{g \in \widetilde{\mathcal{A}} \cup (-\widetilde{\mathcal{A}}) : \|g\|_{C_b(\mathrm{X})} \leq 1\big\}$ as $(f_i)_i$. Then for any $\mu, \nu \in \mathscr{P}(\mathrm{X})$ we define

$$\delta(\mu, \nu) := \sum_{i=0}^{\infty} \frac{1}{2^i} \left| \int f_i \, \mathrm{d}(\mu - \nu) \right|. \tag{1.9}$$

Proposition 1.1.9 *The weak topology on* $\mathscr{P}(\mathrm{X})$ *is induced by the distance* δ.

Proof To prove one implication, we want to show that for any $f \in C_b(\mathrm{X})$ the map $\mu \mapsto \int f \, \mathrm{d}\mu$ is δ-continuous. Fix $\mu, \nu \in \mathscr{P}(\mathrm{X})$. Given any $\varepsilon > 0$, there exists a map $g \in \widetilde{\mathcal{A}}$ such that $g \leq f$ and $\int g \, \mathrm{d}\mu \geq \int f \, \mathrm{d}\mu - \varepsilon$, by Corollary 1.1.8. Let $i \in \mathbb{N}$ be such that $f_i = g/\|g\|_{C_b(\mathrm{X})}$. Then

$$\int f \, \mathrm{d}\nu - \int f \, \mathrm{d}\mu \geq \|g\|_{C_b(\mathrm{X})} \int f_i \, \mathrm{d}(\nu - \mu) - \varepsilon \geq -\|g\|_{C_b(\mathrm{X})} \, 2^i \, \delta(\nu, \mu) - \varepsilon,$$

whence $\varliminf_{\delta(\nu,\mu) \to 0} \int f \, \mathrm{d}(\nu - \mu) \geq 0$ by arbitrariness of ε, i.e. the map $\mu \mapsto \int f \, \mathrm{d}\mu$ is δ-lower semicontinuous. Its δ-upper semicontinuity can be proved in an analogous way.

Conversely, fix $\mu \in \mathscr{P}(\mathrm{X})$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon/2$. Then there is a weak neighbourhood W of μ such that $\left|\int f_i \, \mathrm{d}(\mu - \nu)\right| < \varepsilon/4$ for all $i = 0, \ldots, N$ and $\nu \in W$. Therefore

$$\delta(\mu, \nu) \leq \sum_{i=0}^{N} \frac{1}{2^i}\left|\int f_i \, \mathrm{d}(\mu - \nu)\right| + \sum_{i=N+1}^{\infty} \frac{1}{2^i} \leq \frac{\varepsilon}{2} + \frac{1}{2^N} < \varepsilon \quad \text{for every } \nu \in W,$$

proving that W is contained in the open δ-ball of radius ε centered at μ. □

Remark 1.1.10 Suppose that X is compact. Then $C(\mathrm{X}) = C_b(\mathrm{X})$, thus accordingly $\mathscr{P}(\mathrm{X})$ is weakly compact by (1.5) and Banach-Alaoglu theorem. Conversely, for X non-compact this is in general no longer true. For instance, take $\mathrm{X} := \mathbb{R}$ and $\mu_n := \delta_n$. Suppose by contradiction that a subsequence $(\mu_{n_m})_m$ weakly converges to some limit $\mu \in \mathscr{P}(\mathbb{R})$. For any $k \in \mathbb{N}$ we have that $\mu\big((-k, k)\big) \leq \underline{\lim}_m \, \delta_{n_m}\big((-k, k)\big) = 0$, so that $\mu(\mathbb{R}) = \lim_{k\to\infty} \mu\big((-k, k)\big) = 0$, which leads to a contradiction. This proves that $\mathscr{P}(\mathbb{R})$ is not weakly compact. ■

Definition 1.1.11 (Tightness) A set $\mathcal{K} \subseteq \mathscr{P}(\mathrm{X})$ is said to be *tight* provided for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq \mathrm{X}$ such that $\mu(K_\varepsilon) \geq 1 - \varepsilon$ for every $\mu \in \mathcal{K}$.

Theorem 1.1.12 (Prokhorov) *Let $\mathcal{K} \subseteq \mathscr{P}(\mathrm{X})$ be fixed. Then $\mathcal{K}$ is weakly relatively compact if and only if $\mathcal{K}$ is tight.*

Proof In light of Proposition 1.1.9, compactness and sequential compactness are equivalent. We separately prove the two implications:
SUFFICIENCY. Fix $\mathcal{K} \subseteq \mathscr{P}(\mathrm{X})$ tight. Without loss of generality, suppose that $\mathcal{K} = (\mu_i)_{i\in\mathbb{N}}$. For any $n \in \mathbb{N}$, choose a compact set $K_n \subseteq \mathrm{X}$ such that $\mu_i(K_n) \geq 1 - 1/n$ for all i. By a diagonalization argument we see that, up to a not relabeled subsequence, $\mu_i|_{K_n}$ converges to some measure ν_n in duality with $C_b(K_n)$ for all $n \in \mathbb{N}$, as a consequence of Remark 1.1.10. We now claim that:

$$\begin{aligned} &\nu_n \to \nu \ \text{ in total variation norm,} \quad \text{for some measure } \nu, \\ &\mu_i \rightharpoonup \nu \ \text{ in duality with } C_b(\mathrm{X}). \end{aligned} \tag{1.10}$$

To prove the former, recall (cf. Remark 1.1.15 below) that for any $m \geq n \geq 1$ one has

$$\|\nu_n - \nu_m\|_{\mathsf{TV}} = \sup\left\{\int f \, \mathrm{d}(\nu_n - \nu_m) \;\middle|\; f \in C_b(\mathrm{X}),\ \|f\|_{C_b(\mathrm{X})} \leq 1\right\}.$$

Then fix $f \in C_b(\mathrm{X})$ with $\|f\|_{C_b(\mathrm{X})} \leq 1$. We can assume without loss of generality that $(K_n)_n$ is increasing. We deduce from (1.3) that $\nu_m(K_m \setminus K_n) \leq \underline{\lim}_i \, \mu_i|_{K_m}(\mathrm{X} \setminus K_n) \leq 1/n$. Therefore

$$\int f \, \mathrm{d}(\nu_n - \nu_m) \leq \lim_{i\to\infty}\left(\int f \, \mathrm{d}\mu_i - \int f \, \mathrm{d}\mu_i\right) + \frac{1}{n} + \frac{1}{m} = \frac{1}{n} + \frac{1}{m},$$

proving that $(\nu_n)_n$ is Cauchy with respect to $\|\cdot\|_{\mathsf{TV}}$ and accordingly the first in (1.10). For the latter, notice that for any $f \in C_b(\mathrm{X})$ it holds that

$$\begin{aligned}\left|\int f \,\mathrm{d}(\mu_i - \nu)\right| &= \left|\int_{K_n} f \,\mathrm{d}(\mu_i - \nu_n) - \int_{K_n} f \,\mathrm{d}(\nu - \nu_n)\right. \\ &\qquad \left. + \int_{\mathrm{X}\setminus K_n} f \,\mathrm{d}\mu_i - \int_{\mathrm{X}\setminus K_n} f \,\mathrm{d}\nu\right| \\ &\leq \left|\int_{K_n} f \,\mathrm{d}(\mu_i - \nu_n)\right| + \|f\|_{C_b(\mathrm{X})} \|\nu - \nu_n\|_{\mathsf{TV}} + \frac{2\,\|f\|_{C_b(\mathrm{X})}}{n}.\end{aligned}$$

By first letting $i \to \infty$ and then $n \to \infty$, we obtain that $\lim_i \left|\int f \,\mathrm{d}(\mu_i - \nu)\right| = 0$, showing the second in (1.10). Hence sufficiency is proved.

NECESSITY. Fix $\mathcal{K} \subseteq \mathscr{P}(\mathrm{X})$ weakly relatively sequentially compact. Choose $\varepsilon > 0$ and a sequence $(x_n)_n$ that is dense in X. Arguing by contradiction, we aim to prove that

$$\forall i \in \mathbb{N} \quad \exists N_i \in \mathbb{N} : \quad \mu\left(\bigcup_{j=1}^{N_i} \bar{B}_{1/i}(x_j)\right) \geq 1 - \frac{\varepsilon}{2^i} \quad \forall \mu \in \mathcal{K}. \tag{1.11}$$

If not, there exist $i_0 \in \mathbb{N}$ and $(\mu_m)_m \subseteq \mathcal{K}$ such that $\mu_m\left(\bigcup_{j=1}^m \bar{B}_{1/i_0}(x_j)\right) < 1 - \varepsilon/2^{i_0}$ holds for every $m \in \mathbb{N}$. Up to a not relabeled subsequence $\mu_m \rightharpoonup \mu \in \mathscr{P}(\mathrm{X})$ and accordingly

$$\mu\left(\bigcup_{j=1}^{n} B_{1/i_0}(x_j)\right) \overset{(1.3)}{\leq} \varliminf_{m\to\infty} \mu_m\left(\bigcup_{j=1}^{m} \bar{B}_{1/i_0}(x_j)\right) \leq 1 - \varepsilon/2^{i_0} \quad \text{for any } n \in \mathbb{N},$$

which contradicts the fact that $\lim_{n\to\infty} \mu\left(\bigcup_{j=1}^n B_{1/i_0}(x_j)\right) = \mu(\mathrm{X}) = 1$. This proves (1.11).

Now define $K := \bigcap_{i\in\mathbb{N}} \bigcup_{j=1}^{N_i} \bar{B}_{1/i}(x_j)$. Such set is compact, as it is closed and totally bounded by construction. Moreover, for any $\mu \in \mathcal{K}$ one has that

$$\mu(\mathrm{X} \setminus K) \leq \sum_i \mu\left(\bigcap_{j=1}^{N_i} \mathrm{X} \setminus \bar{B}_{1/i}(x_j)\right) \overset{(1.11)}{\leq} \varepsilon \sum_i \frac{1}{2^i} = \varepsilon,$$

thus proving also necessity. □

Remark 1.1.13 We have that a set $\mathcal{K} \subseteq \mathscr{P}(\mathrm{X})$ is tight if and only if

$$\exists\, \Psi : \mathrm{X} \to [0, +\infty], \text{ with compact sublevels, such that } s := \sup_{\mu\in\mathcal{K}} \int \Psi \,\mathrm{d}\mu < +\infty. \tag{1.12}$$

To prove sufficiency, first notice that Ψ is Borel as its sublevels are closed sets. Now fix $\varepsilon > 0$ and choose $C > 0$ such that $s/C < \varepsilon$. Moreover, by applying Čebyšëv's inequality we obtain that $C\,\mu\{\Psi > C\} \leq \int \Psi \,\mathrm{d}\mu \leq s$ for all $\mu \in \mathcal{K}$, whence $\mu\big(\{\Psi \leq C\}\big) \geq 1 - s/C > 1 - \varepsilon$.

To prove necessity, suppose $\mathcal{K}$ tight and choose a sequence $(K_n)_n$ of compact sets such that $\mu(\mathrm{X} \setminus K_n) \leq 1/n^3$ for all $n \in \mathbb{N}$ and $\mu \in \mathcal{K}$. Define $\Psi(x) := \inf\big\{n \in \mathbb{N} \,:\, x \in K_n\big\}$ for every $x \in \mathrm{X}$. Clearly Ψ has compact sublevels by construction. Moreover, it holds that

$$\sup_{\mu \in \mathcal{K}} \int \Psi \,\mathrm{d}\mu = \sup_{\mu \in \mathcal{K}} \sum_n \int_{K_{n+1} \setminus K_n} \Psi \,\mathrm{d}\mu \leq \sum_n \frac{n+1}{n^3} < +\infty,$$

as required. ■

Remark 1.1.14 Let $\mu \geq 0$ be a finite non-negative Borel measure on X. Then for any Borel set $E \subseteq \mathrm{X}$ one has

$$\mu(E) = \sup\big\{\mu(C) \,:\, C \subseteq E \text{ closed}\big\} = \inf\big\{\mu(\Omega) \,:\, \Omega \supseteq E \text{ open}\big\}. \tag{1.13}$$

To prove it, it suffices to show that the family of all Borel sets E satisfying (1.13), which we shall denote by $\mathcal{E}$, forms a σ-algebra containing all open subsets of X. Then fix $\Omega \subseteq \mathrm{X}$ open. Call $C_n := \big\{x \in \Omega \,:\, \mathsf{d}(x, \mathrm{X} \setminus \Omega) \geq 1/n\big\}$ for all $n \in \mathbb{N}$, whence $(C_n)_n$ is an increasing sequence of closed sets and $\mu(\Omega) = \lim_n \mu(C_n)$ by continuity from below of μ. This grants that $\Omega \in \mathcal{E}$.

It only remains to show that $\mathcal{E}$ is a σ-algebra. It is obvious that $\emptyset \in \mathcal{E}$ and that $\mathcal{E}$ is stable by complements. Now fix $(E_n)_n \subseteq \mathcal{E}$ and $\varepsilon > 0$. There exist $(C_n)_n$ closed and $(\Omega_n)_n$ open such that $C_n \subseteq E_n \subseteq \Omega_n$ and $\mu(\Omega_n) - \varepsilon\, 2^{-n} \leq \mu(E_n) \leq \mu(C_n) + \varepsilon\, 2^{-n}$ for every $n \in \mathbb{N}$. Let us denote $\Omega := \bigcup_n \Omega_n$. Moreover, continuity from above of μ yields the existence of $N \in \mathbb{N}$ such that $\mu\big(\bigcup_{n \in \mathbb{N}} C_n \setminus C\big) \leq \varepsilon$, where we put $C := \bigcup_{n=1}^N C_n$. Notice that Ω is open, C is closed and $C \subseteq \bigcup_n E_n \subseteq \Omega$. Finally, it holds that

$$\mu\bigg(\bigcup_{n=1}^\infty E_n \setminus C\bigg) \leq \sum_{n=1}^\infty \mu(E_n \setminus C_n) + \varepsilon \leq \sum_{n=1}^\infty \frac{\varepsilon}{2^n} + \varepsilon = 2\,\varepsilon,$$

$$\mu\bigg(\Omega \setminus \bigcup_{n=1}^\infty E_n\bigg) \leq \sum_{n=1}^\infty \mu(\Omega_n \setminus E_n) \leq \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon.$$

This grants that $\bigcup_n E_n \in \mathcal{E}$, concluding the proof. ■

Remark 1.1.15 (Total Variation Norm) During the proof of Theorem 1.1.12, we needed the following two properties of the *total variation norm*:

$$\|\mu\|_{\mathsf{TV}} = \sup\bigg\{\int f \,\mathrm{d}\mu \,\bigg|\, f \in C_b(\mathrm{X}),\ \|f\|_{C_b(\mathrm{X})} \leq 1\bigg\} \quad \text{for any signed Borel measure } \mu \text{ on X},$$
$$\big(\mathscr{P}(\mathrm{X}), \|\cdot\|_{\mathsf{TV}}\big) \quad \text{is complete.} \tag{1.14}$$

In order to prove them, we proceed as follows. Given a signed measure μ, let us consider its *Hahn-Jordan decomposition* $\mu = \mu^+ - \mu^-$, where $\mu^\pm$ are non-negative measures with $\mu^+ \perp \mu^-$, which satisfy $\mu(P) = \mu^+(\mathrm{X})$ and $\mu(P^c) = -\mu^-(\mathrm{X})$ for a suitable Borel set $P \subseteq \mathrm{X}$. Hence by definition the total variation norm is defined as

$$\|\mu\|_{\mathsf{TV}} := \mu^+(\mathrm{X}) + \mu^-(\mathrm{X}). \tag{1.15}$$

Such definition is well-posed, since the Hahn-Jordan decomposition (μ^+, μ^-) of μ is unique.

To prove the first in (1.14), we start by noticing that $\int f \,\mathrm{d}\mu \leq \int |f| \,\mathrm{d}(\mu^+ + \mu^-) \leq \|\mu\|_{\mathsf{TV}}$ holds for any $f \in C_b(\mathrm{X})$ with $\|f\|_{C_b(\mathrm{X})} \leq 1$, proving one inequality. To show the converse one, let $\varepsilon > 0$ be fixed. By Remark 1.1.14, we can choose two closed sets $C \subseteq P$ and $C' \subseteq P^c$ such that $\mu^+(P \setminus C), \mu^-(P^c \setminus C') < \varepsilon$. Call $f_n := \big(1 - n\,\mathsf{d}(\cdot, C)\big)^+$ and $g_n := \big(1 - n\,\mathsf{d}(\cdot, C')\big)^+$, so that $f_n \searrow \chi_C$ and $g_n \searrow \chi_{C'}$ as $n \to \infty$. Now define $h_n := f_n - g_n$. Since $|h_n| \leq 1$, we have that $(h_n)_n \subseteq C_b(\mathrm{X})$ and $\|h_n\|_{C_b(\mathrm{X})} \leq 1$ for every $n \in \mathbb{N}$. Moreover, it holds that

$$\begin{aligned}\varliminf_{n\to\infty} \int h_n \,\mathrm{d}\mu &= \varliminf_{n\to\infty} \left[\int f_n \,\mathrm{d}\mu^+ - \int f_n \,\mathrm{d}\mu^- - \int g_n \,\mathrm{d}\mu^+ + \int g_n \,\mathrm{d}\mu^-\right] \\ &= \mu^+(C) + \mu^-(C') \geq \mu^+(P) + \mu^-(P^c) - 2\,\varepsilon = \|\mu\|_{\mathsf{TV}} - 2\,\varepsilon.\end{aligned}$$

By arbitrariness of $\varepsilon > 0$, we conclude that $\varliminf_n \int h_n \,\mathrm{d}\mu \geq \|\mu\|_{\mathsf{TV}}$, proving the first in (1.14).

To show the second, fix a sequence $(\mu_n)_n \subseteq \mathscr{P}(\mathrm{X})$ that is $\|\cdot\|_{\mathsf{TV}}$-Cauchy. Notice that

$$\big|\mu(E)\big| \leq \|\mu\|_{\mathsf{TV}} \quad \text{for every signed measure } \mu \text{ and Borel set } E \subseteq \mathrm{X}.$$

Indeed, $\big|\mu(E)\big| \leq \mu^+(E) + \mu^-(E) \leq \mu^+(\mathrm{X}) + \mu^-(\mathrm{X}) = \|\mu\|_{\mathsf{TV}}$. Therefore

$$\big|\mu_n(E) - \mu_m(E)\big| \leq \|\mu_n - \mu_m\|_{\mathsf{TV}} \quad \text{for every } n, m \in \mathbb{N} \text{ and } E \subseteq \mathrm{X} \text{ Borel.} \tag{1.16}$$

In particular, $\big(\mu_n(E)\big)_n$ is Cauchy for any $E \subseteq \mathrm{X}$ Borel, so that $\lim_n \mu_n(E) = \mathsf{L}(E)$ for some limit $\mathsf{L}(E) \in [0, 1]$. We thus deduce from (1.16) that

$$\forall \varepsilon > 0 \quad \exists\, \bar{n}_\varepsilon \in \mathbb{N} : \quad \big|\mathsf{L}(E) - \mu_n(E)\big| \leq \varepsilon \quad \forall n \geq \bar{n}_\varepsilon \quad \forall E \subseteq \mathrm{X} \text{ Borel.} \tag{1.17}$$

We claim that L is a probability measure. Clearly, $\mathsf{L}(\emptyset) = 0$ and $\mathsf{L}(\mathrm{X}) = 1$. For any E, F Borel with $E \cap F = \emptyset$, we have $\mathsf{L}(E \cup F) = \lim_n \mu_n(E \cup F) = \lim_n \mu_n(E) + \lim_n \mu_n(F) = \mathsf{L}(E) + \mathsf{L}(F)$, which grants that L is finitely additive. To show that it is also σ-additive, fix a sequence $(E_i)_i$ of pairwise disjoint Borel sets. Let us call

$U_N := \bigcup_{i=1}^N E_i$ for all $N \in \mathbb{N}$ and $U := \bigcup_{i=1}^\infty E_i$. Given any $\varepsilon > 0$, we infer from (1.17) that for any $n \geq \bar{n}_\varepsilon$ one has

$$\begin{aligned}\varlimsup_{N\to\infty} \big|\mathsf{L}(U) - \mathsf{L}(U_N)\big| \leq \big|\mathsf{L}(U) - \mu_n(U)\big| \\ + \varlimsup_{N\to\infty} \big|\mu_n(U) - \mu_n(U_N)\big| + \varlimsup_{N\to\infty} \big|\mu_n(U_N) - \mathsf{L}(U_N)\big| \\ \leq 2\,\varepsilon + \varlimsup_{N\to\infty} \big|\mu_n(U) - \mu_n(U_N)\big| = 2\,\varepsilon,\end{aligned}$$

where the last equality follows from the continuity from below of μ_n. By letting $\varepsilon \to 0$ in the previous formula, we thus obtain that $\mathsf{L}(U) = \lim_N \mathsf{L}(U_N) = \sum_{i=1}^\infty \mathsf{L}(E_i)$, so that $\mathsf{L} \in \mathscr{P}(\mathrm{X})$. Finally, we aim to prove that $\lim_n \|\mathsf{L} - \mu_n\|_{\mathsf{TV}} = 0$. For any $n \in \mathbb{N}$, choose a Borel set $P_n \subseteq \mathrm{X}$ satisfying $(\mathsf{L}-\mu_n)(P_n) = (\mathsf{L}-\mu_n)^+(\mathrm{X})$ and $(\mathsf{L}-\mu_n)(P_n^c) = -(\mathsf{L}-\mu_n)^-(\mathrm{X})$. Now fix $\varepsilon > 0$. Hence (1.17) guarantees that for every $n \geq \bar{n}_\varepsilon$ it holds that

$$\|\mathsf{L} - \mu_n\|_{\mathsf{TV}} = (\mathsf{L} - \mu_n)(P_n) - (\mathsf{L} - \mu_n)(P_n^c) = \big|(\mathsf{L} - \mu_n)(P_n)\big| + \big|(\mathsf{L} - \mu_n)(P_n^c)\big| \leq 2\,\varepsilon.$$

Therefore μ_n converges to L in the $\|\cdot\|_{\mathsf{TV}}$-norm. Since $\mathsf{L} \geq 0$ by construction, the proof of (1.14) is achieved. ■

We now present some consequences of Theorem 1.1.12:

Corollary 1.1.16 (Ulam's Theorem) *Any $\mu \in \mathscr{P}(\mathrm{X})$ is concentrated on a σ-compact set.*

Proof Clearly the singleton $\{\mu\}$ is weakly relatively compact, so it is tight by Theorem 1.1.12. Thus for any $n \in \mathbb{N}$ we can choose a compact set $K_n \subseteq \mathrm{X}$ such that $\mu(\mathrm{X} \setminus K_n) < 1/n$. In particular, μ is concentrated on $\bigcup_n K_n$, yielding the statement. □

Corollary 1.1.17 *Let $\mu \in \mathscr{P}(\mathrm{X})$ be given. Then μ is* inner regular, *i.e.*

$$\mu(E) = \sup\big\{\mu(K) \,:\, K \subseteq E \textit{ compact}\big\} \quad \textit{for every } E \subseteq \mathrm{X} \textit{ Borel.} \tag{1.18}$$

In particular, μ is a Radon measure.

Proof By Corollary 1.1.16, there exists an increasing sequence $(K_n)_n$ of compact sets such that $\lim_n \mu\big(\mathrm{X} \setminus K_n\big) = 0$. Any closed subset C of X that is contained in some K_n is clearly compact, whence

$$\begin{aligned}\mu(E) &= \lim_{n\to\infty} \mu(E \cap K_n) = \lim_{n\to\infty} \sup\big\{\mu(C) \,:\, C \subseteq E \cap K_n \text{ closed}\big\} \\ &\leq \sup\big\{\mu(K) \,:\, K \subseteq E \text{ compact}\big\} \qquad \text{for every } E \subseteq \mathrm{X} \text{ Borel,}\end{aligned}$$

proving (1.18), as required. □

Given any function $f : \mathrm{X} \to \mathbb{R}$, let us define

$$\mathrm{Lip}(f) := \sup_{\substack{x,y \in \mathrm{X} \\ x \neq y}} \frac{\big|f(x) - f(y)\big|}{\mathsf{d}(x,y)} \in [0, +\infty]. \tag{1.19}$$

We say that f is *Lipschitz* provided $\mathrm{Lip}(f) < +\infty$ and we define

$$\begin{aligned} \mathrm{LIP}(\mathrm{X}) &:= \big\{ f : \mathrm{X} \to \mathbb{R} \ : \ \mathrm{Lip}(f) < +\infty \big\}, \\ \mathrm{LIP}_{bs}(\mathrm{X}) &:= \big\{ f \in \mathrm{LIP}(\mathrm{X}) \ : \ \mathrm{spt}(f) \text{ is bounded} \big\} \subseteq C_b(\mathrm{X}). \end{aligned} \tag{1.20}$$

We point out that continuous maps having bounded support are not necessarily bounded.

Proposition 1.1.18 (Separability of $L^p(\mu)$ for $p < \infty$) *Let $\mu \in \mathscr{P}(\mathrm{X})$ and $p \in [1, \infty)$. Then the space $\mathrm{LIP}_{bs}(\mathrm{X})$ is dense in $L^p(\mu)$. In particular, the space $L^p(\mu)$ is separable.*

Proof First, notice that $\mathrm{LIP}_{bs}(\mathrm{X}) \subseteq L^\infty(\mu) \subseteq L^p(\mu)$. Call $\mathscr{C}$ the $L^p(\mu)$-closure of $\mathrm{LIP}_{bs}(\mathrm{X})$.

STEP 1. We claim that $\big\{\chi_C \ : \ C \subseteq \mathrm{X} \text{ closed bounded}\big\}$ is contained in the set $\mathscr{C}$. Indeed, called $f_n := \big(1 - n\,\mathsf{d}(\cdot, C)\big)^+ \in \mathrm{LIP}_{bs}(\mathrm{X})$ for any $n \in \mathbb{N}$, one has $f_n \to \chi_C$ in $L^p(\mu)$ by dominated convergence theorem.

STEP 2. We also have that $\big\{\chi_E \ : \ E \subseteq \mathrm{X} \text{ Borel}\big\} \subseteq \mathscr{C}$. Indeed, we can pick an increasing sequence $(C_n)_n$ of closed subsets of E such that $\mu(E) = \lim_n \mu(C_n)$, as seen in (1.13). Then one has that $\|\chi_E - \chi_{C_n}\|_{L^p(\mu)} = \mu(E \setminus C_n)^{1/p} \to 0$, whence $\chi_E \in \mathscr{C}$ by STEP 1.

STEP 3. To prove that $L^p(\mu) \subseteq \mathscr{C}$, fix $f \in L^p(\mu)$, without loss of generality say $f \geq 0$. Given any $n, i \in \mathbb{N}$, let us define $E_{ni} := f^{-1}\big([i/2^n, (i+1)/2^n[\big)$. Observe that $(E_{ni})_i$ is a Borel partition of X, thus it makes sense to define $f_n := \sum_{i \in \mathbb{N}} i\, 2^{-n} \chi_{E_{ni}} \in L^p(\mu)$. Given that we have $f_n(x) \nearrow f(x)$ for μ-a.e. $x \in \mathrm{X}$, it holds $f_n \to f$ in $L^p(\mu)$ by dominated convergence theorem. We aim to prove that $(f_n)_n \subseteq \mathscr{C}$, which would immediately imply that $f \in \mathscr{C}$. Then fix $n \in \mathbb{N}$. Notice that f_n is the $L^p(\mu)$-limit of $f_n^N := \sum_{i=1}^N i\, 2^{-n} \chi_{E_{ni}}$ as $N \to \infty$, again by dominated convergence theorem. Given that each $f_n^N \in \mathscr{C}$ by STEP 2, we get that f_n is in $\mathscr{C}$ as well. Hence $\mathrm{LIP}_{bs}(\mathrm{X})$ is dense in $L^p(\mu)$.

STEP 4. Finally, we prove separability of $L^p(\mu)$. We can take an increasing sequence $(K_n)_n$ of compact subsets of X such that the measure μ is concentrated on $\bigcup_n K_n$, by Corollary 1.1.16. Since $\chi_{K_n} f \to f$ in $L^p(\mu)$ for any $f \in L^p(\mu)$, we see that

$$\bigcup_{n \in \mathbb{N}} \underbrace{\big\{ f \in L^p(\mu) \ : \ f = 0 \ \ \mu\text{-a.e. in } \mathrm{X} \setminus K_n \big\}}_{=:S_n} \quad \text{is dense in } L^p(\mu).$$

To conclude, it is sufficient to show that each S_n is separable. Observe that $C(K_n)$ is separable by Corollary 1.1.6, thus accordingly its subset $\mathrm{LIP}_{bs}(K_n)$ is separable with respect to $\|\cdot\|_{C_b(K_n)}$. In particular, $\mathrm{LIP}_{bs}(K_n)$ is separable with respect to $\|\cdot\|_{L^p(\mu)}$. Moreover, $\mathrm{LIP}_{bs}(K_n)$ is dense in $L^p(\mu_{|K_n}) \cong S_n$ by the first part of the statement, therefore each S_n is separable. □

1.1.2 The Space $L^0(\mathfrak{m})$

By *metric measure space* we mean a triple $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, where

$$\begin{aligned} &(\mathrm{X}, \mathsf{d}) && \text{is a complete and separable metric space,}\\ &\mathfrak{m} \neq 0 && \text{is a non-negative Borel measure on } (\mathrm{X}, \mathsf{d}), \text{ which is finite on balls.} \end{aligned} \tag{1.21}$$

Let us denote by $L^0(\mathfrak{m})$ the vector space of all Borel functions $f : \mathrm{X} \to \mathbb{R}$, which are considered modulo $\mathfrak{m}$-a.e. equality. Then $L^0(\mathfrak{m})$ becomes a topological vector space when endowed with the following distance: choose any Borel probability measure $\mathfrak{m}' \in \mathscr{P}(\mathrm{X})$ such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$ (for instance, pick any Borel partition $(E_n)_n$ made of sets having finite positive $\mathfrak{m}$-measure and set $\mathfrak{m}' := \sum_n \frac{\chi_{E_n}\mathfrak{m}}{2^n\,\mathfrak{m}(E_n)}$) and define

$$\mathsf{d}_{L^0}(f, g) := \int |f-g| \wedge 1 \, \mathrm{d}\mathfrak{m}' \quad \text{for every } f, g \in L^0(\mathfrak{m}). \tag{1.22}$$

Such distance may depend on the choice of $\mathfrak{m}'$, but its induced topology does not, as we are going to show in the next result:

Proposition 1.1.19 *A sequence $(f_n)_n \subseteq L^0(\mathfrak{m})$ is d_{L^0}-Cauchy if and only if*

$$\varlimsup_{n,m\to\infty} \mathfrak{m}\Big(E \cap \big\{|f_n - f_m| > \varepsilon\big\}\Big) = 0 \quad \begin{array}{l} \textit{for every } \varepsilon > 0 \textit{ and } E \subseteq \mathrm{X} \\ \textit{Borel with } \mathfrak{m}(E) < +\infty. \end{array} \tag{1.23}$$

Proof We separately prove the two implications:
NECESSITY. Suppose that (1.23) holds. Fix $\varepsilon > 0$. Choose any point $\bar{x} \in \mathrm{X}$, then there exists $R > 0$ such that $\mathfrak{m}'\big(B_R(\bar{x})\big) \geq 1 - \varepsilon$. Recall that $\mathfrak{m}$ is finite on bounded sets by hypothesis, so that $\mathfrak{m}\big(B_R(\bar{x})\big) < +\infty$. Moreover, since $\mathfrak{m}'$ is a finite measure, we clearly have that $\chi_{B_R(\bar{x})} \frac{\mathrm{d}\mathfrak{m}'}{\mathrm{d}\mathfrak{m}} \in L^1(\mathfrak{m})$. Now let us call $A_{nm}(\varepsilon)$ the set $B_R(\bar{x}) \cap \big\{|f_n - f_m| > \varepsilon\big\}$. Then property (1.23) grants that $\chi_{A_{nm}(\varepsilon)} \to 0$ in $L^1(\mathfrak{m})$ as $n, m \to \infty$, whence an application of the dominated convergence theorem yields

$$\varlimsup_{n,m\to\infty} \mathfrak{m}'\big(A_{nm}(\varepsilon)\big) = \varlimsup_{n,m\to\infty} \int \chi_{A_{nm}(\varepsilon)}\, \chi_{B_R(\bar{x})} \frac{\mathrm{d}\mathfrak{m}'}{\mathrm{d}\mathfrak{m}} \, \mathrm{d}\mathfrak{m} = 0. \tag{1.24}$$

Therefore we deduce that

$$\begin{aligned}\int |f_n - f_m| \wedge 1 \,\mathrm{d}\mathfrak{m}' &= \int_{\mathrm{X}\setminus B_R(\bar{x})} |f_n - f_m| \wedge 1 \,\mathrm{d}\mathfrak{m}' + \int_{B_R(\bar{x})} |f_n - f_m| \wedge 1 \,\mathrm{d}\mathfrak{m}' \\ &\leq \varepsilon + \int_{B_R(\bar{x})\cap\{|f_n - f_m|\leq\varepsilon\}} |f_n - f_m| \wedge 1 \,\mathrm{d}\mathfrak{m}' \\ &\quad + \int_{A_{nm}(\varepsilon)} |f_n - f_m| \wedge 1 \,\mathrm{d}\mathfrak{m}' \\ &\leq 2\,\varepsilon + \mathfrak{m}'\big(A_{nm}(\varepsilon)\big),\end{aligned}$$

from which we see that $\overline{\lim}_{n,m} \mathsf{d}_{L^0}(f_n, f_m) \leq 2\,\varepsilon$ by (1.24). By arbitrariness of $\varepsilon > 0$, we conclude that $\lim_{n,m} \mathsf{d}_{L^0}(f_n, f_m) = 0$, which shows that the sequence $(f_n)_n$ is d_{L^0}-Cauchy.
SUFFICIENCY. Suppose that $(f_n)_n$ is d_{L^0}-Cauchy. Fix any $\varepsilon \in (0, 1)$ and a Borel set $E \subseteq \mathrm{X}$ with $\mathfrak{m}(E) < +\infty$. Hence the Čebyšëv inequality yields

$$\mathfrak{m}'\big(\{|f_n - f_m| > \varepsilon\}\big) = \mathfrak{m}'\big(\{|f_n - f_m| \wedge 1 > \varepsilon\}\big) \leq \frac{1}{\varepsilon} \int |f_n - f_m| \wedge 1 \,\mathrm{d}\mathfrak{m}' = \frac{\mathsf{d}_{L^0}(f_n, f_m)}{\varepsilon},$$

so that $\overline{\lim}_{n,m} \mathfrak{m}'\big(\{|f_n - f_m| > \varepsilon\}\big) = 0$. Finally, observe that $\chi_E \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}\mathfrak{m}'} \in L^1(\mathfrak{m}')$, whence

$$\mathfrak{m}\Big(E \cap \big\{|f_n - f_m| > \varepsilon\big\}\Big) = \int \chi_E \frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}\mathfrak{m}'} \chi_{\{|f_n - f_m| > \varepsilon\}} \,\mathrm{d}\mathfrak{m}' \overset{n,m}{\longrightarrow} 0$$

by dominated convergence theorem. Therefore (1.23) is proved. □

Remark 1.1.20 Recall that two metrizable spaces with the same Cauchy sequences have the same topology, while the converse implication does not hold in general. For instance, consider the real line $\mathbb{R}$ endowed with the following two distances:

$$\left.\begin{aligned}\mathsf{d}_1(x, y) &:= |x - y|, \\ \mathsf{d}_2(x, y) &:= \big|\arctan(x) - \arctan(y)\big|,\end{aligned}\right. \qquad \text{for every } x, y \in \mathbb{R}.$$

Then d_1 and d_2 induce the same topology on $\mathbb{R}$, but the d_2-Cauchy sequence $(x_n)_n \subseteq \mathbb{R}$ defined by $x_n := n$ is not d_1-Cauchy. ■

We now show that the distance d_{L^0} metrizes the 'local convergence in measure':

Proposition 1.1.21 *Let $f \in L^0(\mathfrak{m})$ and $(f_n)_n \subseteq L^0(\mathfrak{m})$. Then the following are equivalent:*

i) *It holds that $\mathsf{d}_{L^0}(f_n, f) \to 0$ as $n \to \infty$.*
ii) *Given any subsequence $(n_m)_m$, there exists a further subsequence $(n_{m_k})_k$ such that the limit $\lim_k f_{n_{m_k}}(x) = f(x)$ is verified for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$.*