Bohmian Mechanics

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The Physics and Mathematics of Quantum Theory

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Preface

This book is about Bohmian mechanics, a non-relativistic quantum theory based on a particle ontology. As such it is a consistent theory of quantum phenomena, i.e., none of the mysteries and paradoxes which plague the usual descriptions of quantum mechanics arise. The most important message of this book is that quantum mechanics, as defined by its most general mathematical formalism, finds its explication in the statistical analysis of Bohmian mechanics following Boltzmann's ideas.

The book connects the physics with the abstract mathematical formalism and prepares all that is needed to achieve a commonsense understanding of the non-relativistic quantum world. Therefore this book may be of interest to both physicists and mathematicians. The latter, who usually aim at unerring precision, are often put off by the mystical-sounding phrases surrounding the abstract mathematics of quantum mechanics. In this book we aim at a precision which will also be acceptable to mathematicians.

Bohmian mechanics, named after its inventor David Bohm,[1] is about the motion of particles. The positions of particles constitute the primitive variables, the primary ontology. For a quantum physicist the easiest way to grasp Bohmian mechanics and to write down its defining equations is to apply the dictum: Whenever you say particle, mean it!

The key insight for analyzing Bohmian mechanics lies within the foundations of statistical physics. The reader will find it worth the trouble to work through Chap. 2 on classical physics and Chap. 4 on chance, which are aimed at the understanding of the statistical analysis of a physical theory as it was developed by the great physicist Ludwig Boltzmann. Typicality is the ticket to get to the statistical import of Bohmian mechanics, which is succinctly captured by $\rho = |\psi|^2$. The justification for

[1] The equations were in fact already written down by the mathematical physicist Erwin Madelung, even before the famous physicist Louis de Broglie suggested the equations at the famous 1927 Solvay conference. But these are unimportant historical details, which have *no significance* for the understanding of the theory. David Bohm was not aware of these early attempts, and moreover he presented the full implications of the theory for the quantum formalism. The theory is also called the pilot wave theory or the de Broglie–Bohm theory, but Bohm himself called it the causal interpretation of quantum mechanics.

this and the analysis of its consequences are two central points in the book. One major consequence is the emergence of the abstract mathematical structure of quantum mechanics, observables as operators on Hilbert space, POVMS, and Heisenberg's uncertainty relation. All this and more follows from the theory of particles in motion.

But this is not the only reason for the inclusion of some purely mathematical chapters. Schrödinger's cat story is world famous. A common argument to diminish the measurement problem is that the quantum mechanical description of a cat in a box is so complicated, the mathematics so extraordinarily involved, that nobody has done that yet. But, so the argument continues, it could in principle be done, and if all the mathematics is done properly and if one introduces all the observables with their domains of definitions in a proper manner, in short, if one does everything in a mathematically correct way, then there is no measurement problem. This answer also appears in disguise as the claim that decoherence solves the measurement problem. But this is false! It is precisely because one can in principle describe a cat in a box quantum mechanically that the problem is there and embarrassingly plain to see. We have included all the mathematics required to ensure that no student of the subject can be tricked into believing that everything in quantum physics would be alright if only the mathematics were done properly.

Bohmian mechanics has been around since 1952. It was promoted by John Bell in the second half of the last century. In particular, it was the manifestly nonlocal structure of Bohmian mechanics that led Bell to his celebrated inequalities, which allow us to check experimentally whether nature is local. Experiments have proved that nature is nonlocal, just as Bohmian mechanics predicted. Nevertheless there was once a time when physicists said that Bell's inequalities proved that Bohmian mechanics was impossible. In fact, all kinds of criticisms have been raised against Bohmian mechanics. Since Bohmian mechanics is so simple and straightforward, only one criticism remains: there must be something wrong with Bohmian mechanics, otherwise it would be taught. And as a consequence, Bohmian mechanics is not taught because there must be something wrong with it, otherwise it would be taught. We try in this book to show how Bohmian mechanics could be taught.

Any physicist who is ready to quantize everything under his pen should know what quantization means in the simplest and established frame of non-relativistic physics, and learn what conclusions should be drawn from that. The one conclusion which cannot be drawn is that nothing exists, or more precisely, that what exists cannot be named within a mathematically consistent theory! For indeed it can! The lesson here is that one should never give up ontology! If someone says: "I do not know what it means to exist," then that is fine. That person can view the theory of Bohmian mechanics as a precise and coherent mathematical theory, in which all that needs to be said is written in the equations, ready for analysis.

Our guideline for writing the book was the focus on the *genesis* of the ideas and concepts, to be clear about *what it is* that we are talking about, and hence to pave the way for the hard technical work of learning *how it is done*. In short, we have tried not to leave out the letter 'h' (see the Melville quote on p. 4).

References

The references we give are neither complete nor balanced. Naturally the references reflect our point of view. They are nevertheless chosen in view of their basic character, sometimes containing many further references which the reader may follow up to achieve a more complete picture.

Acknowledgements

The present book is a translation and revision of the book *Bohmsche Mechanik als Grundlage der Quantenmechanik* by one of the present authors (D.D.), published by Springer in 2001. The preface for this book read as follows: "In this book I wrote down what I learned from my friends and colleagues Sheldon Goldstein and Nino Zanghì. With their cooperation everything in this book would have been said correctly – but only after an unforeseeably long time (extrapolating the times it takes us to write a normal article). For that, I lack the patience (see the Epilogue) and I wanted to say things in my own style. Because of that, and only because of that, things may be said wrongly." This paragraph is still relevant, of course, but we feel the need to express our thanks to these truly great physicists and friends even more emphatically. Without them this book would never have been written (but unfortunately with them perhaps also not). We have achieved our goal if they would think of this as a good book. With them the (unwritten) book would have been excellent. Today we should like to add one more name: Roderich Tumulka, a former student, now a distinguished researcher in the field of foundations who has since helped us to gain further deep insights.

The long chapter on chance resulted from many discussions with Reinhard Lang which clarified many issues. It was read and corrected several times by our student Christian Beck. Selected chapters were read, reread, and greatly improved by our students and coworkers Dirk Deckert, Robert Grummt, Florian Hoffmann, Tilo Moser, Peter Pickl, Sarah Römer, and Georg Volkert. The whole text was carefully read and critically discussed by the students of the reading class on Bohmian mechanics, and they are all thanked for their commitment. In particular, we wish to thank Dustin Lazarovici, Niklas Boers, and Florian Rieger for extensive and thoughtful corrections. We are especially grateful to Sören Petrat, who did a very detailed and expert correction of the whole text.

The foundations of quantum mechanics are built on slippery grounds, and one depends on friends and colleagues who encourage and support (with good criticism) this research, which is clearly not mainstream physics. Two important friends in this undertaking are Giancarlo Ghirardi and Herbert Spohn. But there are more friends we wish to acknowledge, who have, perhaps unknowingly, had a strong impact on the writing of the book: Angelo Bassi, Gernot Bauer, Jürg Fröhlich, Rodolfo Figari, Martin Kolb, Tim Maudlin, Sandro Teta, and Tim Storck.

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be substantially improved by the editorial work done on it. We were extremely fortunate that our editor was Angela Lahee at Springer, who astutely provided us with the best copy-editor we could have wished for, namely Stephen Lyle, who brilliantly transformed our Germanic English constructions into proper English without jeopardizing our attempt to incite the reader to active contemplation throughout. We are very grateful to both Angela and Stephen for helping us with our endeavour to present Bohmian mechanics as a respectable part of physics.

Landsberg and Tübingen, *Detlef Dürr*
October 2008 *Stefan Teufel*

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Introduction

1.1 Ontology: What There Is

1.1.1 Extracts

Sometimes quantization is seen as the procedure that puts hats on classical observables to turn them into quantum observables.

Lewis Carroll on Hatters and Cats

Lewis Carroll (alias Charles Lutwidge Dodgson 1832–1898) was professor of mathematics at Oxford (where Schrödinger wrote his famous cat article):

> The Cat only grinned when it saw Alice. It looked good-natured, she thought: still it had VERY long claws and a great many teeth, so she felt that it ought to be treated with respect. “Cheshire Puss,” she began, rather timidly, as she did not at all know whether it would like the name: however, it only grinned a little wider. “Come, it’s pleased so far,” thought Alice, and she went on. “Would you tell me, please, which way I ought to go from here?” “That depends a good deal on where you want to get to,” said the Cat. “I don’t much care where –” said Alice. “Then it doesn’t matter which way you go,” said the Cat. “– so long as I get SOMEWHERE,” Alice added as an explanation. “Oh, you’re sure to do that,” said the Cat, “if you only walk long enough.” Alice felt that this could not be denied, so she tried another question. “What sort of people live about here?” “In THAT direction,” the Cat said, waving its right paw round, “lives a Hatter: and in THAT direction,” waving the other paw, “lives a March Hare. Visit either you like: they’re both mad.” “But I don’t want to go among mad people,” Alice remarked. “Oh, you can’t help that,” said the Cat: “we’re all mad here. I’m mad. You’re mad.” “How do you know I’m mad?” said Alice. “You must be,” said the Cat, “or you wouldn’t have come here.” Alice didn’t think that proved it at all; however, she went on “And how do you know that you’re mad?” “To begin with,” said the Cat, “a dog’s not mad. You grant that?” “I suppose so,” said Alice. “Well, then,” the Cat went on, “you see, a dog growls when it’s angry, and wags its tail when it’s pleased. Now I growl when I’m pleased, and wag my tail when I’m angry. Therefore I’m mad.” “I call it purring, not growling,” said Alice. “Call it what you like,” said the Cat. “Do you play croquet with the

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> Queen today?" "I should like it very much," said Alice, "but I haven't been invited yet." "You'll see me there," said the Cat, and vanished.
>
> Alice was not much surprised at this, she was getting so used to queer things happening. While she was looking at the place where it had been, it suddenly appeared again.

Alice's Adventures in Wonderland (1865) [1].

Parmenides on What There Is

The Greek philosopher Parmenides of Elea wrote as follows in the sixth century BC:

> Come now, I will tell thee – and do thou hearken to my saying and carry it away – the only two ways of search that can be thought of. The first, namely, that It is, and that it is impossible for it not to be, is the way of belief, for truth is its companion. The other, namely, that It is not, and that it must needs not be, – that, I tell thee, is a path that none can learn of at all. For thou canst not know what is not – that is impossible – nor utter it; for it is the same thing that can be thought and that can be.
>
> It needs must be that what can be spoken and thought is; for it is possible for it to be, and it is not possible for what is nothing to be. This is what I bid thee ponder. I hold thee back from this first way of inquiry, and from this other also, upon which mortals knowing naught wander two-faced; for helplessness guides the wandering thought in their breasts, so that they are borne along stupefied like men deaf and blind. Undiscerning crowds, who hold that it is and is not the same and not the same, and all things travel in opposite directions!
>
> For this shall never be proved, that the things that are not are; and do thou restrain thy thought from this way of inquiry.

The Way of Truth [2].

It is sometimes said that, in quantum mechanics, the observer calls things into being by the act of observation.

Schrödinger on Quantum Mechanics

Erwin Schrödinger wrote in 1935:

> One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, so small, that perhaps in the course of the hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives if meanwhile no atom has decayed. The psi-function of the entire system would express this by having in it the living and dead cat (pardon the expression) mixed or smeared out in equal parts.
>
> It is typical of these cases that an indeterminacy originally restricted to the atomic domain becomes transformed into macroscopic indeterminacy, which can then be resolved by direct observation. That prevents us from so naively accepting as valid a "blurred model" for

representing reality. In itself it would not embody anything unclear or contradictory. There is a difference between a shaky or out-of-focus photograph and a snapshot of clouds and fog banks.

Die gegenwärtige Situation in der Quantenmechanik [3].
Translated by J.D. Trimmer in [4].

The difference between a shaky photograph and a snapshot of clouds and fog banks is the difference between Bohmian mechanics and quantum mechanics. In itself a "blurred model" representing reality would not embody anything unclear, whereas resolving the indeterminacy by direct observation does. As if observation were not part of physics.

Feynman on Quantum Mechanics

Feynman said the following:

Does this mean that my observations become real only when I observe an observer observing something as it happens? This is a horrible viewpoint. Do you seriously entertain the thought that without observer there is no reality? Which observer? Any observer? Is a fly an observer? Is a star an observer? Was there no reality before 10^9 B.C. before life began? Or are you the observer? Then there is no reality to the world after you are dead? I know a number of otherwise respectable physicists who have bought life insurance. By what philosophy will the universe without man be understood?

Lecture Notes on Gravitation [5].

Bell on Quantum Mechanics

According to John S. Bell:

It would seem that the theory is exclusively concerned about "results of measurement", and has nothing to say about anything else. What exactly qualifies some physical systems to play the role of "measurer"? Was the wavefunction of the world waiting to jump for thousands of years until a single-celled living creature appeared? Or did it have to wait a little longer, for some better qualified system [...] with a Ph.D.? If the theory is to apply to anything but highly idealized laboratory operations, are we not obliged to admit that more or less "measurement-like" processes are going on more or less all the time, more or less everywhere? Do we not have jumping then all the time?

Against "measurement" [6].

Einstein on Measurements

But it is in principle quite false to base a theory solely on observable quantities. Since, in fact, it is the other way around. It is the theory that decides what we can observe.

Albert Einstein, cited by Werner Heisenberg[1] in [7].

[1] Quoted from *Die Quantenmechanik und ein Gespräch mit Einstein*. Translation by Detlef Dürr.

Melville on Omissions

> While you take in hand to school others, and to teach them by what name a whale-fish is to be called in our tongue leaving out, through ignorance, the letter H, which almost alone maketh the signification of the word, you deliver that which is not true.

Melville (1851), Moby Dick; or, The Whale, Etymology [8].

1.1.2 In Brief: The Problem of Quantum Mechanics

Schrödinger remarks in his laconic way that there is a difference between a shaky or out-of-focus photograph and a snapshot of clouds and fog banks.

The first problem with quantum mechanics is that it is not about what there is. It is said to be about the microscopic world of atoms, but it does not spell out which physical quantities in the theory describe the microscopic world. Which variables specify what is microscopically there? Quantum mechanics is about the wave function. The wave function lives on configuration space, which is the coordinate space of all the particles participating in the physical process of interest. That the wave function lives on configuration space has been called by Schrödinger *entanglement* of the wave function. The wave function obeys a linear equation – the Schrödinger equation. The linearity of the Schrödinger equation prevents the wave function from representing reality.[2] We shall see that in a moment. The second problem of quantum mechanics is that the first problem provokes many rich answers, which to the untrained ear appear to be of philosophical nature, but which leave the problem unanswered: What is it that quantum mechanics is about?

It is said for example that the virtue of quantum mechanics is that it is *only* about what can be measured. Moreover, that what can be measured is *defined* through the measurement. Without measurement there is nothing there. But a measurement belongs to the macroscopic world (which undeniably exists), and its macroscopic constituents like the measurement apparatus are made out of atoms, which quantum mechanics is supposed to describe, so this entails that the apparatus itself is to be described quantum mechanically. This circularity lies at the basis of the measurement problem of quantum mechanics, which is often phrased as showing incompleteness of quantum mechanics. Some things, namely the objects the theory is about, have been left out of the description, or the description – the Schrödinger equation – is not right. Mathematically the measurement problem may be presented as follows. Suppose a system is described by linear combinations of wave functions φ_1 and φ_2. Suppose there exists a piece of apparatus which, when brought into interaction with the system, measures whether the system has wave function φ_1 or φ_2. Measuring

[2] Even if the equation were nonlinear, as is the case in reduction models, the wave function living on configuration space could not by itself represent reality in physical space. There must still be some "beable" (related to the wave function) in the sense of Bell [9] representing physical reality. But we ignore this fine point here.

means that, next to the 0 pointer position, the apparatus has two pointer positions 1 and 2 "described" by wave functions Ψ_0, Ψ_1, and Ψ_2, for which

$$\varphi_i \Psi_0 \overset{\text{Schrödinger evolution}}{\longrightarrow} \varphi_i \Psi_i \,, \qquad i = 1, 2 \,. \tag{1.1}$$

When we say that the pointer positions are "described" by wave functions, we mean that in a loose sense. The wave function has a support in configuration space which corresponds classically to a set of coordinates of particles which would form a pointer.

The Schrödinger equation is linear, so for the superposition, (1.1) yields

$$\varphi = c_1 \varphi_1 + c_2 \varphi_2 \,, \qquad c_1, c_2 \in \mathbb{C} \,, \qquad |c_1|^2 + |c_2|^2 = 1 \,,$$

$$\varphi \Psi_0 = (c_1 \varphi_1 + c_2 \varphi_2) \Psi_0 \overset{\text{Schrödinger evolution}}{\longrightarrow} c_1 \varphi_1 \Psi_1 + c_2 \varphi_2 \Psi_2 \,. \tag{1.2}$$

The outcome on the right does not concur with experience. It shows rather a "macroscopic indeterminacy". In the words of Schrödinger, observation then resolves this macroscopic indeterminacy, since one only observes either 1 (with probability $|c_1|^2$) or 2 (with probability $|c_2|^2$), i.e., observation resolves the blurred description of reality into one where the wave function is either $\varphi_1 \Psi_1$ or $\varphi_2 \Psi_2$. In Schrödinger's cat thought experiment $\varphi_{1,2}$ are the wave functions of the non-decayed and decayed atom and $\Psi_{0,1}$ are the wave functions of the live cat and Ψ_2 is the wave function of the dead cat. Schrödinger says that this is unacceptable. But why? Is the apparatus not supposed to be the observer? What qualifies us better than the apparatus, which we designed in such a way that it gives a definite outcome and not a blurred one?

The question is: what in the theory describes the actual facts? Either those variables which describe the actual state of affairs have been left out of the description (Bohmian mechanics makes amends for that) or the Schrödinger equation which yields the unrealistic result (1.2) from (1.1) is false. (GRW theories, or more generally, dynamical reduction models, follow the latter way of describing nature. The wave function collapses by virtue of the dynamical law [10].)

The evolution in (1.2) is an instance of the so-called decoherence. The apparatus decoheres the superposition $c_1 \varphi_1 + c_2 \varphi_2$ of the system wave function. Decoherence means that it is in a practical sense impossible to get the two wave packets $\varphi_1 \Psi_1$ and $\varphi_2 \Psi_2$ superposed in $c_1 \varphi_1 \Psi_1 + c_2 \varphi_2 \Psi_2$ to interfere. It is sometimes said that, taking decoherence into account, there would not be any measurement problem. Decoherence is this practical impossibility, which Bell referred to as fapp-impossibility (where fapp means for all practical purposes), of the interference of the pointer wave functions. It is often dressed up, for better looks, in terms of density matrices. In Dirac's notation, the density matrix is

$$\rho_G = |c_1|^2 |\varphi_1\rangle |\Psi_1\rangle \langle \Psi_1| \langle \varphi_1| + |c_2|^2 |\varphi_2\rangle |\Psi_2\rangle \langle \Psi_2| \langle \varphi_2| \,.$$

This can be interpreted as describing a statistical mixture of the two states $|\varphi_1\rangle|\Psi_1\rangle$ and $|\varphi_2\rangle|\Psi_2\rangle$, and one can say that in a fapp sense the pure state given by the right-hand side of (1.2), namely

$$\rho = |c_1\varphi_1\Psi_1 + c_2\varphi_2\Psi_2\rangle\langle c_1\Psi_1\varphi_1 + c_2\Psi_2\varphi_2| \; ,$$

is close to ρ_G. They are fapp-close because the off-diagonal element

$$c_1c_2^*|\Psi_1\varphi_1\rangle\langle\Psi_2\varphi_2|$$

where the asterisk denotes complex conjugation, would only be observable if the wave parts could be brought into interference, which is fapp-impossible. The argument then concludes that the meaning of the right-hand side of (1.2) is fapp-close to the meaning of the statistical mixture. To emphasise the fact that such arguments miss the point of the exercise, Schrödinger made his remark that there is a difference between a shaky or out-of-focus photograph and a snapshot of clouds and fog banks.

To sum up then, decoherence does not create the facts of our world, but rather produces a sequence of fapp-redundancies, which physically increase or stabilize decoherence. So the cat decoheres the atom, the observer decoheres the cat that decoheres the atom, the environment of the observer decoheres the observer that decoheres the cat that decoheres the atom and so on. In short, what needs to be described by the physical theory is the behavior of real objects, located in physical space, which account for the facts.

1.1.3 In Brief: Bohmian Mechanics

Bohmian mechanics is about point particles in motion. The theory was invented in 1952 by David Bohm (1917–1992) [11] (there is a bit more on the history in Chap. 7). In a Bohmian universe everything is made out of particles. Their motion is guided by the wave function. That is why the wave function is there. That is its role. The physical theory is formulated with the variables $\mathbf{q}_i \in \mathbb{R}^3$, $i = 1,2,\ldots,N$, the positions of the N particles which make up the system, and the wave function $\psi(\mathbf{q}_1,\ldots,\mathbf{q}_N)$ on the configuration space of the system. If the wave function consists of two parts which have disjoint supports in configuration space, then the system configuration is in one or the other support. In the measurement example, the pointer configuration is either in the support $\text{supp}\,\Psi_1$ of Ψ_1 (pointing out 1) or in the support $\text{supp}\,\Psi_2$ of Ψ_2 (pointing out 2).

Bohmian mechanics happens to be deterministic. A substantial success of Bohmian mechanics is the explanation of quantum randomness or Born's statistical law, on the basis of Boltzmann's principles of statistical mechanics, i.e., Born's law is not an axiom but a theorem in Bohmian mechanics. Born's statistical law concerning $\rho = |\psi|^2$ says that, if the wave function is ψ, the particle configuration is $|\psi|^2$-distributed. Applying this to (1.2) implies that the result i comes with probability

$|c_i|^2$. Suppose the enlarged system composed of system plus apparatus is described by the configuration coordinates $\mathbf{q} = (\mathbf{x}, \mathbf{y})$, where $\mathbf{x} \in \mathbb{R}^m$ is the system configuration and $\mathbf{y} \in \mathbb{R}^n$ the pointer configuration. We compute the probability for the result 1 by Born's law. This means we compute the probability that the true (or actual) pointer configuration Y lies in the support $\operatorname{supp} \Psi_1$. In the following computation, we assume that the wave functions are all normalized to unity. Then

$$\mathbb{P}(\text{pointer on } 1) = \int_{\operatorname{supp}\Psi_1} |c_1 \varphi_1 \Psi_1 + c_2 \varphi_2 \Psi_2|^2 \mathrm{d}^m x \, \mathrm{d}^n y \tag{1.3a}$$

$$= |c_1|^2 \int_{\operatorname{supp}\Psi_1} |\varphi_1 \Psi_1|^2 \mathrm{d}^m x \, \mathrm{d}^n y + |c_2|^2 \int_{\operatorname{supp}\Psi_1} |\varphi_2 \Psi_2|^2 \mathrm{d}^m x \mathrm{d}^n y$$

$$+ 2\Re \left[c_1^* c_2 \int_{\operatorname{supp}\Psi_1} (\varphi_1 \Psi_1)^* \varphi_2 \Psi_2 \mathrm{d}^m x \, \mathrm{d}^n y \right] \tag{1.3b}$$

$$= |c_1|^2 \int |\varphi_1 \Psi_1|^2 \mathrm{d}^m x \, \mathrm{d}^n y = |c_1|^2 \,, \tag{1.3c}$$

where $\Re$ in (1.3b) denotes the real part of a complex quantity. The terms involving Ψ_2 yield zero because of the disjointness[3] of the supports of the pointer wave functions Ψ_1, Ψ_2. This result holds fapp-forever, since it is fapp-impossible for the wave function parts Ψ_1 and Ψ_2 to interfere in the future, especially when the results have been written down or recorded in any way – the ever growing decoherence.

Suppose one removes the positions of the particles from the theory, as for example Heisenberg, Bohr, and von Neumann did. Then to be able to conclude from the fapp-impossibility of interference that only one of the wave functions remains, one needs to add an "observer" who by the act of observation collapses the wave function with probability $|c_i|^2$ to the Ψ_i part, thereby "creating" the result i. Once again, we may ask what qualifies an observer as better than a piece of apparatus or a cat. Debate of this kind has been going on since the works of Heisenberg and Schrödinger in 1926. The debate, apart from producing all kinds of philosophical treatises, revolves around the collapse of the wave function. But when does the collapse happen and who is entitled to collapse the wave function? Does the collapse happen at all? It is, to put it mildly, a bit puzzling that such an obvious shortcoming of the theory has led to such a confused and unfocussed debate. Indeed, it has shown physics in a bad light. In Bohmian mechanics, the collapse does not happen at all, although there is a fapp-collapse, which one may introduce when one analyzes the theory. In collapse theories (like GRW), the collapse does in fact happen. Bohmian mechanics and collapse theories differ in that way. They make different predictions for macroscopic interference experiments, which may, however, be very difficult to perform, if not fapp-impossible.

We did not say how the particles are guided by the wave function. One gets that by simply taking language seriously. Whenever you say "particle" in quantum mechanics, mean it! That is Bohmian mechanics. All problems evaporate on the

[3] The wave functions will in reality overlap, but the overlap is negligible. Therefore the wave functions are well approximated by wave functions with disjoint supports.

spot. In a nutshell, the quantity $|\psi_t|^2$, with ψ_t a solution of Schrödinger's equation, satisfies a continuity equation, the so-called quantum flux equation. The particles in Bohmian mechanics move along the flow lines of the quantum flux. In other words the quantum flux equation is the continuity equation for transport of probability along the Bohmian trajectories. Still not satisfied? Is that too cheap? Why the wave function? Why $|\psi_t|^2$? We shall address all these questions and more in the chapters to follow.

Bohmian mechanics is defined by two equations: one is the Schrödinger equation for the guiding field ψ_t and one is the equation for the positions of the particles. The latter equation reads

$$\dot{\mathbf{Q}} = \mu \Im \frac{\nabla \psi_t(\mathbf{Q})}{\psi_t(\mathbf{Q})} , \tag{1.4}$$

where $\Im$ denotes the imaginary part, and μ is an appropriate dimension factor. The quantum formalism in its most general form, including all rules and axioms, follows from this by *analysis* of the theory. In particular, Heisenberg's uncertainty relation for position and momentum, from which it is often concluded that particle trajectories are in conflict with quantum mechanics, follows directly from Bohmian mechanics.

Bohmian mechanics is nonlocal in the sense of Bell's inequalities and therefore, according to the experimental tests of Bell's inequalities, concurs with the basic requirements that any correct theory of nature must fulfill.

A Red Herring: The Double Slit Experiment

This is a quantum mechanical experiment which is often cited as conflicting with the idea that there can be particles with trajectories. One sends a particle (i.e., a wave packet ψ) through a double slit. Behind the slit at some distance is a photographic plate. When the particle arrives at the plate it leaves a black spot at its place of arrival. Nothing yet speaks against the idea that the particle moves on a trajectory. But now repeat the experiment. The next particle marks a different spot of the photographic plate. Repeating this a great many times the spots begin to show a pattern. They trace out the points of constructive interference of the wave packet ψ which, when passing the two slits, shows the typical Huygens interference of two spherical waves emerging from each slit. Suppose the wave packet reaches the photographic plate after a time T. Then the spots show the $|\psi(T)|^2$ distribution,[4] in the sense that this is their empirical distribution. Analyzing this using Bohmian mechanics, i.e., analyzing Schrödinger's equation and the guiding equation (1.4), one immediately understands why the experiment produces the result it does. It is clear that in each run the particle goes either through the upper or through the lower slit. The wave function goes through both slits and forms after the slits a wave function with an

[4] In fact, it is the quantum flux across the surface of the photographic plate, integrated over time (see Chap. 16).

interference pattern. Finally the repetition of the experiment produces an ensemble which checks Born's statistical law for that wave function. That is the straightforward physical explanation.

So where is the argument which reveals a conflict with the notion of particle trajectories? Here it is:

$$\text{Close slit 1 and open slit 2.} \tag{1.5a}$$

$$\text{The particle goes through slit 2.} \tag{1.5b}$$

$$\text{It arrives at } \mathbf{x} \text{ on the plate with probability } |\psi_2(\mathbf{x})|^2, \tag{1.5c}$$

where ψ_2 is the wave function which passed through slit 2. Next

$$\text{close slit 2 and open slit 1.} \tag{1.6a}$$

$$\text{The particle goes through slit 1.} \tag{1.6b}$$

$$\text{It arrives at } \mathbf{x} \text{ on the plate with probability } |\psi_1(\mathbf{x})|^2, \tag{1.6c}$$

where ψ_1 is the wave function which passed through slit 1. Now open both slits.

$$\text{Both slits are open.} \tag{1.7a}$$

$$\text{The particle goes through slit 1 or slit 2.} \tag{1.7b}$$

$$\text{It arrives at } \mathbf{x} \text{ with probability } |\psi_1(\mathbf{x}) + \psi_2(\mathbf{x})|^2\ . \tag{1.7c}$$

Now observe that in general

$$|\psi_1(\mathbf{x}) + \psi_2(\mathbf{x})|^2 = |\psi_1(\mathbf{x})|^2 + |\psi_2(\mathbf{x})|^2 + 2\Re\psi_1^*(\mathbf{x})\psi_2(\mathbf{x}) \neq |\psi_1(\mathbf{x})|^2 + |\psi_2(\mathbf{x})|^2\ .$$

The $\neq$ comes from interference of the wave packets ψ_1, ψ_2 which passed through slit 1 and slit 2. The argument now proceeds in the following way. Situations (1.5b) and (1.6b) are the exclusive alternatives entering (1.7b), so the probabilities (1.5c) and (1.6c) must add up. But they do not. So is logic false? Is the particle idea nonsense? No, the argument is a red herring, since (1.5a), (1.6a), and (1.7a) are *physically* distinct.

1.2 Determinism and Realism

It is often said that the aim of Bohmian mechanics is to restore determinism in the quantum world. That is false. Determinism has nothing to do with ontology. What is "out there" could just as well be governed by stochastic laws, as is the case in GRW or dynamical reduction models with, e.g., flash ontology [12, 13]. A realistic quantum theory is a quantum theory which spells out what it is about. Bohmian mechanics is a realistic quantum theory. It happens to be deterministic, which is fine, but not an ontological necessity. The merit of Bohmian mechanics is not determinism, but the refutation of all claims that quantum mechanics cannot be reconciled with a realistic description of reality. In physics, one needs to know what is going on. Bohmian mechanics tells us what is going on and it does so in the most straightforward way imaginable. It is therefore the fundamental description of Galilean physics.

The following passage taken from a letter from Pauli to Born, concerning Einstein's view on determinism, is in many ways reminiscent of the present situation in Bohmian mechanics:

> Einstein gave me your manuscript to read; he was not at all annoyed with you, but only said that you were a person who will not listen. This agrees with the impression I have formed myself insofar as I was unable to recognise Einstein whenever you talked about him in either your letter or your manuscript. It seemed to me as if you had erected some dummy Einstein for yourself, which you then knocked down with great pomp. In particular, Einstein does not consider the concept of "determinism" to be as fundamental as it is frequently held to be (as he told me emphatically many times), and he denied energetically that he had ever put up a postulate such as (your letter, para. 3): "the sequence of such conditions must also be objective and real, that is, automatic, machine-like, deterministic." In the same way, he disputes that he uses as a criterion for the admissibility of a theory the question: "Is it rigorously deterministic?" Einstein's point of departure is "realistic" rather than "deterministic".

Wolfgang Pauli, in [14], p. 221.

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Chapter 2
Classical Physics

What is classical physics? In fact it has become the name for non-quantum physics. This begs the question: What is quantum physics in contrast to classical physics? One readily finds the statement that in classical physics the world is described by classical notions, like particles moving around in space, while in modern physics, i.e., quantum mechanics, the classical notions are no longer adequate, so there is no longer such a "naive" description of what is going on. In a more sophisticated version, quantum mechanics is physics in which the position and momentum of a particle are operators. But such a statement as it stands is meaningless. One also reads that the difference between quantum physics and classical physics is that the former has a smallest quantum of action, viz., Planck's constant $\hbar$, and that classical physics applies whenever the action is large compared to $\hbar$, and in many circumstances, this is a true statement.

But our own viewpoint is better expressed as follows. Classical physics is the description of the world when the interference effects of the Schrödinger wave, evolving according to Schrödinger's equation, can be neglected. This is the case for a tremendously wide range of scales from microscopic gases to stellar matter. In particular it includes the scale of direct human perception, and this explains why classical physics was found before quantum mechanics. Still, the viewpoint just expressed should seem puzzling. For how can classical motion of particles emerge from a wave equation like the Schrödinger equation? This is something we shall explain. It is easy to understand, once one writes down the equations of motion of Bohmian mechanics. But first let us discuss the theory which governs the behavior of matter across the enormous range of classical physics, namely, Newtonian mechanics. In a letter to Hooke, Newton wrote: "If I have seen further it is by standing on the shoulders of giants."

D. Dürr, S. Teufel, *Bohmian Mechanics*, DOI 10.1007/978-3-540-89344-8_2,

2.1 Newtonian Mechanics

Newtonian mechanics is about point particles. What is a point particle? It is "stuff" or "matter" that occupies a point in space called its position, described mathematically by $\mathbf{q} \in \mathbb{R}^3$. The theory describes the motion of point particles in space. Mathematically, an N-particle system is described by the positions of the N particles:

$$\mathbf{q}_1, \dots, \mathbf{q}_N , \qquad \mathbf{q}_i \in \mathbb{R}^3 ,$$

which change with time, so that one has trajectories $\mathbf{q}_1(t), \dots, \mathbf{q}_N(t)$, where the parameter $t \in \mathbb{R}$ is the time.

Newtonian mechanics is given by equations – the physical law – which govern the trajectories, called the equations of motion. They can be formulated in many different (but more or less equivalent) ways, so that the physical law looks different for each formulation, but the trajectories remain the same. We shall soon look at an example. Which formulation one prefers will be mainly a matter of taste. One may find the arguments leading to a particular formulation more satisfactory or convincing than others.

To formulate the law of Newtonian mechanics one introduces positive parameters, called masses, viz., $m_1, \dots, m_N$, which represent "matter", and the law reads

$$m_i \ddot{\mathbf{q}}_i = \mathbf{F}_i(\mathbf{q}_1, \dots, \mathbf{q}_N) . \tag{2.1}$$

$\mathbf{F}_i$ is called the force. It is in general a function of all particle positions. Put another way, it is a function of the *configuration*, i.e., the family of all coordinates $(\mathbf{q}_1, \dots, \mathbf{q}_N) \in \mathbb{R}^{3N}$. The set of all such N-tuples is called *configuration space*. The quantity $\dot{\mathbf{q}}_i = \mathrm{d}\mathbf{q}_i/\mathrm{d}t = \mathbf{v}_i$ is the velocity of the ith particle, and its derivative $\ddot{\mathbf{q}}_i$ is called the acceleration.

Newtonian mechanics is romantic in a way. One way of talking about it is to say that particles accelerate each other, they interact through forces exerted upon each other, i.e., Newtonian mechanics is a theory of interaction. The fundamental interaction is gravitation or mass attraction given by

$$\mathbf{F}_i(\mathbf{q}_1, \dots, \mathbf{q}_N) = \sum_{j \neq i} G m_i m_j \frac{\mathbf{q}_j - \mathbf{q}_i}{\|\mathbf{q}_j - \mathbf{q}_i\|^3} , \tag{2.2}$$

with G the gravitational constant.

All point particles of the Newtonian universe interact according to (2.2). In effective descriptions of subsystems (when we actually use Newtonian mechanics in everyday life), other forces like harmonic forces of springs can appear on the right-hand side of (2.1). Such general forces need not (and in general will not) arise from gravitation alone. Electromagnetic forces will also play a role, i.e., one can sometimes describe electromagnetic interaction between electrically charged particles by the Coulomb force using Newtonian mechanics. The Coulomb force is similar to (2.2), but may have a different sign, and the masses m_i are replaced by the charges e_i which may be positive or negative.

One may wonder why Newtonian mechanics can be successfully applied to subsystems like the solar system, or even smaller systems like systems on earth. That is, why can one ignore all the rest of the universe? One can give various reasons. For example, distant matter which surrounds the earth in a homogeneous way produces a zero net field. The force (2.2) falls off with large distances and the gravitational constant is very small. In various general situations, and depending on the practical task in hand, such arguments allow a good effective description of the subsystem in which one ignores distant matter, or even not so distant matter.

Remark 2.1. Initial Value Problem

The equation (2.1) is a differential equation and thus poses an initial value problem, i.e., the trajectories $\mathbf{q}_i(t)$, $t \in \mathbb{R}$, which obey (2.1) are only determined once initial data of the form $\mathbf{q}_i(t_0)$, $\dot{\mathbf{q}}_i(t_0)$ are given, where t_0 is some time, called the initial time. This means that the future and past evolution of the trajectories is determined by the "present" state $\mathbf{q}_i(t_0)$, $\dot{\mathbf{q}}_i(t_0)$. Note that the position alone is not sufficient to determine the state of a Newtonian system.

It is well known that differential equations need not have unique and global solutions, i.e., solutions which exist for all times for all initial values. What does exist, however, is – at least in the case of gravitation – a local unique solution for a great many initial conditions, i.e., a solution which exists uniquely for some short period of time, if the initial values are reasonable. So (2.1) and (2.2) have no solution if, for example, two particles occupy the same position. Further, for the solution to exist, it must not happen that two or more particles collide, i.e., that they come together and occupy the same position. It is a famous problem in mathematical physics to establish what is called the existence of dynamics for a gravitating many-particle system, where one hopes to show that solutions fail to exist globally only for exceptional initial values. But what does "exceptional" mean? We shall answer this in a short while. ■

We wish to comment briefly on the manner of speaking about interacting particles, which gives a human touch to Newtonian mechanics. We say that the particles *attract each other*. Taking this notion to heart, one might be inclined to associate with the notion of particle more than just an object which has a position. But that might be misleading, since no matter how one justifies or speaks about Newtonian mechanics, when all is said and done, there remains a physical law about the motion of point particles, and that is a mathematical expression about changes of points in space with time. We shall explore one such prosaic description next.

2.2 Hamiltonian Mechanics

One can formulate the Newtonian law differently. Different formulations are based on different fundamental principles, like for example the principle of least action. But never mind such principles for the moment. We shall simply observe that it is

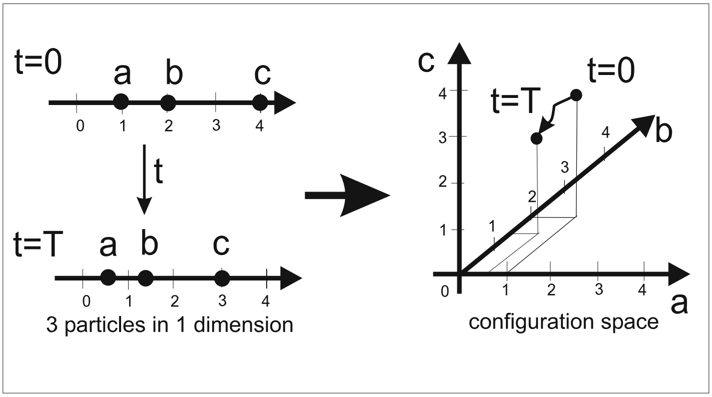

Fig. 2.1 Configuration space for 3 particles in a one-dimensional world

mathematically much nicer to rewrite everything in terms of configuration space variables:

$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_N \end{pmatrix} \in \mathbb{R}^{3N} ,$$

that is, we write the differential equation for all particles in a compact form as

$$m\ddot{\mathbf{q}} = \mathbf{F} , \tag{2.3}$$

with

$$\mathbf{F} = \begin{pmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_N \end{pmatrix} ,$$

and the mass matrix $m = (\delta_i^j m_j)_{i,j=1,\dots,N}$.

Configuration space cannot be depicted (but see Fig. 2.1 for a very special situation), at least not for a system of more than one particle, because it is 6-dimensional for 2 particles in physical space. It is thus not so easy to think intuitively about things going on in configuration space. But one had better build up some intuition for configuration space, because it plays a fundamental role in quantum theory.

A differential equation is *by definition* a relation between the *flow* and the *vector field*. The flow is the mapping along the solution curves, which are integral curves along the vector field (the tangents of the solution curves). If a physical law is given by a differential equation, the vector field encodes the physical law. Let us see how this works.

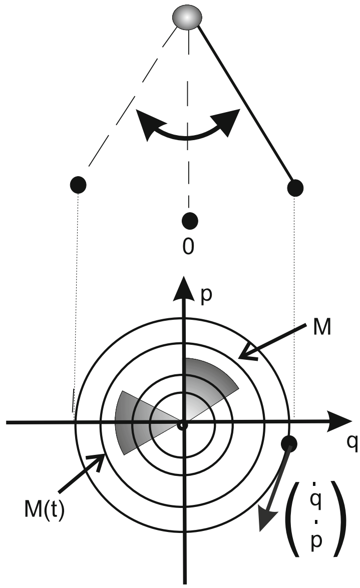

Fig. 2.2 Phase space description of the mathematically idealized harmonically swinging pendulum. The possible trajectories of the mathematically idealized pendulum swinging in a plane with frequency 1 are concentric circles in phase space. The sets M und $M(t)$ will be discussed later

The differential equation (2.3) is of second order and does not express the relation between the integral curves and the vector field in a transparent way. We need to change (2.3) into an equation of first order, so that the vector field becomes transparent. For this reason we consider the *phase space* variables

$$\begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix} = \begin{pmatrix}\mathbf{q}_1\\ \vdots\\ \mathbf{q}_N\\ \mathbf{p}_1\\ \vdots\\ \mathbf{p}_N\end{pmatrix} \in \mathbb{R}^{3N} \times \mathbb{R}^{3N} = \Gamma ,$$

which were introduced by Boltzmann,[1] where we consider positions and velocities. However, for convenience of notation, the latter are replaced by momenta $\mathbf{p}_i = m_i\mathbf{v}_i$. One point in Γ represents the present state of the entire N-particle system. The phase space has twice the dimension of configuration space and can be depicted for one particle moving in one dimension, e.g., the pendulum (see Fig. 2.2).

Clearly, (2.3) becomes

[1] The notion of phase space was taken by Ludwig Boltzmann (1844–1906) as synonymous with the state space, the phase being the collection of variables which uniquely determine the physical state. The physical state is uniquely determined if its future and past evolution in time is uniquely determined by the physical law.

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} m^{-1}\mathbf{p} \\ \mathbf{F}(\mathbf{q}) \end{pmatrix} . \tag{2.4}$$

The state of the N-particle system is completely determined by $\begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}$, because (2.4) and the initial values $\begin{pmatrix} \mathbf{q}(t_0) \\ \mathbf{p}(t_0) \end{pmatrix}$ uniquely determine the phase space trajectory (if the initial value problem allows for a solution).

For (2.2) and many other effective forces, there exists a function V on $\mathbb{R}^{3N}$, the so called potential energy function, with the property that

$$\mathbf{F} = -\mathrm{grad}_q V = -\frac{\partial V}{\partial \mathbf{q}} = -\nabla V .$$

Using this we may write (2.4) as

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial H}{\partial \mathbf{p}}(\mathbf{q},\mathbf{p}) \\ -\dfrac{\partial H}{\partial \mathbf{q}}(\mathbf{q},\mathbf{p}) \end{pmatrix} , \tag{2.5}$$

where

$$\begin{aligned} H(\mathbf{q},\mathbf{p}) &= \frac{1}{2}(\mathbf{p}\cdot m^{-1}\mathbf{p}) + V(\mathbf{q}) \\ &= \frac{1}{2}\sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{m_i} + V(\mathbf{q}_1,\dots,\mathbf{q}_N) . \end{aligned} \tag{2.6}$$

Now we have the Newtonian law in the form of a transparent differential equation (2.5), expressing the relation between the integral curves (on the left-hand side, differentiated to yield tangent vectors) and the vector field on the right-hand side (which are the tangent vectors expressing the physics). The way we have written it, the vector field is actually generated by a function H (2.6) on phase space. This is called the Hamilton function, after its inventor William Rowan Hamilton (1805–1865), who in fact introduced the symbol H in honor of the physicist Christiaan Huygens (1629–1695). We shall see later what the "wave man" Huygens has to do with all this. The role of the Hamilton function $H(\mathbf{q},\mathbf{p})$ is to give the vector field

$$\mathbf{v}^H(\mathbf{q},\mathbf{p}) = \begin{pmatrix} \dfrac{\partial H}{\partial \mathbf{p}} \\ -\dfrac{\partial H}{\partial \mathbf{q}} \end{pmatrix} , \tag{2.7}$$

and the Hamiltonian dynamics is simply given by

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \mathbf{v}^H(\mathbf{q},\mathbf{p}) . \tag{2.8}$$

The function H allows us to focus on a particular structure of Newtonian mechanics, now rewritten in Hamiltonian terms. Almost all of this section depends solely on this structure, and we shall see some examples shortly. Equations (2.5) and (2.6) with the Hamilton function $H(\mathbf{q},\mathbf{p})$ define a Hamiltonian system.

The integral curves along this vector field (2.7) represent the possible system trajectories in phase space, i.e., they are solutions $\begin{pmatrix} \mathbf{q}\big(t,(\mathbf{q},\mathbf{p})\big) \\ \mathbf{p}\big(t,(\mathbf{q},\mathbf{p})\big) \end{pmatrix}$ of (2.8) for given initial values $\begin{pmatrix} \mathbf{q}\big(0,(\mathbf{q},\mathbf{p})\big) \\ \mathbf{p}\big(0,(\mathbf{q},\mathbf{p})\big) \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}$. Note that this requires existence and uniqueness of solutions of the differential equations (2.8). One possible evolution of the entire system is represented by one curve in phase space (see Fig. 2.3), which is called a flow line, and one defines the Hamiltonian flow by the map $(\Phi_t^H)_{t\in\mathbb{R}}$ from phase space to phase space, given by the prescription that, for any t, a point in phase space is mapped to the point to which it moves in time t under the evolution (as long as that evolution is defined, see Remark 2.1):

$$\Phi_t^H\left(\begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}\right) = \begin{pmatrix} \mathbf{q}\big(t,(\mathbf{q},\mathbf{p})\big) \\ \mathbf{p}\big(t,(\mathbf{q},\mathbf{p})\big) \end{pmatrix} .$$

We shall say more about the flow map later on. The flow can be thought of pictorially as the flow of a material fluid in Γ, with the system trajectories as flow lines.

Hamiltonian mechanics is another way of talking about Newtonian mechanics. It is a prosaic way of talking about the motion of particles. The only romance left is the secret of how to write down the physically relevant H. Once that is done, the romance is over and what lies before one are the laws of mechanics written in mathematical language. So that is all that remains. The advantage of the Hamiltonian form is that it directly expresses the law as a differential equation (2.8). And it has the further advantage that it allows one to talk simultaneously about all possible trajectories of a system. This will be helpful when we need to define a *typical* trajectory of the system, which we must do later.

However, this does not by any means imply that we should forget the Newtonian approach altogether. To understand which path a system takes, it is good to know how the particles in the system interact with each other, and to have some intuition about that. Moreover, we should not lose sight of what we are interested in, namely, the behavior of the system in physical space. Although we have not elaborated on the issue at all, it is also important to understand the physical reasoning which leads to the mathematical law (for example, how Newton found the gravitational potential), as this may give us confidence in the correctness of the law. Of course we also achieve confidence by checking whether the theory correctly describes what we see, but since we can usually only see a tiny fraction of what a theory says, confidence is mainly grounded on theoretical insight.

The fundamental properties of the Hamiltonian flow are conservation of energy and conservation of volume. These properties depend only on the form of the equa-

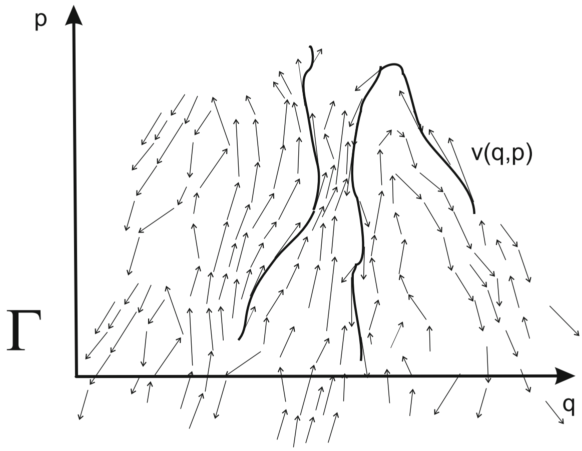

Fig. 2.3 The Hamilton function generates a vector field on the $6N$-dimensional phase space of an N-particle system in physical space. The integral curves are the possible trajectories of the entire system in phase space. Each point in phase space is the collection of all the positions and velocities of all the particles. One must always keep in mind that the trajectories in phase space are not trajectories in physical space. They can never cross each other because they are integral curves on a vector field, and a unique vector is attached to every point of phase space. Trajectories in phase space do not interact with each other! They are not the trajectories of particles

tions (2.8) with (2.7), i.e., $H(\mathbf{q},\mathbf{p})$ can be a completely general function of $(\mathbf{q},\mathbf{p})$ and need not be the function (2.6). When working with this generality, one calls $\mathbf{p}$ the canonical momentum, which is no longer simply velocity times mass. Now, conservation of energy means that the value of the Hamilton function does not change along trajectories. This is easy to see. Let $\big(\mathbf{q}(t),\mathbf{p}(t)\big)$, $t\in\mathbb{R}$, be a solution of (2.8). Then

$$\frac{\mathrm{d}}{\mathrm{d}t}H\big(\mathbf{q}(t),\mathbf{p}(t)\big)=\dot{\mathbf{q}}\frac{\partial H}{\partial \mathbf{q}}+\dot{\mathbf{p}}\frac{\partial H}{\partial \mathbf{p}}=\frac{\partial H}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{q}}-\frac{\partial H}{\partial \mathbf{q}}\frac{\partial H}{\partial \mathbf{p}}=0\,. \tag{2.9}$$

More generally, the time derivative along the trajectories of any function $f\big(\mathbf{q}(t),\mathbf{p}(t)\big)$ on phase space is

$$\frac{\mathrm{d}}{\mathrm{d}t}f\big(\mathbf{q}(t),\mathbf{p}(t)\big)=\dot{\mathbf{q}}\frac{\partial f}{\partial \mathbf{q}}+\dot{\mathbf{p}}\frac{\partial f}{\partial \mathbf{p}}=\frac{\partial H}{\partial \mathbf{p}}\frac{\partial f}{\partial \mathbf{q}}-\frac{\partial H}{\partial \mathbf{q}}\frac{\partial f}{\partial \mathbf{p}}=:\{f,H\}\,. \tag{2.10}$$

The term $\{f,H\}$ is called the Poisson bracket of f and H. It can also be defined in more general terms for any pair of functions f,g, viewing g as the Hamilton function and Φ_t^g the flow generated by g:

$$\{f,g\}=\frac{\mathrm{d}}{\mathrm{d}t}f\circ\Phi_t^g=\frac{\partial g}{\partial \mathbf{p}}\frac{\partial f}{\partial \mathbf{q}}-\frac{\partial g}{\partial \mathbf{q}}\frac{\partial f}{\partial \mathbf{p}}\,. \tag{2.11}$$

Note, that $\{f,H\}=0$ means that f is a *constant of the motion*, i.e., the value of f remains unchanged along a trajectory ($\mathrm{d}f/\mathrm{d}t=0$), with $f=H$ being the simplest example.

Now we come to the conservation of volume. Recall that the Hamiltonian flow $(\Phi_t^H)_{t\in\mathbb{R}}$ is best pictured as a fluid flow in Γ, with the system trajectories as flow lines. These are the integral curves along the Hamiltonian vector field $\mathbf{v}^H(\mathbf{q},\mathbf{p})$ (2.7). These flow lines have neither sources nor sinks, i.e., the vector field is divergence-free:

$$\operatorname{div}\mathbf{v}^H(\mathbf{q},\mathbf{p}) = \left(\frac{\partial}{\partial\mathbf{q}},\frac{\partial}{\partial\mathbf{p}}\right)\begin{pmatrix}\dfrac{\partial H}{\partial\mathbf{p}}\\ -\dfrac{\partial H}{\partial\mathbf{q}}\end{pmatrix} = \frac{\partial^2 H}{\partial\mathbf{q}\partial\mathbf{p}} - \frac{\partial^2 H}{\partial\mathbf{p}\partial\mathbf{q}} = 0\,. \tag{2.12}$$

This important (though rather trivial) mathematical fact is known as Liouville's theorem for the Hamiltonian flow, after Joseph Liouville (1809–1882). (It has nothing to do with Liouville's theorem in complex analysis.) A fluid with a flow that is divergence-free is said to be incompressible, a behavior different from air in a pump, which gets very much compressed. Consequently, and as we shall show below, the "volume" of any subset in phase space which gets transported via the Hamiltonian flow remains unchanged. Before we express this in mathematical terms and give the proof, we shall consider the issue in more general terms.

Remark 2.2. On the Time Evolution of Measures.
The notion of volume deserves some elaboration. Clearly, since phase space is very high-dimensional, the notion of volume here is more abstract than the volume of a three-dimensional object. In fact, we shall later use a notion of volume which is not simply the trivial extension of three-dimensional volume. Volume here refers to a *measure*, the size or weight of sets, where one may in general want to consider a biased weight. The most famous measure, and in fact the mother of all measures, is the generalization of the volume of a cube to arbitrary subsets, known as the Lebesgue measure λ. We shall say more about this later.[2] If one feels intimidated by the name Lebesgue measure, then take $|A|=\int_A \mathrm{d}^n x$, the usual Riemann integral, as the (fapp-correct) Lebesgue measure of A. The measure may in a more general sense be thought of as some kind of weight distribution, where the Lebesgue measure gives equal (i.e., unbiased) weight to every point. For a continuum of points, this is a somewhat demanding notion, but one may nevertheless get a feeling for what is meant. For the time being we require that the measure be an additive nonnegative set function, i.e., a function which attributes positive or zero values to sets, and which is additive on disjoint sets: $\mu(A\cup B)=\mu(A)+\mu(B)$. The role of the measure will

[2] We need to deal with the curse of the continuum, which is that not all subsets of $\mathbb{R}^n$ actually have a volume, or as we now say, a measure. There are non-measurable sets within the enormous multitude of subsets. These non-measurable sets exist mathematically, but are not constructible in any practical way out of unions and intersections of simple sets, like cubes or balls. They are nothing we need to worry about in practical terms, but they are nevertheless there, and so must be dealt with properly. This we shall do in Sect. 4.3.1.