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Andrey Polyakov

Generalized Homogeneity in Systems and Control

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To my family

Preface

Homogeneity is a certain invariance of an object (a function, a set, etc) with respect to a class of transformations called dilations. All linear and a lot of essentially nonlinear models of mathematical physics are homogeneous (symmetric) in some sense. Homogeneous models can be utilized as local approximations of dynamical systems if, for example, linearizion is too conservative, non-informative, or simply impossible.

Homogeneous control laws appear as solutions to many control problems such as a minimum time feedback control for the chain of integrators or the high-order sliding mode design. The homogeneity allows some time constraints in control systems to be fulfilled by means of a proper selection of the so-called homogeneity degree. Similar to the linear case, an asymptotic stability of a homogeneous system implies its robustness (input-to-state stability) with respect to a certain class of parametric uncertainties and exogenous perturbations.

This monograph studies both finite-dimensional and infinite-dimensional models of control systems. Part I of the book surveys some mathematical tools required for an analysis of dynamical models while the second one is devoted to the analysis and design of homogeneous systems. Elements of set, measure and operator theories as well as some classical results of functional analysis are presented in Appendix in order to make the monograph self-contained.

An introduction to homogeneous systems is presented in Chap. 1, where some important features of homogeneous control and estimation algorithms are discussed. Chapter 2 considers the finite-dimensional models of control systems. In particular, ordinary differential equations with discontinuous right-hand sides and differential inclusions are studied. Elements of the theory of evolution equations in Banach and Hilbert spaces are presented in Chap. 3. Stability notions and the common tool of stability analysis (the Lyapunov function method) are surveyed in Chaps. 4 and 5, where the concepts of *finite-time and fixed-time stability* (typical for many homogeneous systems) are studied with details.

Part II of the book deals with the analysis and design of homogeneous control systems. Linear dilations (Chap. 6) and homogeneous mappings (Chap. 7) are introduced in finite-dimensional and infinite-dimensional spaces. Some features of stability and robustness analysis of homogeneous dynamical systems are discussed (Chap. 8). Homogeneous control algorithms are designed in Chap. 9. A technique for a simple upgrade of an existing linear control to a nonlinear homogeneous one is presented and demonstrated in real experiments. Some issues of a digital implementation of the homogeneous control systems are studied in Chap. 10. The homogeneity-based state estimation algorithms are developed in Chap. 11. Finally, homogeneous optimal control problems are considered in Chap. 12.

I acknowledge my colleagues and friends for their strong support, for fruitful ideas, suggestions, criticism, and always interesting discussions. I am also very grateful to my family for their patience and personal sacrifices which they have given to my work.

Lille, France Andrey Polyakov
November 2019

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Part I Models of Control Systems and Stability Analysis

Part II Homogeneous Control Systems

Notation

- $\mathbb{N}$ is the set of natural numbers; $\mathbb{Z}$ is the set of integers; $\mathbb{R}$ is the set of reals; $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ and $\mathbb{R}_+ = [0, +\infty)$; $\mathbb{C}$ is the set of complex numbers.
- $\mathcal{I}$ denotes one of the following intervals: $[a, b]$, (a, b), $[a, b)$, or $(a, b]$, where $a < b$, $a, b \in \mathbb{R}$, or $a = -\infty$, and/or $b \in +\infty$.
- $A \times B$ denotes the Cartesian product of sets A and B.
- The inner product of vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ from an n-dimensional Euclidean space is given by $x \cdot y = \sum_{i=1}^{n} x_i y_i$, where x_i and y_i are coordinates of the vectors x and y in an orthonormal basis.
- $\mathbb{B}$ is a real Banach space with a norm $\|\cdot\|$ and $\mathbb{H}$ is a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$.
- $\text{span}\{e_1, e_2, \ldots, e_k\} := \{\alpha_1 e_1 + \ldots + \alpha_k e_k : \alpha_i \in \mathbb{R}, \; i = 1, 2, \ldots, k\}$, where $e_i \in \mathbb{B}$.
- $S = \{x \in \mathbb{B} : \|x\| = 1\}$ is the unit sphere in $\mathbb{B}$.
- The notation $\|\cdot\|_X$ is utilized if it is necessary to indicate that this is a norm in a space X.
- $\mathcal{L}(X, Y)$ is the space of linear bounded operators $X \to Y$, where X and Y are Banach spaces, and
  $$\|A\|_{\mathcal{L}(X,Y)} = \sup_{u \neq 0} \frac{\|Au\|_Y}{\|u\|_X} \quad \text{and} \quad \lfloor A \rfloor_{\mathcal{L}(X,Y)} = \inf_{u \neq 0} \frac{\|Au\|_Y}{\|u\|_X}, \quad A \in \mathcal{L}(X, Y).$$
  We also use the notations $\|A\|$ and $\lfloor A \rfloor$ for shortness if a context is clear.
- $f_1(f_2)$ and $f_1 \circ f_2$ denote the composition of nonlinear operators (functions) f_1 and f_2. In the case of linear operators A and B, for simplicity, the brackets and the sign "$\circ$" can be omitted, i.e. AB denote the composition of linear operators A and B.
- If $P = P^\top \in \mathbb{R}^{n \times n}$ then $P \succ 0$ (resp. $\succeq 0$) means that the matrix P is positive definite (resp. semidefinite) and $P \prec 0$ (resp. $\preceq 0$) means that the matrix P is negative definite (resp. semidefinite).
- $\lambda_{min}(P)$ and $\lambda_{max}(P)$ denotes minimum and maximum eigenvalues of a symmetric matrix $P = P^\top \in \mathbb{R}^n$.

- $\operatorname{rank}(A)$ denotes the rank of $A \in \mathbb{R}^{m \times n}$.
- $\operatorname{tr}(A)$ denotes the trace of $A \in \mathbb{R}^{n \times n}$.
- $I \in \mathcal{L}(\mathbb{B}, \mathbb{B})$ denotes the identity operator in $\mathbb{B}$ and I_n is the identity matrix in $\mathbb{R}^{n \times n}$.
- $\operatorname{div}(u) = \sum_{i=1}^{n} \frac{\partial u_i}{\partial z_i}$ for a function $u : \mathbb{R}^n \to \mathbb{R}^n$
- $\nabla := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, and $\Delta := \nabla \cdot \nabla = \operatorname{div}(\nabla) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$—Laplace operator.
- $\overline{X}$ denotes the closure of the set X of a metric space.
- $C(X, Y)$ is the space of uniformly continuous functions $X \to Y$ with the supremum norm, $\|f\| = sup_{x \in X} \|f(x)\|_Y$, where $f \in C(X, Y)$ and X, Y are normed vector spaces.
- $C_c^\infty(\Omega, \mathbb{R}^m)$ is the space of infinitely differentiable (smooth) functions having compact support in Ω, where $\Omega \subset \mathbb{R}^m$ is an open set.
- $C_0^\infty(\Omega, \mathbb{R}^m)$ is the space of infinitely differentiable (smooth) functions vanishing on the boundary of Ω.
- Let $L^{p,\mu}(\Omega, \mathbb{R}^m)$ denotes the Banach space of functions $\Omega \to \mathbb{R}^m$

$$L^{p,\mu}(\Omega, \mathbb{R}^m) := \left\{u : \|u\|_{p,\mu} < +\infty\right\}, \quad \mu \in \mathbb{R},$$

$$\|u\|_{p,\mu} := \left(\int_\Omega |x|^{\mu p} |u(x)|^p \, dx\right)^{1/p}, \quad 0 < p < \infty,$$

$$\|u\|_{\infty,\mu} := \operatorname{ess\,sup}(|x|^\mu u(x)), \quad p = \infty.$$

  We refer the reader to Appendix for more details about function spaces.
- The Lebesgue space L^p is given by

$$L^p := L^{p,0}$$

  with the norm

$$\|u\|_p := \|u\|_{p,0}.$$

- $L^2(\Omega, \mathbb{R}^m)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_2 = \int_\Omega u(x) \cdot v(x) \, dx.$$

- The Sobolev space $H^p(\Omega, \mathbb{R}^m) := H^{p,0}(\Omega, \mathbb{R}^m)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{H^p} := \sum_{i=0}^{p} \langle \nabla^i u, \nabla^i v\rangle_2$$

and the norm $\|\cdot\|_{H^p}=\sqrt{\langle\cdot,\cdot\rangle_{H^p}}$, which is equivalent to the norm $\|\cdot\|_{H^{p,0}}$.

- $H_0^p(\Omega,\ \mathbb{R}^m)$ is the completion of $C_c^\infty(\Omega,\ \mathbb{R}^m)$ with respect to $\|\cdot\|_{H^p}$.
- If $\Omega_1 \subset \mathbb{B}$ and $\Omega_2 \subset \mathbb{B}$ then, by definition, the identity $\Omega_1 = \Omega_2$ means $\Omega_1 \subset \Omega_2$ and $\Omega_2 \subset \Omega_1$.
- The geometric sum of two sets is denoted by "$\dot{+}$" and given by

$$\mathrm{M}_1 \dot{+} \mathrm{M}_2 = \bigcup_{x_1\in \mathrm{M}_1,\ x_2\in \mathrm{M}_2} x_1 + x_2, \tag{1}$$

where $\mathrm{M}_1 \subset \mathbb{B},\ \mathrm{M}_2 \subset \mathbb{B}$.

- The product of a scalar $\alpha \in \mathbb{R}$ and a set $\mathrm{M} \subset \mathbb{B}$ is defined as follows

$$\alpha\mathrm{M} = \mathrm{M}\alpha = \bigcup_{x\in \mathrm{M}} \alpha x. \tag{2}$$

- The product of a set $\mathrm{N} \subset \mathbb{R}$ and a set $\mathrm{M} \subset \mathbb{B}$ is defined as follows

$$\mathrm{N}\cdot\mathrm{M} = \bigcup_{\alpha\in N} \alpha\mathrm{M}. \tag{3}$$

- An application of an operator $f\ :\ \mathbb{B} \to \mathbb{X}$ to a set $\mathrm{M} \subset \mathbb{B}$ is given by

$$f(\mathrm{M}) := \bigcup_{x\in \mathrm{M}} f(x). \tag{4}$$

- $B(r) := \{x \in \mathbb{B} : \|x\| < r\}$ is the open ball in $\mathbb{B}$ of the radius $r \in \mathbb{R}_+$ with the center at the origin. Under introduced notations, $B(y,r) = y \dot{+} B(\varepsilon)$ is an open ball of the radius $\varepsilon > 0$ centered at $y \in \mathbb{B}$.
- $\partial\Omega$ is the boundary of a set $\Omega \subset \mathbb{R}^n$.
- $\mathrm{int}(\Omega)$ denotes the interior of a set $\Omega \subset \mathbb{R}^n$, i.e. $x \in \mathrm{int}(\Omega)$ if and only if $\exists r \in \mathbb{R}_+ : x \dot{+} \mathbb{B}(r) \subset \Omega$.
- The set consisting of elements $x_1, x_2, \ldots, x_n$ is denoted by $\{x_1, x_2, \ldots, x_n\}$.
- The power set (i.e. the set of all subsets) of a set $M \subset \mathbb{R}^n$ is 2^M.
- $\mathrm{span}\{e_1, \ldots e_n\}$ denotes a linear hull, i.e.

$$\mathrm{span}\{e_1, \ldots e_n\} = \left\{ y : y = \sum_{i=1}^{n} \alpha_i e_i,\ \forall\alpha = (\alpha_1, \ldots, \alpha_n)^{\mathrm{T}} \in \mathbb{R}^n \right\}.$$

- The convex hull of $A \subset \mathbb{B}$, denoted by $co(A)$, is the smallest convex set containing A. The closed convex hull of A, denoted by $\overline{co}(A)$, is the smallest closed convex set containing A.
- The sign function is defined by

$$\mathrm{sign}_{\sigma}(\rho) := \begin{cases} 1 & \text{if} \quad \rho > 0, \\ -1 & \text{if} \quad \rho < 0, \\ \sigma & \text{if} \quad \rho = 0, \end{cases} \tag{5}$$

where $\sigma \in \mathbb{R} : -1 \leq \sigma \leq 1$. If a concrete value of σ is not important for considerations, we use the notation sign (ρ).

- The set-valued extension of the sign function is given by

$$\overline{\mathrm{sign}}(\rho) := \begin{cases} \{1\} & \text{if} \quad \rho > 0, \\ \{-1\} & \text{if} \quad \rho < 0, \\ [-1,\ 1] & \text{if} \quad \rho = 0. \end{cases} \tag{6}$$

- $x^{\lfloor \alpha \rceil} := |x|^{\alpha}\mathrm{sign}[x]$ is a power operation, which preserves the sign of the number $x \in \mathbb{R}$. For example, $(-2)^{\lfloor 2 \rceil} = -4$ and $2^{\lfloor 2 \rceil} = 4$.
- The inequalities $y < 0$, $y \leq 0$, $y > 0$ and $y \geq 0$ for $y \in \mathbb{R}^n$ are understood in the component-wise sense.
- A function $\sigma \in C([0, +\infty), [0, +\infty))$ belongs to the class $\mathcal{K}$ if $\sigma(0) = 0$ and σ is increasing, i.e. $t_1 \leq t_2 \Rightarrow \sigma(t_1) \leq \sigma(t_2)$.
- A function $\sigma \in \mathcal{K}$ belongs to the class $\mathcal{K}^{\infty}$ if $\sigma(t) \to +\infty$ as $t \to +\infty$.
- A continuous function $\xi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\xi(\cdot,\ t) \in \mathcal{K}$ for any fixed $t \geq 0$ and $\xi(s,\ \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing to zero for any fixed $s \geq 0$.
- $\mathcal{H}_{\mathbf{d}}(\mathbb{B})$ is a set of **d**-homogeneous functionals $\mathbb{B} \to \mathbb{R}$ and $deg_{\mathbf{d}}(h) \in \mathbb{R}$ is a homogeneity degree of $h \in \mathcal{H}_{\mathbf{d}}(\mathbb{B})$.
- $\mathcal{F}_{\mathbf{d}}(\mathbb{B})$ is a set of **d**-homogeneous functionals $\mathbb{B} \to \mathbb{R}$ and $deg_{\mathbf{d}}(f) \in \mathbb{R}$ is a homogeneity degree of $f \in \mathcal{F}_{\mathbf{d}}(\mathbb{B})$.

For more details about the given notations, we refer the reader to Appendix.

Chapter 1
Introduction

This chapter introduces some basics of the theory of homogeneous dynamical systems. A symmetry of trajectories and convergence rates of homogeneous differential equations are studied. An evolution of the homogeneity theory is briefly surveyed. Possible advantages of the homogeneity-based approach to control systems design are discussed.

1.1 Homogeneity Versus Linearity

1.1.1 Dilation Symmetry

Symmetry is a type of invariance when some characteristics of an object do not change under a certain set of transformations. It occurs in many branches of mathematics. The simplest example of a symmetry can be found in the geometry as an invariance of geometric figures with respect to rotations, translations, or dilations. It is well known that both the size and the shape of the figure are invariant with respect to rotations and translations while the dilation does not change only the shape (see Fig. 1.1).

The symmetry of shapes with respect to dilations can be discovered for level sets of the so-called homogeneous functions. The symmetry of a function f with respect to the *uniform dilation* of its argument

$$x \to \lambda x,$$

where $\lambda > 0$ is the scaling factor, is known as *homogeneity*:

$$f(\lambda x) = \lambda f(x), \quad \forall \lambda > 0, \quad \forall x.$$

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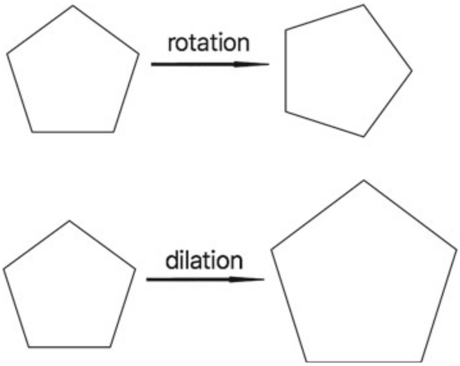

Fig. 1.1 Dilation and rotation symmetries

In other words, **homogeneity is a dilation symmetry**. All linear functions are homogeneous with respect to the uniform dilation. In this chapter, it is shown that homogeneous nonlinear mappings are rather similar to linear ones. However, they have their own specific features which could be useful for an advanced control system design.

In the eighteenth century, a homogeneity with respect to the uniform dilation (known also as the *standard homogeneity*) was studied by Leonhard Euler. His notion of homogeneity is well known today in the context of the so-called homogeneous polynomials.

Definition 1.1 A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be standard homogeneous if there exists a number $\nu \in \mathbb{R}$ such that

$$f(\lambda x) = \lambda^{\nu} f(x), \quad \forall \lambda > 0, \quad \forall x \in \mathbb{R}^n.$$

The number ν is called the *homogeneity degree* of the function f.

According to this definition, any linear function has the homogeneity degree 1, but the quadratic one

$$x = (x_1, x_2)^{\top} \xrightarrow{f} x_1^2 + x_1 x_2 + x_2^2$$

is homogeneous of the degree 2. The level sets of f are ellipsoids centered at the origin. They are symmetric with respect to the uniform dilation.

Euler's homogeneous function theorem given below is one of the famous results underlying the modern theory of homogeneous systems.

Theorem 1.1 *A continuously differentiable function* $f : \mathbb{R}^n \to \mathbb{R}$ *is homogeneous of a degree* k *if and only if* $\frac{\partial f}{\partial x} x = k f(x)$, $x \in \mathbb{R}^n$.

From this theorem, in particular, we conclude that the homogeneity of a function is inherited by its derivatives. In fact, the dilation symmetry can be inherited by other mathematical objects induced by homogeneous functions. For example, solutions of homogeneous differential equations and inclusions are also symmetric (homogeneous) in a certain sense.

1.1.2 Homogeneous Differential Equations

Let us consider the simplest scalar ordinary differential equation (ODE) with the standard homogeneous right-hand side

$$\dot{x} = -x^{\nu}, \quad t > 0, \quad \nu = p/q,$$

where p is an odd integer and q is an even natural number. Its solution with the initial condition $x(0) = x_0 \in \mathbb{R}$ is given by

$$x(t, x_0) = \tfrac{x_0}{\left(1+(\nu-1)t|x_0|^{\nu-1}\right)^{1/(\nu-1)}}.$$

Hence, we easily derive the symmetry of solutions with respect to the simultaneous dilation of the initial condition and the time variable t:

$$x\left(\lambda^{1-\nu} t, \lambda x_0\right) = \lambda x\left(t, x_0\right), \quad \lambda > 0.$$

Notice that the time scaling factor depends on the homogeneity degree ν. In fact, it is easy to check that the mentioned symmetry of solutions can be established for any standard homogeneous differential equation. In Chap. 8 this result is proven for a more general class of systems, which are homogeneous in a generalized sense.

Theorem 1.2 *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous standard homogeneous vector field of a degree* $\nu \in \mathbb{R}$ *such that the Cauchy problem*

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n$$

admits a solution $x(t, x_0)$ *defined for all* $t > 0$. *Then*

$$x(\lambda^{1-\nu}t, \lambda x_0) = \lambda x(t, x_0), \quad \lambda > 0,$$

where $x(\cdot, \lambda x_0)$ *is a solution to the same problem with the scaled initial condition* $x(0) = \lambda x_0$.

Homogeneity simplifies analysis and design of nonlinear control systems since the homogeneous vector fields have many properties similar to linear ones. For example, $x = 0$ is always an equilibrium of a continuous homogeneous system. If there exists another equilibrium $x^* \neq 0$, then the set $M = \{x \in \mathbb{R}^n : x = \lambda x^*, \lambda > 0\}$ is, at least, weakly invariant and $f(x) = 0$ for all $x \in M$. In the case of linear systems, the set M is a linear subspace.

From the latter theorem, we immediately conclude that any local property (e.g. local stability and the existence of solutions for small initial data) can always be expanded globally. Similar to linear systems, the robustness (input-to-state stability) of a homogeneous system can be granted by its stability in the disturbance-free case (see Chap. 8 for more details). Table 1.1 compares some properties of linear and nonlinear homogeneous control systems. All mentioned (and many other) properties are studied in this monograph for homogeneous systems in both finite-dimensional and infinite-dimensional spaces.

Homogeneity degree allows some additional qualitative analysis of ODEs to be done easily. For instance, *finite-time* or practical *fixed-time stability* can be derived from the homogeneity degree. Indeed, if $\nu = 1/3$ then the solution of the scalar homogeneous system $\dot{x} = -x^{\nu}$ is given by

Table 1.1 Some properties of linear and homogeneous systems

	Linear system $\dot{x} = Ax$ $A \in \mathbb{R}^{n\times n}$	Homogeneous system $\dot{x} = f(x)$ $f(\lambda x) = \lambda^{\nu} f(x)$
Trajectory scaling	$x(t, \lambda x_0) = \lambda x(t, x_0)$	$x(t, \lambda x_0) = \lambda x(\lambda^{\nu-1}t, x_0)$
Stability **Local ⇔ Global**	✓	✓
Lyapunov function	Quadratic $V = x^{\top}Px,$ $P \succ 0$	Homogeneous (quadratic-like) $V = x^{\top}\Xi^{\top}(x)P\,\Xi(x)x,$ $P \succ 0,\ \Xi(\lambda x) = \Xi(x) \in \mathbb{R}^{n\times n}$
Stability ⇒ Robustness (Input-to-state stability)	$\dot{x} = Ax + Dw$ $w \in L^{\infty}$	$\dot{x} = f(x, w),\ w \in L^{\infty},$ $\tilde{f} = \binom{f}{0}$—homogeneous
Convergence rate	Exponential	Finite-time ($\nu < 1$) Exponential ($\nu = 1$) Fixed-time ($\nu > 1$)
Consistent discretization preserves the convergence rate	✓	✓

$$x(t, x_0) = \begin{cases} \left(|x_0|^{\frac{2}{3}} - \frac{2}{3}t\right)^{\frac{3}{2}} & \text{if } t \in \left[0, \frac{3}{2}|y_0|^{\frac{2}{3}}\right), \\ 0 & \text{if } t \geq \frac{3}{2}|x_0|^{\frac{2}{3}}. \end{cases}$$

Obviously, it converges to zero in a finite time $T(x_0) = \frac{3}{2}|x_0|^{2/3}$. However, if $\nu > 1$ then each trajectory of the system converges into any neighborhood of the origin in a fixed time independent of the initial condition, namely,

$$|x(t, x_0)| < r, \quad \forall t > \frac{1}{r^{\nu-1}(\nu - 1)}$$

for any $x_0 \in \mathbb{R}$ and any $r > 0$. In fact, these properties can be established in a more general case. Namely, if the origin of a homogeneous system is asymptotically stable then each trajectory of the system

(a) vanishes in a finite time provided that $\nu < 1$;
(b) converges to zero exponentially if $\nu = 1$;
(c) converges to a neighborhood of the origin in a fixed time independent of the initial condition.

1.1.3 Homogeneous Approximations

An approximation by a homogeneous function (*homogeneous approximation*) can be utilized for a local analysis of a dynamical system if linearization is non-informative or simply impossible. For example, the system $\dot{x} = -x^3 + x^5$ is locally uniformly asymptotically stable. This system does not admit the local asymptotic stability analysis by the first-order approximation since the linearization at the origin is $\dot{x} = 0$. However, the homogeneous approximation $\dot{x} = -x^3$ allows us to make the correct conclusion about the local asymptotic stability of the original system.

As an example of a locally homogeneous model, which does not admit a linearization, let us consider a mechanical system consisting of a rigid body moving laterally on a contact surface and in a viscous environment (fluid). The simplest real-life example of such a mechanical system is a car moving on a flat road.

Let $z(t)$ be the position of the center of mass of the body in an inertial frame at time $t \in \mathbb{R}$. The equation describing a motion of this system has the form

$$\dot{z}(t) = v(t), \quad m\,\dot{v}(t) = F(t), \quad t > 0, \quad z(t) \in \mathbb{R},$$

where $v(t)$ is the velocity, m is the mass of the body, and F is the sum of external forces.

Let us consider only the deceleration phase of the motion assuming that at the initial instant of time this mechanical system has some nonzero velocity $\dot{z}(0) = v(0) \neq 0$. Dissipation of the energy is caused, basically, by two external forces:

- the *drag force* (fluid resistance) is proportional to the velocity squared [1]
$$F_{drag}(t) = -k_{drag}\, v^2(t)\, \mathrm{sign}(v(t)),$$
where $k_{drag} > 0$ is the coefficient of fluid (air) resistance and the sign function is given by
$$\mathrm{sign}(\rho) = \begin{cases} 1 & \text{if } \rho > 0, \\ 0 & \text{if } \rho = 0, \\ -1 & \text{if } \rho < 0; \end{cases}$$
- the *dry friction force* is nearly independent of the velocity and can be modeled as follows (see e.g. [2])
$$F_{dry}(t) = -k_{dry}\, \mathrm{sign}(v(t)),$$
where $k_{dry} > 0$ is the coefficient of dry friction.

A more general friction model also may contain some linear terms (proportional to the velocity). We skip them for simplicity of analysis since they will not change any conclusion about local homogeneity degrees and convergence rates of the system.

The sum of external forces $F(t)$ can be represented as follows

$$F(t) = F_{drag}(t) + F_{dry}(t) = -\left(k_{dry} + k_{drag}\, v^2(t)\right) \mathrm{sign}(v(t)),$$

and the differential equation describing an evolution of the velocity of the body has the form:

$$m\dot{v}(t) = -\left(k_{dry} + k_{drag}\, v^2(t)\right) \mathrm{sign}(v(t)).$$

It is not difficult to show that $v = 0$ is the equilibrium of the latter equation, which is globally asymptotically stable, $v(t) \to 0$ as $t \to +\infty$. The solution of this ODE can be found explicitly as

$$v(t) = \tan\left(\arctan(|v(0)|) - \frac{\sqrt{k_{dry}\, k_{drag}}}{m} t\right) \mathrm{sign}(v(0)).$$

This immediately implies $v(t) = 0$ for $t \geq \frac{m \arctan(|v(0)|)}{\sqrt{k_{dry}\, k_{drag}}}$. The function arctan is globally uniformly bounded. We conclude that *independently of the initial velocity, the motion of the body terminates no later than the following instant of time*

$$T_{\max} = \frac{m\pi}{2\sqrt{k_{dry}\, k_{drag}}}.$$

The model of the friction force F is not homogeneous function of v, but locally (close to the origin or close to infinity) it is. Namely,

$$F \approx -k_{dry}\mathrm{sign}(v) \text{ as } v \to 0 \quad \text{and} \quad F \approx -k_{drag} v^2 \mathrm{sign}(v) \text{ as } v \to \infty.$$

In other words, the approximation of F at zero is a homogeneous function with the degree 0, but the approximation at infinity is a homogeneous function with the degree 2. The first term allows us to conclude a finite-time convergence of $v(t)$ to zero for small initial conditions while the second one guarantees a fixed-time convergence into a neighborhood of zero for large initial conditions. Such a combination yields a fixed-time deceleration of the mechanical system independently of the initial velocity. This property is also known as the *fixed-time stability* (or the *fixed-time convergence*) [3].

1.2 Generalized Homogeneity

1.2.1 Weighted Dilations

The standard homogeneity considered above has been introduced by means of the uniform dilation $x \to \lambda x$, $\lambda > 0$. It is clear that if we change the dilation rule then another type of homogeneity can be defined. The *weighted dilation* (introduced by V. I. Zubov in 1958, [4]) of the vector $x = (x_1, x_2, ..., x_n)^\top \in \mathbb{R}^n$ is the simplest case of the so-called generalized (nonuniform) dilation:

$$(x_1, x_2, \ldots, x_n) \quad \to \quad (\lambda^{r_1} x_1, \lambda^{r_2} x_2, \ldots, \lambda^{r_n} x_n),$$

where $\lambda > 0$, as before, is the scaling factor and the positive numbers $r_1, r_2, \cdots, r_n$ are the weights, which specify dilation rates along different coordinates. If $r_1 = r_2 = ... = r_n = 1$ then the weighted dilation becomes uniform. The introduced transformation of coordinates

$$x \to \Lambda x$$

is a linear mapping $\mathbb{R}^n \to \mathbb{R}^n$ defined by the dilation matrix

$$\Lambda = \begin{pmatrix} \lambda^{r_1} & 0 & ... & 0 \\ 0 & \lambda^{r_2} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda^{r_n} \end{pmatrix}.$$

The symmetry (homogeneity) of a scalar-valued function with respect to the weighted dilation can be identified analogously to the uniform case.

Definition 1.2 A function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous with respect to the weighted dilation Λ if

$$f(\Lambda x) = \lambda^\nu f(x), \quad x \in \mathbb{R}^n, \quad \lambda > 0,$$

where $\nu \in \mathbb{R}$ is the homogeneity degree.

The weighted dilation extends the class of homogeneous functions under consideration. For example, the polynomial function

$$(x_1, x_2) \xrightarrow{f} x_1^2 + x_1 x_2^2 + x_2^4$$

is homogeneous with respect to the weighted dilation $(x_1, x_2) \to (\lambda^2 x_1, \lambda x_2)$, but it is not homogeneous with respect to the uniform one $(x_1, x_2) \to (\lambda x_1, \lambda x_2)$.

Definition 1.3 A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be weighted homogeneous if

$$f(\Lambda x) = \lambda^\mu \Lambda f(x), \quad x \in \mathbb{R}^n, \quad \lambda > 0,$$

where $\mu \in \mathbb{R}$ is a homogeneity degree of the vector field f.

Notice that, due to the nonuniformity of the weighted dilation, the definition of the weighted homogeneous vector field $\mathbb{R}^n \to \mathbb{R}^n$ differs from Definition 1.1. However, it can be shown (see Chap. 7) that *any weighted (in fact, even a more generalized) homogeneous system is topologically equivalent (homeomorphic) to a standard homogeneous one*. Consequently, it demonstrates the same properties like the symmetry of solutions, the equivalence of local and global properties, the finite-time convergence for $\mu < 0$, and the practical fixed-time convergence for $\mu > 0$.

Weighted homogeneous models frequently appear in control theory and applications. For example, the classical minimum time control problem

$$T \to \min_u$$

subject to

$$\begin{cases} \dot{x}_1 = x_2, & u \in L^\infty((0, T), \mathbb{R}) \text{ such that } |u(t)| \leq 1, \\ \dot{x}_2 = u, & x_1(T) = x_2(T) = 0 \end{cases}$$

has the solution (see, e.g. [5] or Chap. 12) in the form of the weighted homogeneous feedback

$$u = -\mathrm{sign}(|x_2|x_2 + 2x_1).$$

Indeed, $u(\lambda^2 x_1, \lambda x_2) = u(x_1, x_2)$, i.e. u is the weighted homogeneous function with the zero degree.

The weighted dilations and the weighted homogeneity in $\mathbb{R}^n$ allow some important results to be obtained about

- the global expansion of a local stability and the existence of homogeneous Lyapunov functions [4, 6];
- controllability, stabilizability, and observability of nonlinear systems [7–10];
- controllers and observers design [11–18];
- robustness analysis of both delay-free [15, 19, 20] and time-delay systems [21, 22];
- the high-order sliding mode algorithms [23–25].

Below, for simplicity, the scaling factor $\lambda > 0$ is given by $\lambda = e^s$ with $s \in \mathbb{R}$, where $e = 2.71828\ldots$ is the Euler number.

In this case, the weighted dilation becomes

$$x \to \mathbf{d}(s)x, \quad s \in \mathbb{R},$$

where

$$\mathbf{d}(s) = \begin{pmatrix} e^{r_1 s} & 0 & \cdots & 0 \\ 0 & e^{r_2 s} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{r_n s} \end{pmatrix}, \quad r_i > 0.$$

1.2.2 Linear Dilations in Banach Spaces

As shown above, once a homogeneity of the right-hand side of ODE with respect to a group of transformations (dilations) is established, many properties of the nonlinear system can be studied easily. In fact, similar conclusions can be made for systems of ordinary differential inclusions [23, 24] and infinite-dimensional dynamical models [26, 27].

The crucial point of the homogeneity-based analysis is a selection of a dilation group $\mathbf{d}(s)$, $s \in \mathbb{R}$. Generalized dilations in $\mathbb{R}^n$ are studied in [28–30], where the dilation is generated by a C^1 vector field. In this book we deal with groups of *linear dilations*, which can be easily defined even in a Banach space $\mathbb{B}$ as a strongly continuous group $\mathbf{d}$ of *linear bounded operators*

$$x \to \mathbf{d}(s)x, \quad x \in \mathbb{B}, \quad s \in \mathbb{R},$$

where $\mathbf{d}(s) \in \mathcal{L}(\mathbb{B}, \mathbb{B})$ and $s \in \mathbb{R}$ is the group parameter. The theory of strongly continuous semigroups and groups is well developed for evolution systems in Banach and Hilbert spaces (see e.g. [31, 32] or Chap. 3 for more details).

Groups of linear dilations are studied with details in Chap. 6. Here we just mention that, to be a dilation, the group $\mathbf{d}$ must satisfy some limit property [33], e.g.

$$\|\mathbf{d}(s)x\| \to 0 \text{ as } s \to -\infty$$

and

$$\|\mathbf{d}(s)x\| \to +\infty \text{ as } s \to +\infty,$$

where $x \in \mathbb{B}\backslash\{\mathbf{0}\}$.

Notice also that any continuous group of linear dilations in $\mathbb{R}^n$ can be defined by means of the matrix exponential

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!},$$

where $s \in \mathbb{R}$ is the group parameter and $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is an anti-Hurwitz matrix, which is called the *generator* of the dilation $\mathbf{d}$.

Linear dilation in $\mathbb{R}^n$ includes the uniform dilation and the weighted dilation (considered above) as particular cases. In the two-dimensional case, the differences between uniform, weighted, and linear dilations are illustrated in Fig. 1.2, where the so-called homogeneous curves $\{\mathbf{d}(s)x : s \in \mathbb{R}\}$ for the following three dilation groups

$$\mathbf{d}_1(s) = e^s I, \qquad \mathbf{d}_2(s) = \begin{pmatrix} e^{2s} & 0 \\ 0 & e^s \end{pmatrix}, \qquad \mathbf{d}_3(s) = e^{sG_{\mathbf{d}}}$$

are schematically depicted.

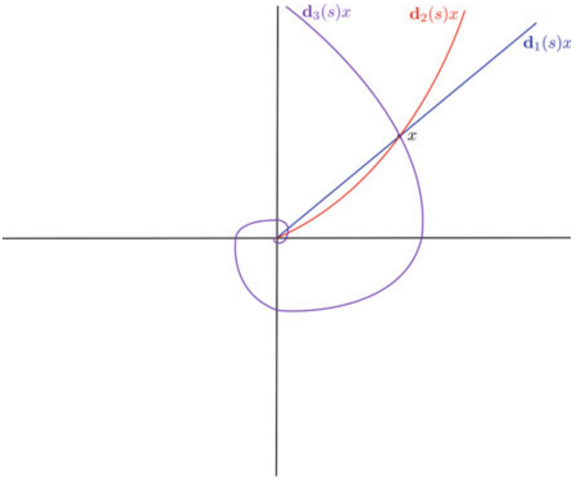

Fig. 1.2 $\mathbf{d}_1(s)$—uniform dilation, $\mathbf{d}_2(s)$—weighted dilation, and $\mathbf{d}_3$—linear dilation

In the general case, the generator $G_{\mathbf{d}}$ of a dilation in $\mathbb{B}$ may be a *linear unbounded operator*.

Example 1.1 As an example of the linear dilation in the Banach space $\mathbb{B} = L_2(\mathbb{R}, \mathbb{R})$, we consider

$$(\mathbf{d}(s)u)(x) = e^s u(e^{-s}x), \quad s \in \mathbb{R}$$

for $u \in \mathbb{B}$ and $x \in \mathbb{R}$. In Chap. 7 it is shown that $\mathbf{d}$ is, indeed, a dilation in $\mathbb{B}$. Moreover, simple calculations show that the differential operator $A = \frac{\partial}{\partial x}$ is homogeneous (symmetric) with respect to the dilation $\mathbf{d}$ in the following sense

$$A\mathbf{d}(s)u = e^{-s}\mathbf{d}(s)Au.$$

All linear and many nonlinear models of mathematical physics are homogeneous. The heat, wave, Saint-Venant, Burgers, Korteweg–de Vries (KdV), and Navier–Stokes equations are examples of homogeneous (in a generalized sense) systems in Banach spaces.

In this monograph, we show that an analysis of evolution systems can be based on the concept of the generalized homogeneity, which is useful for the control systems design and purely theoretical problems of the systems science (e.g. a global expansion of regularity of nonlinear evolution equations).

1.3 From Linearity to Homogeneity in Control Systems

A quality of any control system is estimated by some *quantitative* indexes (see e.g. [34–36]), which reflects control precision, optimality of transient motions, energetic effectiveness, robustness of the closed-loop system with respect to disturbances, etc. From a mathematical point of view, the design of a "good" control law is a multi-objective optimization problem. The mentioned objectives frequently contradict to each other, e.g. a time optimal feedback control could not be energetically optimal but it may be efficient for a disturbance rejection [5]. Moreover, some criteria cannot be even clearly mathematically formulated (e.g. simplicity of the practical implementation and tuning). The control practice frequently relaxes mathematical criteria for simplicity. An adjustment of a guaranteed (small enough) convergence time can be considered instead of the minimum time control problem. An exact convergence of systems states to a set-point could be relaxed to a convergence into a sufficiently small neighborhood of this point.

A well-tuned linear controller, such as PID (Proportional–Integral–Differential) algorithm, guarantees a good enough control quality in many practical cases [34]. However, the theory of linear control systems reaches its peak of maintenance and further improvements of a control performance using the same linear strategy seems impossible. Being a certain generalization of the linearity, the homogeneity could provide some additional tools for improvement of the control quality. In this context, it is important to know which features of homogeneous systems may be useful for advanced analysis and design of control systems.

1.3.1 Convergence Rates of Homogeneous Algorithms

Finite-Time and Fixed-Time Stabilization

Finite-time and fixed-time stability could be utilized if a control or estimation algorithm must guarantee a convergence in a prescribed time. The typical example is a missile (or antimissile) control problem since the control plant simply does not exist after the missile explosion. A control algorithm, which guarantees an asymptotic convergence without any tuning of the convergence time, is not appropriate for this case.

Stability properties of homogeneous and locally homogeneous systems discussed above allow us to propose a simple way to fulfill some time constraints in control and estimation systems. For example, to stabilize a state in a fixed time independently of the initial condition, a stable closed-loop system needs to be homogeneous of a positive degree “close to infinity” and homogeneous of a negative degree close to the origin. This idea can be illustrated on the simplest scalar example

$$\dot{x}(t) = u(t), \quad t > 0, \quad x(0) = x_0,$$

where $x(t) \in \mathbb{R}$ is the state variable and $u(t) \in \mathbb{R}$ is the control signal. The control aim is to stabilize this system at the origin such that the condition $|u(x)| \leq 1$ must be fulfilled for $|x| \leq 1$.

- The classical approach gives the standard *linear* proportional feedback algorithm

  $$u_{lin}(x) = -x,$$

  which guarantees an asymptotic (in fact, the *exponential*) convergence to the origin of any trajectory of the closed-loop system:

  $$|x(t)| = e^{-t}|x_0|.$$

- The globally homogeneous feedback of the form

  $$u_{ft}(x) = -\sqrt{|x|}\text{sign}[x]$$

  stabilizes the system at the origin in a *finite time*:

  $$x(t) = 0, \quad \text{for } t \geq T(x_0).$$

The convergence time T depends on the initial condition $x(0) = x_0$, in particular, $T(x_0) = 2\sqrt{|x_0|}$ for the considered control law. Obviously, $T(x_0)$ tends to infinity as $|x_0|$ tends to infinity.

- The *fixed-time* stabilizing controller can be selected locally homogeneous in the form:

$$u_{fxt}(x) = \begin{cases} -|x|^{1/2}\mathrm{sign}[x] & \text{if} \quad |x| \leq 1, \\ -|x|^{3/2}\mathrm{sign}[x] & \text{if} \quad |x| > 1. \end{cases}$$

It stabilizes the system in a fixed time, namely,

$$x(t) = 0, \quad t \geq 4$$

for any $x_0 \in \mathbb{R}$.

The results of numerical simulations for the considered controllers are presented in Figs. 1.3, 1.4, and 1.5. They confirm that the fixed-time controller is always faster than the linear one while the finite-time controller shows a faster convergence rate only close to the origin (in the zone $|x| < 1$).

Finite-Time and Fixed-Time Estimation

The finite-time stability of homogeneous systems can also be utilized for an observer's design. Indeed, let us consider the simplest observation problem

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, & \quad t > 0, \\ y = x_1 \end{cases}$$

where $x_1, x_2 \in \mathbb{R}$ are system states, $u \in \mathbb{R}$ is a known input, and $y \in \mathbb{R}$ is a measured output. The aim is to estimate the unknown state variable x_2 in a finite time.

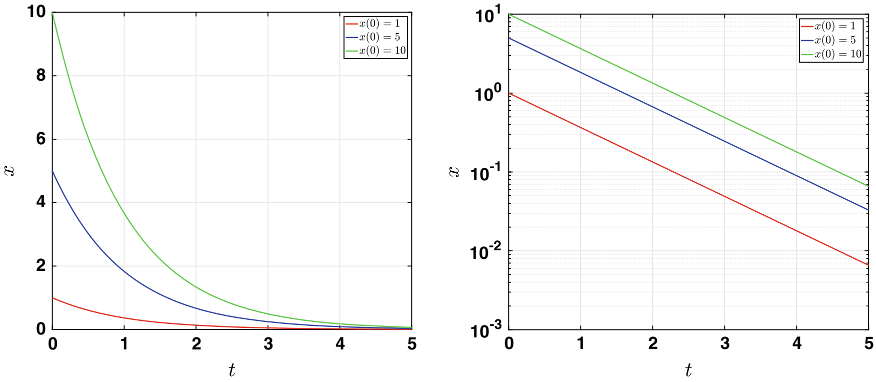

Fig. 1.3 Trajectories of the *exponentially* stable system with $u(t) = u_{lin}(x(t))$ in linear and logarithmic scales