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Praveen Agarwal · Dumitru Baleanu · YangQuan Chen · Shaher Momani · José António Tenreiro Machado *Editors*

Fractional Calculus

ICFDA 2018, Amman, Jordan, July 16–18

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Preface

These are the keynote and invited talks of the *International Conference on Fractional Differentiation and its Applications* (ICFDA-2018), which was held at the Amman Marriot Hotel, Sheissani in Amman, The Hashemite Kingdom of Jordan from July 16 to July 18, 2018. The ICFDA18 is a specialized conference on fractional-order calculus and its applications. It is a generalization of the integer-order ones. The fractional-order differentiation of arbitrary orders takes into account the memory effect of most systems. The order of the derivatives may also be variable, distributed, or complex. Recently, fractional-order calculus became a more accurate tool to describe systems in various fields in mathematics, biology, chemistry, medicine, mechanics, electricity, control theory, economics, and signal and image processing.

For this edition, we were happy to have 23 invited speakers who gave talks on a subject for which they are internationally known experts. Thirteen of these talks are collected in this volume. Throughout this book, the fractional calculus concepts have been explained very carefully in the simplest possible terms, and illustrated by a number of complete solved examples. This book contains some theorems and their proofs.

The book is organized as follows. In chapter "Closed-Form Discretization of Fractional-Order Differential and Integral Operators", a closed-form concretization of fractional-order differential or integral Laplace operators is introduced. The proposed method depends on extracting the necessary phase requirements from the phase diagram. The magnitude frequency response follows directly due to the symmetry of the poles and zeros of the finite z-transfer function. Unlike the continued fraction expansion technique, or the infinite impulse response of second-order IIR-type filters, the proposed technique generalizes the Tustin operator to derive a first-, second-, third-, and fourth-order discrete-time operators (DTO) that were stable and of minimum phase. The proposed method depends only on the order of the Laplace operator. The resulted discrete-time operators enjoy flat phase response over a wide range of discrete-time frequency spectrum. The closed-form DTO enables one to identify the stability regions of fractional-order discrete-time systems or even to design discrete-time-fractional-order $PI^{\lambda}D^{\mu}$ controllers.

The effectiveness of this work was demonstrated via several numerical simulations. In chapter "On Fractional-Order Characteristics of Vegetable Tissues and Edible Drinks", we are concerned about frequency response techniques to characterize vegetable tissues and edible drinks. In the first phase, the impedance of the distinct samples is measured and fractional-order models are applied to the resulting data. In the second phase, hierarchical clustering and multidimensional scaling tools are adopted for comparing and visualizing the similarities between the specimen.

In chapter "Some Relations Between Bounded Below Elliptic Operators and Stochastic Analysis", we apply Malliavin calculus tools to the case of a bounded below elliptic right-invariant pseudo-differential operators on a Lie group. We give examples of bounded below pseudo-differential elliptic operators on $\mathbb{R}^d$ by using the theory of the Poisson process and the Garding inequality. In the two cases, there are no stochastic processes because the considered semi-groups do not preserve positivity. In chapter "Discrete Geometrical Invariants: How to Differentiate the Pattern Sequences from the Tested Ones?" based on the new method (defined below as the discrete geometrical invariants—DGI(s)), one can show that it enables to differentiate the statistical differences between random sequences that can be presented in the form of 2D curves. We generalized and considered the Weierstrass–Mandelbrot function and found the desired invariant of the fourth order that connects the WM-functions with different fractal dimensions. Besides, we consider an example based on real experimental data. A high correlation of the statistically significant parameters of the DGI obtained from the measured data (associated with reflection optical spectra of olive oil) with the sample temperature is shown. This new methodology opens wide practical applications in the differentiation of the hidden interconnections between measured by the environment and external factors.

In chapter "Nonlocal Conditions for Semi-linear Fractional Differential Equations with Hilfer Derivative", we study the existence of solutions and some topological proprieties of solution sets for nonlocal semi-linear fractional differential equations of Hilfer type in Banach space by using noncompact measure method in the weighted space of continuous functions. The main result is illustrated with the aid of an example. In chapter "Offshore Wind System in the Way of Energy 4.0: Ride Through Fault Aided by Fractional PI Control and VRFB", we present a simulation about a study to improve the ability of an offshore wind system to recover from a fault due to a rectifier converter malfunction. The system comprises: a semi-submersible platform; a variable speed wind turbine; a synchronous generator with permanent magnets; a five-level multiple point diode clamped converter; a fractional PI controller using the Carlson approximation. Recovery is improved by shielding the DC link of the converter during the fault using as further equipment a redox vanadium flow battery, aiding the system operation as desired in the scope of Energy 4.0. Contributions are given for: (i) the fault influence on the behavior of voltages and currents in the capacitor bank of the DC link; (ii) the drivetrain modeling of the floating platform by a three-mass modeling; (iii) the vanadium flow battery integration in the system.

In chapter "Soft Numerical Algorithm with Convergence Analysis for Time-Fractional Partial IDEs Constrained by Neumann Conditions", a soft numerical algorithm is proposed and analyzed to fitted analytical solutions of PIDEs with appropriate initial and Neumann conditions in Sobolev space. Meanwhile, the solutions are represented in series form with accurately computable components. By truncating the n-term approximate solutions of analytical solutions, the solution methodology is discussed for both linear and nonlinear problems based on the nonhomogeneous term. Analysis of convergence and smoothness is given under certain assumptions to show the theoretical structures of the method. Dynamic features of the approximate solutions are studied through an illustrated example. The yield of numerical results indicates the accuracy, clarity, and effectiveness of the proposed algorithm as well as provide a proper methodology in handling such fractional issues. Chapter "Approximation of Fractional-Order Operators" deals with the several comparisons in the time response and Bode results between four well-known methods; Oustaloup's method, Matsuda's method, AbdelAty's method, and El-Khazali's method are made for the rational approximation of fractional-order operator (fractional Laplace operator). The various methods along with their advantages and limitations are described in this chapter. Simulation results are shown for different orders of the fractional operator. It has been shown in several numerical examples that the El-Khazali's method is very successful in comparison with Oustaloup's, Matsuda's, and AbdelAty's methods.

In chapter "Multistep Approach for Nonlinear Fractional Bloch System Using Adomian Decomposition Techniques", we discuss a superb multistep approach, based on the Adomian decomposition method (ADM), which is successfully implemented for solving nonlinear fractional Bolch system over a vast interval, numerically. This approach is demonstrated by studying the dynamical behavior of the fractional Bolch equations (FBEs) at different values of fractional order α in the sense of Caputo concept over a sequence of the considerable domain. Further, the numerical comparison between the proposed approach and implicit Runge–Kutta method is discussed by providing an illustrated example. The gained results reveal that the MADM is a systematic technique in obtaining a feasible solution for many nonlinear systems of fractional order arising in natural sciences.

The chapter "Simulation of the Space–Time-Fractional Ultrasound Waves with Attenuation in Fractal Media" deals with the simulation of the space–time-fractional ultrasound waves with attenuation in fractal media. In chapter "Certain Properties of Konhauser Polynomial via Generalized Mittag-Leffler Function", we establish several new properties of generalized Mittag-Leffler function via Konhauser polynomials. Properties like mixed recurrence relations, differential equations, pure recurrence relations, finite summation formulae, and Laplace transform have been obtained. In chapter "An Effective Numerical Technique Based on the Tau Method for the Eigenvalue Problems", we consider the (presumably new) effective numerical scheme based on the Legendre polynomials for approximate solution of eigenvalue problems. First, a new operational matrix, which can be represented by sparse matrix is defined by using the Tau method and orthogonal functions. Sparse data is by nature more compressed and thus require significantly less storage.

A comparison of the results for some examples reveals that the presented method is convenient and effective, also we consider the problem of column buckling to show the validity of the proposed method. Finally, in chapter "On Hermite–Hadamard-Type Inequalities for Coordinated Convex Mappings Utilizing Generalized Fractional Integrals", we obtain the Hermite–Hadamard-type inequalities for coordinated convex function via generalized fractional integrals, which generalize some important fractional integrals such as the Riemann–Liouville fractional integrals, the Hadamard fractional integrals, and Katugampola fractional integrals. The results given in this chapter provide a generalization of several inequalities obtained in earlier studies.

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Contents

Closed-Form Discretization of Fractional-Order Differential and Integral Operators

Reyad El-Khazali and J. A. Tenreiro Machado

Abstract This paper introduces a closed-form discretization of fractional-order differential or integral Laplace operators. The proposed method depends on extracting the necessary phase requirements from the phase diagram. The magnitude frequency response follows directly due to the symmetry of the poles and zeros of the finite z-transfer function. Unlike the continued fraction expansion technique, or the infinite impulse response of second-order IIR-type filters, the proposed technique generalizes the Tustin operator to derive a first-, second-, third-, and fourth-order discrete-time operators (DTO) that are stable and of minimum phase. The proposed method depends only on the order of the Laplace operator. The resulted discrete-time operators enjoy flat-phase response over a wide range of discrete-time frequency spectrum. The closed-form DTO enables one to identify the stability regions of fractional-order discrete-time systems or even to design discrete-time fractional-order $PI^{\lambda}D^{\mu}$ controllers. The effectiveness of this work is demonstrated via several numerical simulations.

Keywords Fractional calculus · Transfer function · Discrete-time operator · Discrete-time integro-differential operators · Frequency response

1 Introduction

Fractional calculus is a generalization of the integer-order one. Most practical systems exhibit fractional-order dynamics, which could be of real or complex values. Fractional-order systems enjoy the hereditary effect that is approximated by infinite-dimensional models [8, 20]. It is used in many fields such as in economy, physics,

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biology, chemistry, medicine, social sciences, and engineering. To analyze fractional-order systems, one has to look for finite-dimensional and realizable models that approximate such systems [11, 12, 19, 21–25].

The use of microprocessors nowadays are necessary for signal processing and system analysis. Thus, a straightforward method is required to discretize a continuous-time fractional-order system into a discrete-time one. This can be accomplished by discretizing the fractional-order Laplacian operator s^α and replacing it with a finite-order DTO. In general, there are two methods that are used to discretize s^α; i.e., a direct and an indirect one. In the indirect discretization method, a rational continuous-time operator (CTO) is first obtained and then discretized using techniques such as the bilinear transformation, the Al-Alaoui operator, the Euler's backward method, or the stable Simpson's method [1–3].The direct method, however, allows one to generate discrete-time operators that converts a continuous-time operator (CTO) into a DTO [4, 5, 17].

The indirect discretization method is achieved in two steps; the first one is to approximate the Laplacian operator s^α by a rational transfer function in the s-domain, which is then simplified using the continued fraction expansion (CFE), and the second step is to discretize the expanded form using either the bilinear transformation, Simpson's method, Euler's method, or a linear combination of them or other existing forms [6, 24]. It is important to realize that the CFE method could yield an unstable non-minimum phase discrete-time operator. An alternative approach to the CFE was discussed in [19], where infinite impulse response (IIR) autoregressive moving-average (ARMA) models are used to develop DTO operators, which may result in developing higher order approximation. Notice that the Al-Alaoui operator is obtained as a linear combination of the trapezoidal and the rectangular integration rules [2, 14, 15, 26]. The interpolation and inversion processes may induce, in some cases, unstable fractional-order operators.

This work introduces a straightforward discretization direct method to discretize continuous differential and/or integral operators. It can be considered as a dynamic (or adaptive) discretization technique, where the poles and the zeros of the generated z-transfer function are all located inside the unit disc and their values depend only on the fractional-order α. The proposed method yields finite-order DTO that exhibits a competitive frequency response to higher order operators developed in [2, 6, 16].

The paper is organized as follows. Section 2 summarizes some preliminary concepts and background. Section 3 introduces the main results of first-, second-, third-, and fourth-order operators, while Sect. 4 summarizes the numerical simulation and a comparison between different operators. Section 5 outlines the main conclusions.

2 Preliminary Concepts and Background

The general fractional-order differential (integral) operator is denoted by ${}_aD_t^{\pm\alpha}$ (${}_aI_t^{\alpha}$), respectively [18], where a and t represent the starting time and $\alpha \in \mathbb{R}$ is the order of the operator. For example, if one wishes to implement a discrete-time fractional-order controller, then it is necessary to look for a stable non-minimum phase DTO operator of low order. The design and implementation of fractional-order discrete-time controllers cannot accommodate higher order operators since this will increase the complexity of the controlled system, and could yield unstable ones. Therefore, the proposed technique provides a competitive DTO that benefits from the IIR structure of such operators; i.e., a second-order DTO is competitive to that of a ninth-order one introduced in [6, 19, 20].

As mentioned in Sect. 1, the indirect discretization method starts by developing a rational finite-order transfer functions, that is, $s^{\pm\alpha} \approx \frac{N(s,\alpha)}{D(s,\alpha)}$ [10, 13, 21], and it is followed by using any existing discretization technique, or a linear combination of such methods. For example, the Al-Alaoui discrete-time integral operator is simply a linear interpolation of the backward rectangular rule and the trapezoidal rule, namely $H(z) = aH_{Rect}(z) + (1-a)H_{Trap}(z)$, where $0 < a < 1$ [1–3]. A similar approach was used to derive a hybrid digital integrator using a linear combination of Trapezoidal and Simpson integrator [6, 10]. Such interpolation reduces the frequency warping over a limited frequency band, and their phase frequency response is not constant. For comparison, Fig. 1 displays the frequency response of the Tustin operator, $s = H(z) = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$, Al-Alaoui operator, $s = H(z) = \frac{8}{7T}\frac{1-z^{-1}}{1+\frac{1}{7}z^{-1}}$, and Chen discrete-time operator [5]. Another discrete-time operator that approximates an integer-order integrator was also introduced in [6] and given here for completeness:

$$H(z) = \frac{6\left(z^2-1\right)}{T(3-a)(z+p_1)(z+p_2)}, \tag{1a}$$

$$p_1 = \frac{3+a+2\sqrt{3a}}{3-a}, \tag{1b}$$

$$p_2 = \frac{3+a-2\sqrt{3a}}{3-a}, \tag{1c}$$

where T is the sampling time and $0 < a < 1$ is a scaling factor. Equation (1) can then be used to generate several quadratic forms that discretize $s^{\pm 1}$.

Figure 1 shows the frequency response of the aforementioned three DTO operators that approximate s^1 for $T = 0.001$. The magnitude response of Tustin operator exhibits large errors at both ends of the frequency spectrum. The magnitude response of the Al-Alaoui operator, however, is almost identical to that of the Tustin operator at low frequency, but provides a better response at high frequency. Moreover, it yields a linear phase response due to the asymmetric pole-zero location, while the hybrid ninth-order operator reported in [6] yields a perfect phase behavior. However, one cannot afford this size of an operator since a discrete-time fractional-order phase-locked loop, for example, will be modeled by an 18th-order discrete-time z-transfer function.

Since the goal is to look for a closed-form discrete-time model for $s^{\pm\alpha}$, the direct approach is adopted here to develop a straightforward discretization method.

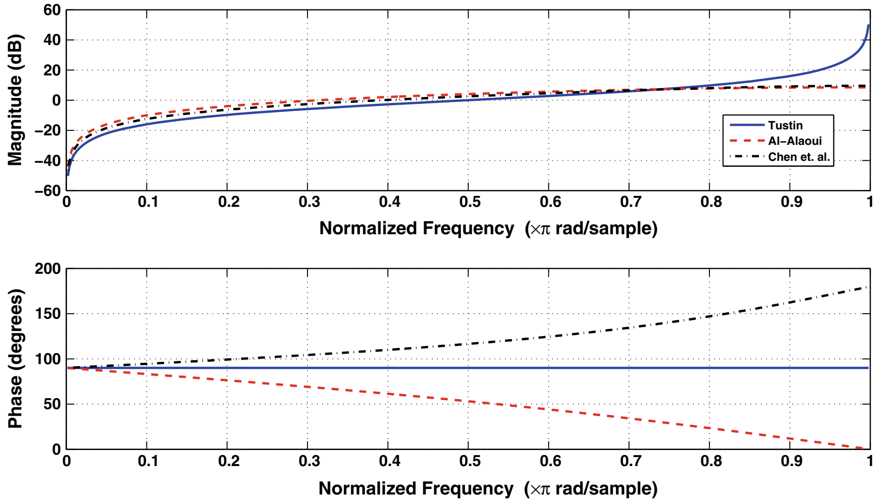

Fig. 1 Frequency response of Tustin, Al-Alaoui, and the DTO of Eq. (1) for $a = 1$

In all direct methods, the continuous frequency operator is replaced by a generating function, that is, $s^{\pm\alpha} = \left(\omega\left(z^{-1}\right)\right)^{\pm\alpha}$. To gain more insight, one may start with the Grünwald–Letnikov (GL) definition of the fractional-order differential (integral) operator [7, 13, 14, 21, 22]:

$$ {}_aD_t^{\pm\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\pm\alpha}} \sum_{j=0}^{\infty} C_j^{\pm\alpha} f((t-j)h). \tag{2}$$

where

$$ C_j^{\pm\alpha} = (-1)^j \binom{\pm\alpha}{j} = \left(1 - \frac{1\pm\alpha}{j}\right) C_{j-1}^{\pm\alpha},\ j = 1,\ldots,n, \tag{3a}$$

$$ C_0^{\pm\alpha} = 1. \tag{3b}$$

Taking the $\mathcal{Z}$-transform of (2) and using the short memory principle [14], the following generating function may discretize $s^{\pm\alpha}$:

$$ \left(\omega\left(z^{-1}\right)\right)^{\pm\alpha} = T^{\mp\alpha} \left(\sum_{j=0}^{\left[\frac{L}{T}\right]} C_j^{\pm\alpha} z^{-j} \right), \tag{4}$$

where $T = h$ is the sampling time, and $\frac{L}{T} = \left[\frac{nh-a}{h}\right]$ is an increasing memory size $L - nh - a$.

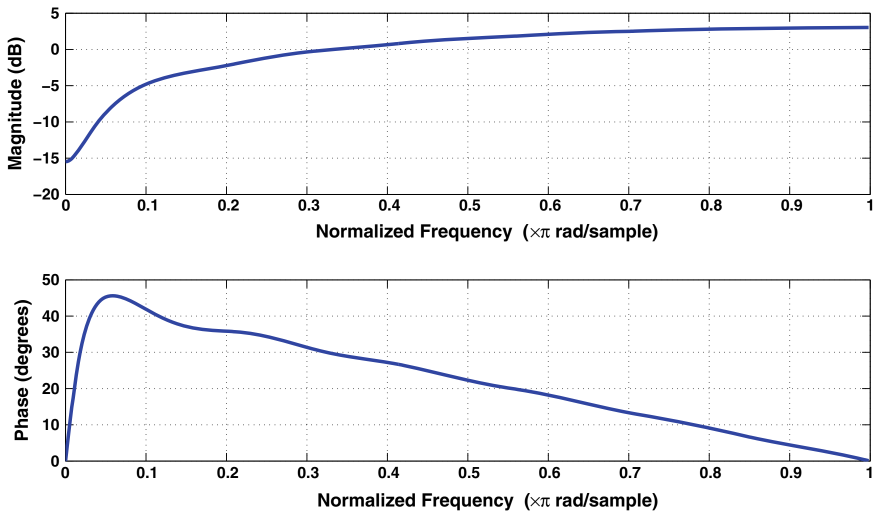

Fig. 2 Frequency response of a FIR-type discrete-time differentiator for $s^{0.5}$ with $T = 0.001$ s, and $L = 0.011$

Equation (4) defines a transfer function of a finite impulse response (FIR) discrete-time system of $s^{\pm\alpha}$. The memory size, L, determines the accuracy of the approximation. Hence, a compromise has to be made between the accuracy and the size of the operator. Figure 2 shows the frequency response of (4) for $\alpha = 0.5$, $T = 0.001$, and $L = 11$. Clearly, the phase diagram is close to the expected angle of $\frac{\pi}{4}$ over a very narrow frequency band $\omega \in (0.06, 0.08)$ rad/s, which may not be suitable for realization techniques.

Obviously, in spite of its large size, the frequency response of the FIR discrete-time form of Eq. (4) does not provide the expected constant phase response. Therefore, an alternative discrete-time IIR-type rational z-transfer function, of lower size than the FIR form, to discretize $s^{\pm\alpha}$ will be the choice to overcome such problem.

Since, the CFE approach does not always yield a minimum phase and stable system, or a flat-phase response [2, 6, 7, 11, 13], a compromise has to be made between the size of the expansion and the type of the generating functions used for approximation. The following generating functions can be used to discretize $s^{\pm\alpha}$ and replace it with DTO operators [5, 6, 14, 17]:

(a) Backward-Euler method: $\left(\omega\left(z^{-1}\right)\right)^{\pm\alpha} = \left(\frac{1-z^{-1}}{T}\right)^{\pm\alpha}$

(b) Trapezoidal (Tustin) discretization rule: $\left(\omega\left(z^{-1}\right)\right)^{\pm\alpha} = \left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\pm\alpha}$

(c) Al-Alaoui Operator: $\left(\omega\left(z^{-1}\right)\right)^{\pm\alpha} = \left(\frac{8}{7T}\frac{1-z^{-1}}{1+\frac{1}{7}z^{-1}}\right)^{\pm\alpha}$

(d) A Hybrid interpolation of Simpson and Trapezoidal discrete-time integrators:

$$H(z) = aH_S(z) + (1-a)\,H_T(z)\,,\ 0 < a < 1, \tag{5}$$

where $H_S(z) = \frac{T}{3}\frac{1+4z^{-1}+z^{-2}}{1-z^{-2}}$ and $H_T(z) = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$.

The interpolation in (5) represents a generalization of the first three methods. Since the magnitude frequency response of the integer-order integrator, s^{-1}, lies between the Simpson rule and that of the Trapezoidal discrete-time integrator [2, 3], the linear combination in (5) for $0 < a < 1$ can be used to generate a typical IIR-type discrete-time operator as follows [6]:

$$\left(\omega\left(z^{-1}\right)\right)^{\pm\alpha} = k_0\left(\frac{1-z^{-2}}{\left(1+bz^{-1}\right)^2}\right)^{\alpha}, \tag{6}$$

where $\alpha \in [0, 1]$, $k_0 = \left(\frac{6z_2}{T(3-a)}\right)^{\alpha}$ and $b = z_2 = \frac{3+a-2\sqrt{3a}}{3-a}$.

Several transfer functions of different sizes can be obtained to approximate $\left(\omega\left(z^{-1}\right)\right)^{\pm\alpha}$. For example, when $\alpha = 0.5$ and $T = 0.001$, Eq. (6), yields the following z-transfer functions, $G_{(n,a)}(z)$, that discretize $s^{0.5}$, where n and a represent the order and the weighting factor of the approximation, respectively [6]:

$$G_{(2,0,5)}\left(z^{-1}\right) = \frac{127 + 41.26z^{-1} - 112.6z^{-2}}{4 + 2.98z^{-1} - z^{-2}}, \tag{7a}$$

$$G_{(3,0,5)}\left(z^{-1}\right) = \frac{1501 - 503.6z^{-1} - 1298z^{-2} + 446.5z^{-3}}{47.26 + 4z^{-1} - 23.63z^{-2} - z^{-3}}, \tag{7b}$$

$$G_{(4,0,5)}\left(z^{-1}\right) = \frac{508.1 - 1501z^{-1} - 4.478z^{-2} + 1298z^{-3} - 382.9z^{-4}}{16 - 40.54z^{-1} - 12z^{-2} + 20.27z^{-3} + z^{-4}}. \tag{7c}$$

Figure 3 shows the frequency response of (7) for $\omega \in (-\pi, \pi)$. The magnitude frequency response of the second-order approximation yields a warping effect at high frequency, while the phase diagram of the three forms exhibit a decreasing phase value over most of the spectrum.

Remark 1 The approximation given by (7c), reported in [6], represents an unstable non-minimum phase DTO since it has a pole and a zero outside the unit circle at $p = 2.6298$, and $z = 2.6328$, respectively. Even though $p \approx z$, that almost cancel each other, implementing such an operator would cause system instability. Furthermore, according to [6], one must improve the phase performance of $G_{(4,0,5)}(z)$ by cascading a causal lead compensator $z^{0.5} = \frac{z^{-0.5}}{z^{-1}}$, which requires the implementation of a fractional-order sampler.

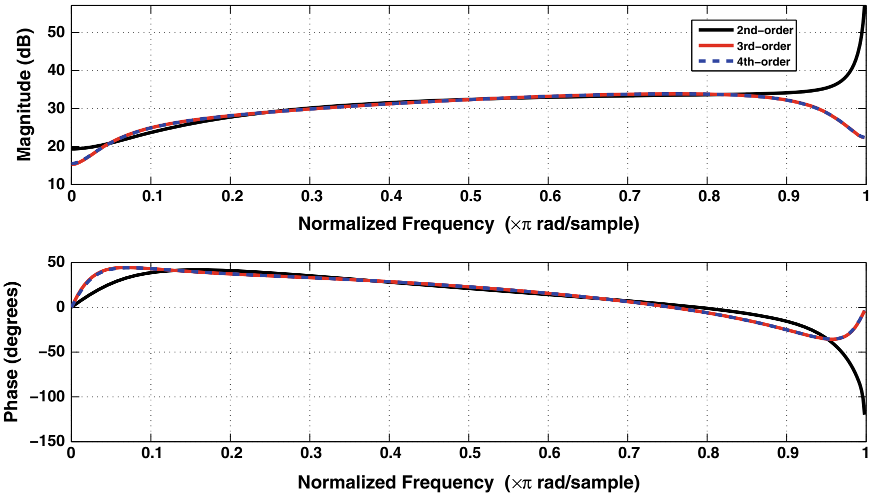

Fig. 3 Frequency response of $s^{0.5}$ using (7a), (7b), (7c)

3 New Fractional-Order Discrete-Time Operators

As discussed in Sect. 2, the discretization technique of generating functions using the CFE yields high order and an unstable non-minimum phase discrete-time approximation. The aim of this work is to avoid such subtleties by developing an adaptive closed-form DTO that effectively discretizes the fractional-order operators, $s^{\pm\alpha}$, which only depend on its order $\pm\alpha$. Furthermore, one can also define the stability region of the discrete form of $s^{\pm\alpha}$.

3.1 *First-Order Operators*

The following first-order operator based on a closed-form solution was first introduced in [8, 9]. It represents an approximation of a first-order discrete-time differential operator (DTDO), where its reciprocal also defines a discrete-time integral operator (DTIO):

$$s^{\pm\alpha} \approx H_{1_K}(z) = \left(\frac{2}{T}\right)^{\pm\alpha} \frac{z \mp z_1(\alpha)}{z \mp p_1(\alpha)}, \tag{8}$$

where

$$z_1(\alpha) = -p_1(\alpha) = \frac{1}{\tan\left((2-\alpha)\frac{\pi}{4}\right)},\ 0 < \alpha < 1, \tag{9}$$

and where $z_1(\alpha) = -p_1(\alpha) \in \mathbb{R}$.

Obviously, for $0 < \alpha < 1$, $|z_1(\alpha)| = |p_1(\alpha)| < 1$ are located inside the unit circle.

3.2 Second-Order Operator

The second-order discrete-time operator was also introduced in [8, 9]. It yields a normalized biquadratic discrete-time transfer function that approximates $s^{\pm\alpha}$ and is given by (Fig. 4):

$$s^{\pm\alpha} \approx H_{2_K}(z) = \left(\frac{2}{T}\right)^{\pm\alpha} \frac{(z \mp z_1(\alpha))(z \mp z_2(\alpha))}{(z \mp p_1(\alpha))(z \mp p_2(\alpha))}, \tag{10}$$

where

$$z_1(\alpha) = \frac{\eta_2 - 2 + \sqrt{5\eta_2^2 + 4}}{2\eta_2}, \quad \eta_2 = \tan\left(\alpha\frac{\pi}{4}\right), \tag{11}$$

and

$$\begin{cases} z_2(\alpha) = z_1(\alpha) - 1 \\ p_1(\alpha) = -z_2(\alpha) \\ p_2(\alpha) = -z_1(\alpha) \end{cases}. \tag{12}$$

Clearly, for large values of α, the first-order DTO yields a competitive frequency response to that of the second-order DTO as shown in Fig. 5.

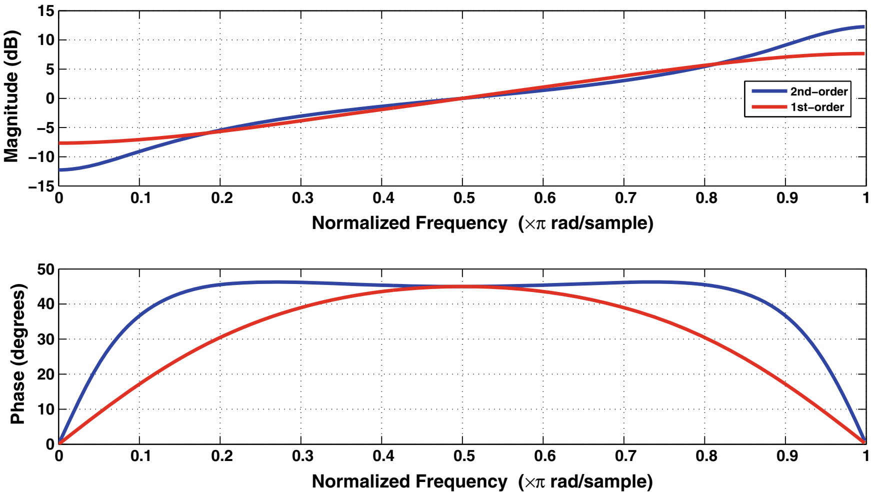

Fig. 4 Frequency response of discrete-time first- and second-order operators for $s^{0.5}$

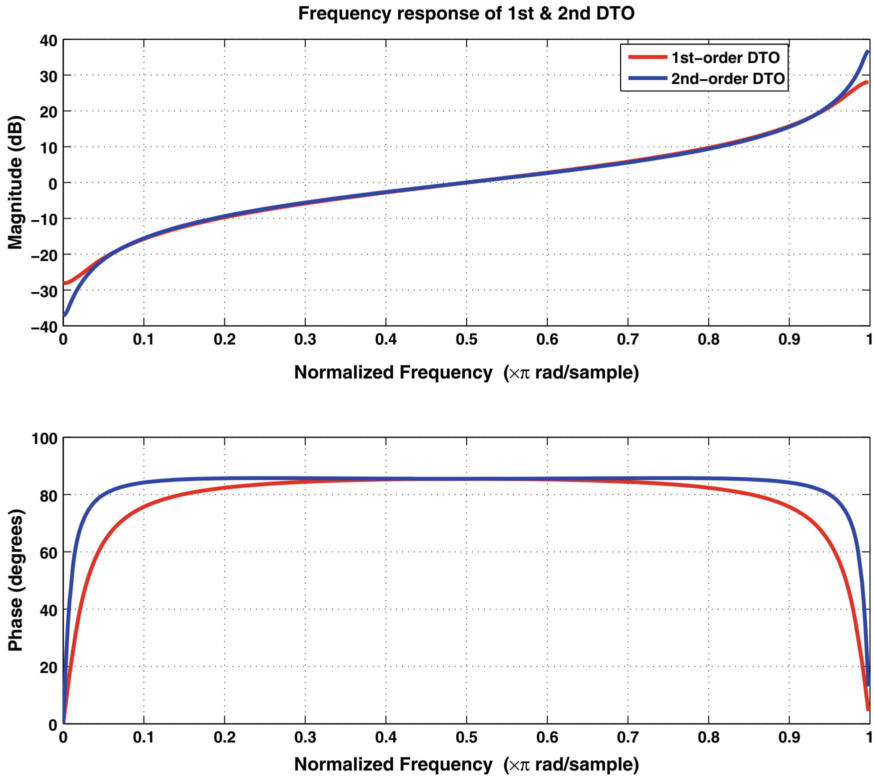

Fig. 5 Frequency response of (8) and (10) for $s^{0.95}$

3.3 Third-Order Operator

The third-order operator is developed to improve the accuracy of the discrete-time approximation over a wider frequency range. Similar to (10), the third-order operator is given by

$$s^{\pm\alpha} \approx H_{3_K}(z) = \left(\frac{2}{T}\right)^{\pm\alpha} \frac{(z \mp z_1(\alpha))(z \mp z_2(\alpha))(z \mp z_3(\alpha))}{(z \mp p_1(\alpha))(z \mp p_2(\alpha))(z \mp p_3(\alpha))}, \quad (13)$$

where

$$\begin{cases} p_3(\alpha) = -z_1(\alpha) \\ z_2(\alpha) = 1 - z_1(\alpha) \\ p_2(\alpha) = -z_2(\alpha) \\ z_3(\alpha) = -p_1(\alpha) \end{cases} \quad (14)$$

The pole-zero map of (14) is shown in Fig. 6, which represents a distribution of alternating real poles and zeros.

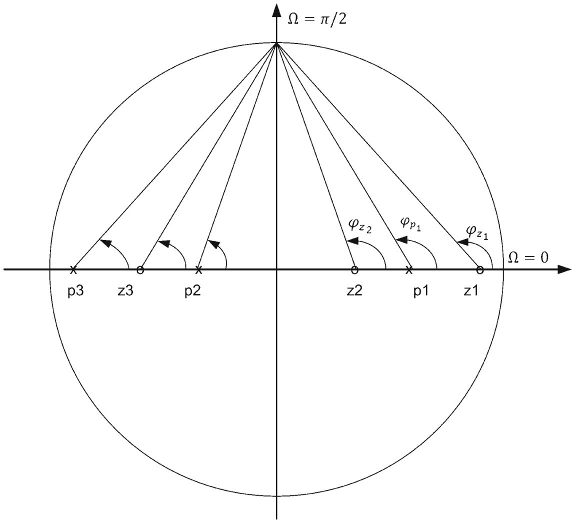

Fig. 6 Pole-zero map of third-order DTO operator

Due to the symmetry of the poles and zeros and since $z_i(\alpha) = -p_i(\alpha)$, $i = 1, 2, 3$, the phase requirement is assumed to meet the phase contribution of the fractional-order operator at the discrete-time frequency $\Omega = \alpha\frac{\pi}{2}$:

$$\left(\varphi_{z_1} + \varphi_{z_2} + \varphi_{z_3}\right) - \left(\varphi_{p_1} + \varphi_{p_2} + \varphi_{p_3}\right) = \alpha\frac{\pi}{2} \tag{15}$$

Substituting (14) into (15) yields

$$z_1 = \max\left(roots\left(z_1^2 - z_1 + q(\alpha)\right)\right), \tag{16}$$

where

$$q(\alpha) = \frac{2 - \alpha(1 + \eta_3)}{1 + \eta_3(1 - \alpha)} \tag{17}$$

and

$$\eta_3 = \tan\left(\alpha\frac{\pi}{4}\right). \tag{18}$$

Hence z_1 is found, the rest of poles and zeros are determined from (17) and (15). For example, for $\alpha = 0.5$ Eq. (19) gives $z_1 = 0.8425$, $z_2 = 0.1575$, and $z_3 = -0.5$, while $p_1 = -z_3$, $p_2 = -z_2$ and $p_3 = -z_1$. Therefore, the third-order DTO that discretizes $s^{0.5}$ for $T = 2$ is given by

$$s^{0.5} \approx H_{3_K}(z) = \frac{z^3 - 0.5z^2 - 0.3673z + 0.06635}{z^3 + 0.5z^2 - 0.3673z - 0.06635}. \tag{19}$$

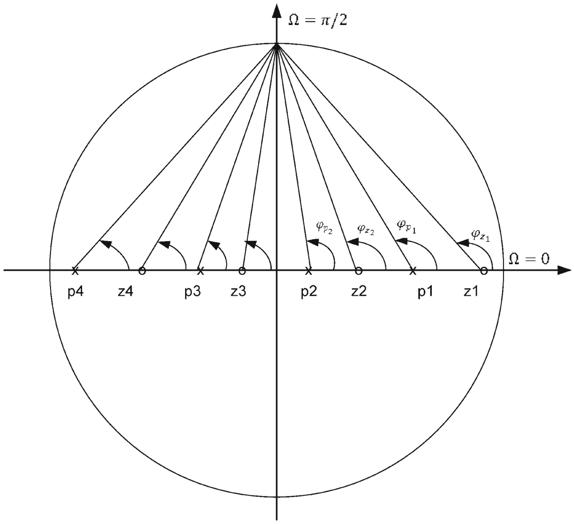

Fig. 7 Pole-zero map of the fourth-order DTO operator

Remark 2 When $\alpha = 1$, then from (17), $q(1) = 0$, and Eq. (16) reduces to $z_1^2 - z_1 = 0$, which yields a nontrivial solution $z_1 = 1$, and the third-order DTO operator given by (13) and (14) for this case reduces to the well-known bilinear transformation $H_{3_K} = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$.

3.4 Fourth-Order Operator

The fourth-order z-transfer function that discretizes the fractional-order operators is similarly developed as the previous three operators and given by the following finite z-transfer function:

$$s^{\pm\alpha} \approx H_{4_K}(z) = \left(\frac{2}{T}\right)^{\pm\alpha} \frac{(z \mp z_1(\alpha))(z \mp z_2(\alpha))(z \mp z_3(\alpha))(z \mp z_4(\alpha))}{(z \mp p_1(\alpha))(z \mp p_2(\alpha))(z \mp p_3(\alpha))(z \mp p_4(\alpha))}, \tag{20}$$

where

$$\begin{cases} p_4(\alpha) = -z_1(\alpha) \\ p_2(\alpha) = 1 - z_1(\alpha) \\ z_3(\alpha) = -p_2(\alpha) \\ z_4(\alpha) = -p_1(\alpha) \\ p_3(\alpha) = -z_2(\alpha) \end{cases} \tag{21}$$

The pole-zero map of (21) is shown in Fig. 7, which also represents a distribution of alternating real poles and zeros of (20).

The phase contribution of (20) is given by

$$\left(\varphi_{z_1} + \varphi_{z_2} + \varphi_{z_3} + \varphi_{z_4}\right) - \left(\varphi_{p_1} + \varphi_{p_2} + \varphi_{p_3} + \varphi_{p_4}\right) = \alpha \frac{\pi}{2} \tag{22}$$

Since there is a symmetry between the poles and zeros as depicted in Fig. 7, one may focus on the phase contribution of the poles and zeros that lie on the positive real axis. By other words, from the symmetry, and without loss of generality, one may conclude from (22), that,

$$\left(\varphi_{z_1} + \varphi_{z_2}\right) - \left(\varphi_{p_1} + \varphi_{p_2}\right) = \alpha \frac{\pi}{4} \tag{23}$$

where

$$\varphi_{z_i} = \pi - \arctan\left(\frac{1}{z_i\left(\alpha\right)}\right),\ \varphi_{p_i} = \pi - \arctan\left(\frac{1}{p_i\left(\alpha\right)}\right),\ i = 1, 2. \tag{24}$$

Assumption 2 Let $p_1(\alpha)$ and $z_2(\alpha)$ lie in the geometric mean of their adjacent zeros and poles, respectively,

$$p_1\left(\alpha\right) = \sqrt{z_1\left(\alpha\right) z_2\left(\alpha\right)}, \tag{25a}$$

$$z_2\left(\alpha\right) = \sqrt{p_1\left(\alpha\right) p_2\left(\alpha\right)}. \tag{25b}$$

Substituting (24) and (25) into (23) yields the following nonlinear function in $z_1(\alpha)$:

$$\begin{aligned} f\left(z\right) = {} & \eta z_1^4 + 2\left(1-\eta\right) z_1^3 - \left(\eta+3\right) z_1^2 + \left(2\eta-1\right) z_1 + \left(\eta+1\right) \\ & + \eta\left[z_1^{\frac{5}{3}}\left(1-z_1\right)^{\frac{1}{3}} + z_1^{\frac{1}{3}}\left(1-z_1\right)^{\frac{5}{3}} - z_1^{\frac{4}{3}}\left(1-z_1\right)^{\frac{2}{3}} - z_1^{\frac{2}{3}}\left(1-z_1\right)^{\frac{4}{3}}\right] \\ & + z_1^{\frac{5}{3}}\left(1-z_1\right)^{\frac{4}{3}} - z_1^{\frac{4}{3}}\left(1-z_1\right)^{\frac{5}{3}} + z_1^{\frac{2}{3}}\left(1-z_1\right)^{\frac{1}{3}} - z_1^{\frac{1}{3}}\left(1-z_1\right)^{\frac{2}{3}} = 0, \end{aligned} \tag{26}$$

where $\eta = \tan\left(\alpha\frac{\pi}{4}\right)$.

Obviously, the nonlinearities in (26) are due to placing the inner pole/zero at the geometric mean of its surrounding zeros/poles. Solving (26) numerically with an accuracy $|f(z)| < \epsilon$, for small $\epsilon > 0$ yields a desired solution $0 < z_1 < 1$. For $\alpha = 0.5$, the fourth- order operator that discretizes $s^{0.5}$ is found to be

$$s^{0.5} \approx H_{4k}\left(z\right) = \left(\frac{2}{T}\right)^{\alpha} \frac{z^4 - 0.5295z^3 - 0.3835z^2 + 0.05843z + 0.01218}{z^4 + 0.5295z^3 - 0.3835z^2 - 0.05843z + 0.01218}. \tag{27}$$

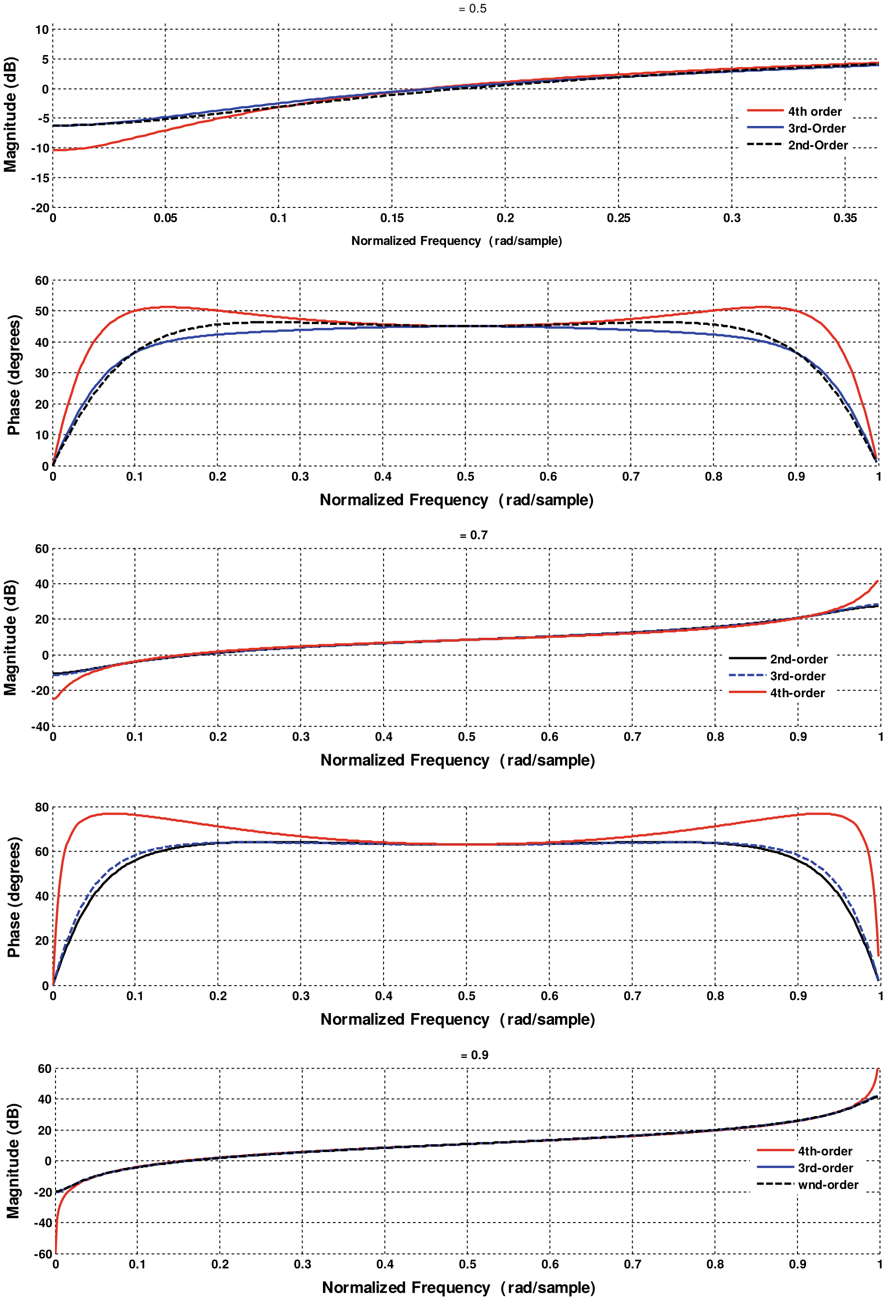

Fig. 8 Frequency response of the second-, third-, and fourth-order operators for $\alpha = 0.5$, $\alpha = 0.7$ and $\alpha = 0.9$

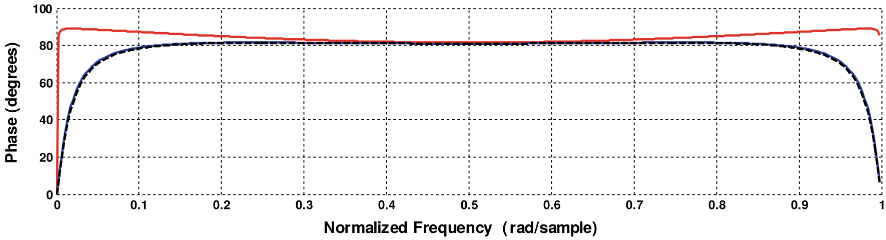

Fig. 8 (continued)

4 Numerical Simulation

Figure 8a–c shows the frequency response of the second-, third-, and the fourth-order operators for $\alpha = 0.5, \alpha = 0.7$, and $\alpha = 0.9$, respectively. As noted, the second-order operator is a good competitor to the third-order one, especially for $\alpha > 0.7$, while

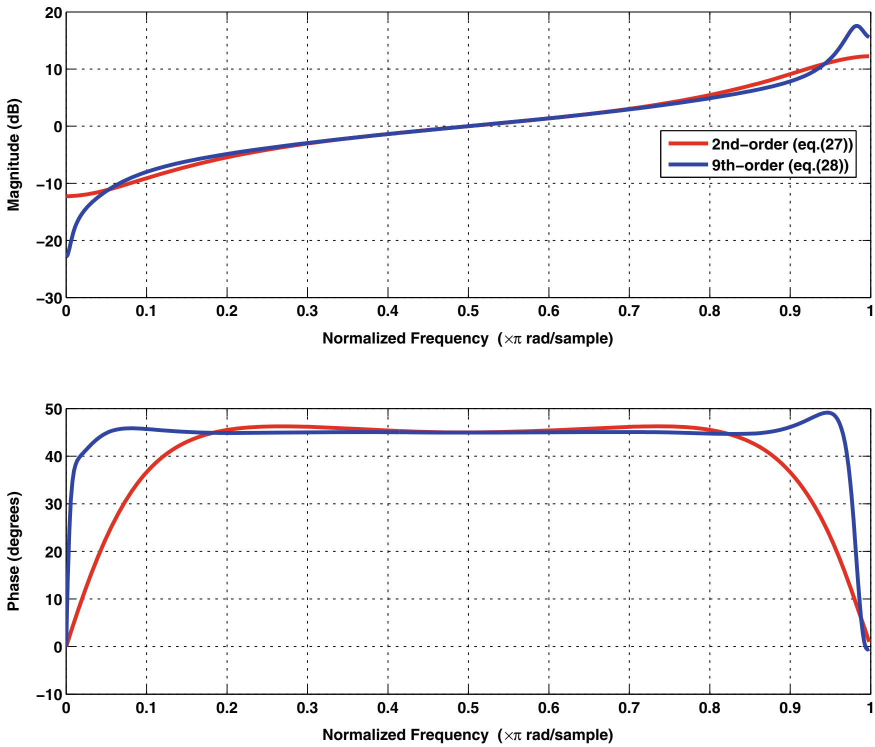

Fig. 9 Frequency response of (27) and (28) for $s^{0.5}$

the fourth-order operator exhibits a much better frequency response with a constant phase at the middle frequency with some overshoot at both ends of the spectrum.

To appreciate the proposed DTO, the frequency response of the second- order operator described by (10) is compared with other forms of DTO reported in [2, 6]. The case when $\alpha = 0.5$ for $T = 0.001$ is taken as a benchmark. Equations (10)–(12) then yield

$$s^{0.5} \approx \frac{44.7214 - 22.0313z^{-1} - 8.4670z^{-2}}{1.0 + 0.4926z^{-1} - 0.1893z^{-2}}. \tag{28}$$

The following ninth-order DTO that discretizes $s^{0.5}$ using the CFE and reported in [6] is investigated against the one given by (28)

$$G_9(z) = 44.72\frac{z^9 - 0.5z^8 - 2z^7 + 0.875z^6 + 1.313z^5 - 0.4688z^4}{z^9 + 0.5z^8 - 2z^7 - 0.875z^6 + 1.313z^5 + 0.4688z^4}\cdots$$
$$\frac{-0.3125z^3 + 0.07813z^2 + 0.01953z - 0.001953}{-0.3125z^3 - 0.07813z^2 + 0.01953z - 0.001953}. \tag{29}$$

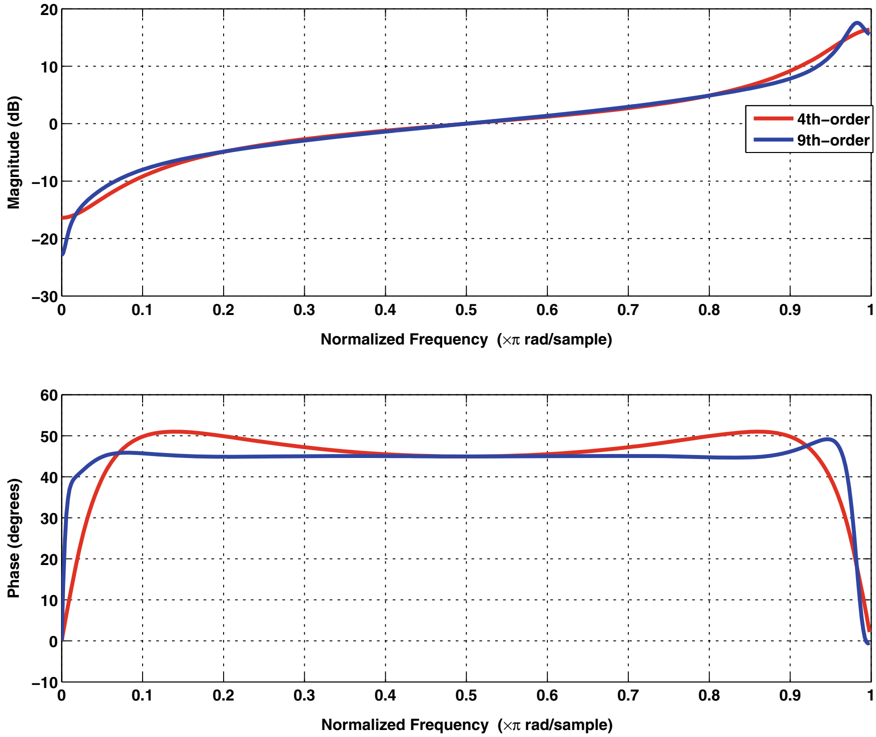

Fig. 10 Comparison between the approximations of (27) and (29) for $s^{0.5}$