Titu Andreescu
Kathy Cordeiro
Alina Andreescu

AWESOME MATH

Teaching Mathematics WITH Problem Based Learning

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"If you are an educator and want to efficiently teach collaborative mathematics to your class, this is the book for you! It is a fun, challenging, and playful way to introduce problem-based learning by providing all the tools and problems necessary to get started."

Michaela Hlasek,
Math Teacher and Combinatorics Instructor,
Awesome Math Summer Program 2017

"Awesome Math, appears to be about math, but really has lessons for education in general and even for re-skilling in the corporate world—an effective approach to educate and prepare the next generation for a YouTube + machine learning world. It gives new meaning to the phrase 'the journey is the reward.' The biggest danger? This book could convince you that math can be fun."

Raj Varadarajan,
Senior Partner and Managing Director, Boston Consulting Group

"This book is a brilliant road map that delights in its own theorem of authenticity and relevance. Full of the philosophical groundwork, expert insights, and plenty of practice problems, *Awesome Math: Teaching Mathematics with Problem-Based Learning* is a must-read for any Math or STEM educator concerned with the relevance and joy of a beautiful and expansive discipline."

Ben Koch,
Co-founder and CEO, Numinds Enrichment

"*Awesome Math* makes a strong case for ditching rote memorization and turning to collaborative problem-solving and mastery-based learning instead. This book is a must-read for parents and educators in all subject areas who wish to develop their students' creative and critical thinking skills."

Jaime Smith,
Founder and CEO of OnlineG3.com

"The book is an excellent source for educators interested in problem-based learning through student-centric approach. Students and teachers will find some 'secrets' of how math circles, math competitions, experiences of other math educators, and even math games along with wonderful and challenging problems can be used for an entire lesson or just as a mini-unit."

Dimitar Grantcharov,
Professor at University of Texas at Arlington

"Through playful problem solving, mastery learning, the three C's, and more, *Awesome Math* challenges the idea of a traditional, teacher-centric classroom. Kathy, Alina, and Titu are visionaries in the field of math education, and their book has sparked new inspiration for strategies that I am eager to utilize in my own math classroom."

Hannah Keener

"*Awesome Math* emphasizes the importance of collaborative problem solving in a classroom setting, featuring interesting and carefully chosen concepts and problems that can be used in a regular classroom and enrichment academic mathematics programs such as math circles or summer camps."

Zvezdelina Stankova,
Teaching Professor of Mathematics at University of California at Berkeley

"This inclusive book speaks in voices of the many. It has the irresistible flow of a well-curated social feed. There are shiny treasures to repost, 'today-I-learned' surprises to ponder, wise checklists to save, heartfelt polemics to debate—and so many kind math friends to meet!"

Dr. Maria Droujkova,
Founding Director of Natural Math

"I believe the most important goal of education is acquiring the ability to learn on your own. This book is mainly aimed at this goal and will help teachers and students improve their logical thinking, making them more independent learners and scholars."

Dr. Krassimir Penev,
Bergen County Academies

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To our awesome community of colleagues, family,
and friends who inspire us daily and made this publication possible.

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– Titu and Alina

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– Kathy

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– Titu, Kathy, and Alina

About the Authors

Dr. Titu Andreescu has been coaching, teaching, and training students and teachers for most of his exemplary career. Starting as a high school mathematics teacher in Romania and later in the United States, Titu became coach and leader of the United States International Mathematics Olympiad team, director of the Mathematical Association of America's AMC tests (American Mathematics Competitions), and an associate professor at University of Texas at Dallas in the Science and Mathematics Education department training mathematics teachers. His passion for problem solving and mathematics teaching has extended to the following noteworthy accomplishments.

AwesomeMath Summer Program is a premier mathematics camp held on the campuses of the University of Texas at Dallas, Cornell University, and the University of Puget Sound. Awesomemath.org

AwesomeMath Academy provides enrichment opportunities for students seeking a strong problem-solving–based curriculum with classes offered in North Texas and online. AwesomeMathacademy.org

AwesomeMath Year-Round is a correspondence-based program that provides students with further opportunities to broaden their mathematical horizons, particularly in those fields from where Olympiad problems are drawn. https://www.awesomemath.org/year-round-program/

XYZ Press (separate business entity affiliated with AwesomeMath) is the publication company that was started in 2008 to more efficiently bring problem-solving books to market. https://www.awesomemath.org/shop/about-xyz-press/

Mathematical Reflections is a free online journal aimed at high school students, undergraduates, and everyone interested in mathematics. https://www.awesomemath.org/mathematical-reflections/

Purple Comet! Math Meet has been "fun and free since 2003." This annual, international, online, team mathematics competition is designed for middle and high school students. http://purplecomet.org/

Metroplex Math Circle is a free program that was designed to attract gifted students and educators in the Dallas/Fort Worth area to provide an avenue outside the standard curriculum to develop their mathematical and problem-solving skills. Further, the circle offers access to math competitions for students (in 2017–2018 school year, approximately 150 students participated in the AMC 8, 10, 12, and AIME competitions) each year who may not be able to participate in their schools. Metroplexmathcircle.org.

The **Math Rocks** curriculum, developed by Dr. Andreescu in 2008–2010, is still going strong in the Plano, Texas, school district for elementary and middle school students. The success of the curriculum has resulted in its extension to over 45 public elementary and 15 middle schools. http://k-12.pisd.edu/currinst/elemen/math/MathRocksInformation.pdf.

For **Kathy Cordeiro**, innovation, problem solving, and team collaboration have been the leading constants throughout her varied career. A degree in communications, coupled with an MBA, has given Kathy a unique skill set to create and market customized education initiatives, in business and/or academia, which allows her customers and students to reach their goals and realize success. Kathy began her own enrichment school, Eudaimonia Academy (2006–2012), where she coached math teams, taught a philosophy/creative writing course, and co-led speech and debate teams.

Kathy is the marketing and communications director for the AwesomeMath organization. In this role, she has had various speaking engagements as well as managed multiple communication channels online, where she discusses mathematics education with parents, teachers, students, and businesses.

Beyond being connected with multiple math groups, Kathy is also a part of a network that includes parents, teachers, and students, such as

- AwesomeMath parents, students, alumni
- Purple Comet supervisors/teachers
- Davidson Young Scholars, parents, and alumni
- Mathematics organizations
- Homeschool groups

Alina Andreescu was born and raised in Romania, at a time when mathematics education was exceptionally strong. She participated successfully in Romanian mathematics competitions. She completed her finance degree in the United States and later obtained an M.A. in management with emphasis on leadership. Alina was never afraid of change and challenges, embarking on lifetime journeys from moving to the United States to becoming a successful cofounder and leader of the AwesomeMath and XYZ Press organizations.

As operations director of the AwesomeMath programs for the past 12 years, Alina has been integral in every facet of creating the opportunities/resources that fulfill the mission of providing enriching experiences in mathematics for intellectually curious learners. She fosters a community of staff, students, and instructors that values critical thinking, creativity, passionate problem solving, and lifetime mathematical learning. Since the AwesomeMath community is international, she must meld a diverse background of individuals into a thriving learning environment.

Introduction

In writing this book, we hope to lead you to what you already know: that problem-based learning is an effective method for raising tomorrow's thinkers by collaborating over interesting and relevant problems. Through the AwesomeMath Summer Program,[1] the inspiration for this book, we've had the privilege to work with thousands of the brightest minds from around the globe for over 10 years. We've seen first-hand the leaps in skills, growth of curiosity, and joy of problem solving that arises when individuals are immersed in a kind, collaborative, and challenging environment where students create positive life-long memories and form valuable friendships.

So, how do you raise out-of-the-box thinkers in a check-the-box world? Teaching is an opportunity to inspire and guide, but that means diverging from the conformity required in today's education system and allowing students to take intellectual risks and, yes, fail. The outdated criterion of identifying top students through grades is flawed; it's evaluating someone's worth based on an outcome and not the process, which sets up situations where students avoid intellectual risks so they can maximize grades. Students aren't learning how to *think, work together,* or *find challenging opportunities*.

Furthermore, they aren't being prepared to face the current challenges in today's workforce, which values innovation, leadership, collaboration, resilience, and critical thinking. We need students who can do more than solve mere exercises for a check mark; they need to be able to tackle difficult problems and also be able to notice problems worthy of solving by seeking patterns, reframing information, and asking the right questions. Students are all different and have different strengths to offer in every setting. We need to value them for who they are with a student-centric approach as opposed to evaluating them with standardized conformity and false metrics.

When Randy Pausch gave his Last Lecture at Carnegie Mellon University,[2] he explained that the moment someone lowers their expectations for what you can accomplish, they've stopped caring about you. Students need to have challenges that we, as educators, know they can overcome and master. When we allow students to work toward mastery instead of grades, then the journey becomes about the process and not the outcome. This approach,

however, requires facilitators, helpers, and guides along the way so that each student can recognize their value and be their best version.

Problem-based learning approaches education with a deep respect for the value, abilities, and strengths of each student by raising expectations beyond the standard and providing guidance in a supportive environment.

The main goals of this book are as follows:

- To show that a problem-based curriculum is an effective way to teach mathematics to students of all levels and backgrounds and prepares them to be creative thinkers in an ever-changing world.
- To train educators on how to employ a problem-based curriculum in their classrooms by creating a collaborative, kind, and engaging environment where each student can be guided to be their best version.
- To provide the curriculum plans and interesting problems that allow educators to successfully train their students to think with a problem-solving mindset.

Here are the top five characteristics of a problem-based learning curriculum as detailed in this book:

1. It is *student-centric* as opposed to teacher-centric. Lectures are kept as brief as possible and students are the drivers in the process while teachers are the facilitators of learning.
2. It is *highly collaborative* because when you engage in the trade of ideas, everyone improves.
3. It is *scalable* so that problems are in a range to reach all levels of students and promote their individualized growth.
4. It relies heavily on *range, rigor, and resilience* to encourage curiosity, critical thinking, and creativity.
5. It is *FUN*! If the teacher and students have the correct mindset of playful mathematics and growth in a supportive environment, then they look forward to the lessons and don't resist extra challenge.

A typical adult gorilla is 5 ft tall and weighs 400 pounds. If King Kong is 20 ft tall, how much does he weigh approximately? This problem is about basic measurements; however, many students get it wrong by rushing to provide an answer without *thinking* about what really is being asked. Many will quickly respond, "1,600 pounds," which is completely illogical if they have a sense of weight.

Only 1,600 pounds for such an enormous gorilla? A typical black and white cow weighs that much! A hippopotamus can weigh up to 4,000 pounds. While this is a simple exercise rather than a complicated problem, it illustrates a larger problem in mathematical thinking.

When mathematics pedagogy is reduced to checking a box, guessing an answer, or completing repetitive exercises, then students are rewarded for quickly reaching a solution over thoughtfully working through a problem.

Solution

You know the height difference (one dimension), but you need to translate that to the difference in volume (three dimensions) of the gorillas. If King Kong is four times bigger in each of the three dimensions (4 times taller, 4 times wider, and 4 times longer) than the average gorilla, that equals 400 multiplied by 4 multiplied by 4 multiplied by 4, i.e., $400 \times (20/5)^3 = 25\,600$ pounds.

Notes

1. https://www.awesomemath.org/what-is-awesomemath
2. Randy Pausch, *Last Lecture: Achieving Your Childhood Dreams*, Carnegie Mellon University, December 20, 2007, https://www.youtube.com/watch?v=ji5_MqicxSo.

SECTION I

Why Problem Solving?

In this section:

- Rewards for a Problem-Based Approach: Range, Rigor, and Resilience
- Maximize Learning: Relevance, Authenticity, and Usefulness
- Creating a Math Learning Environment
- What Is the Telos?
- Gains and Pains with a Problem-Based Curriculum

Today's kids are busier than ever! Juggling their schedules inside and outside of school requires major planning, and as a result, enticing them to focus in a mathematics class can be difficult. That is not to say that they are incapable of deep thought, but rather, asks how mathematics can compete with all the other distractions that life throws their way. What makes activities such as sports or video games so much more appealing? How can we construct a mathematics environment so that students are engaged with the subject and work together to achieve a superior understanding for mathematics?

The common thread is playful problem solving. *Play* is an integral part of life. Even as adults, we love to play and compete and solve problems with friends. You can challenge yourself to move up levels and share your experiences with peers – plus, there is no fear of losing, whereas in mathematics, there is fear. Fear of appearing stupid, fear that if you are slow to understand that you just aren't *good at math,* fear that doing poorly in math means you won't get into college. We need to erase that fear and help kids take thought risks with problem solving.

Let's say that Steve is playing a video game with a friend and loses a boss battle. Will he give up and say, "Well, I guess I'm bad at video games, so I'll stop playing"? Of course not. He'll try a different strategy or ask his friend for advice or go online and watch YouTube videos. What makes the difference between perseverance and giving up? Mathematics education *can* be just as playful and allow students to compete by solving meaningful problems while working as a team, but that means the stakes need to change, and instead of teachers as judges, saying a student's individual work is good or bad, is missing steps, is B work and not A work, they need to shift into being coaches who guide their students to being the best versions of themselves.

While teachers want each student to excel, in reality, great teachers work on improving the abilities of their entire class every day, spotting areas that are weak, celebrating strengths, and being a cohesive unit. When all of those areas come together, then success will happen. *Children are not outcomes* and need to be guided by a great educator to think critically and creatively.

Currently, math education in middle and high schools is a series of exercises with easily obtained answers, e.g., find the perimeter of a square, training students to do what a computer can do better. Problem solving goes much deeper and taps into what makes us human, namely, multiple creative approaches with a string of steps to solving meaningful and interesting problems. It takes the shift away from *outcome*-based learning (grades/test scores, rank, grade point average [GPA]), which is a fixed-mindset approach, to learning for *mastery*, where students challenge themselves to improve every day (growth mindset).

What exactly *is* problem solving? Even mathematicians and researchers haven't come up with a definitive answer, but in this book, we believe problem solving has the following characteristics:

- Problems take several steps to solve.
- More than one approach can be used to arrive at a complete solution.
- Good problems lend themselves well to collaboration with peers.
- Meaningful problem solving promotes flexibility of thought and innovation.
- Mathematical learning and reasoning are integral to the process of problem solving.
- Problem solving is about working around obstacles to understand the unknown.

Problem solving is the strategy, and math competitions are the vehicle to train your math class to be stellar thinkers. Since the current school curriculum delivers a narrow path of mathematics knowledge, climbing aboard the math competition train will expose students to a greater array of topics, including *discrete mathematics*, an area that incorporates both number theory and combinatorics (counting and probability). Discrete math, along with finite mathematics and linear algebra, are necessary to work in the modern world

of computing. Mathematical modeling and a strong understanding of statistics is also critical. The level of deep thinking required to solve hard problems in the areas of discrete mathematics, algebra, geometry, and the areas in between (e.g., geometric inequalities), transfers to future careers in STEM (science, technology, engineering, mathematics) fields, and beyond. Mathematics competitions provide exposure to all these topics while working with peers to solve challenging problems.

Just as every football player cannot be the quarterback, not every student is going to excel in the same way with mathematics competitions, but this brings us back to the focus being placed on the process and not the outcomes. The reason to engage in math competitions is to have something to work toward where each student can get a little better every day and be motivated in a collaborative and supportive environment. Some students may enjoy working through lots of different types of problems while others may prefer to look at the methods employed and want to write their own problems based on their discoveries. Every type of student can play an important role in your mathematics class, and as the teacher, you want to look at every student as a collection of strengths as opposed to a collection of weaknesses that need to be fixed.

What Were Your School Experiences Like in Your Country That Contributed to Your Love of Problem Solving?

Dr. Branislav Kisačanin: When I was growing up in former Yugoslavia, during the 1980s, math and physics competitions were well organized and students were encouraged to participate. Competitions were held at school, city, regional, and national level, and from there teams were sent to the International Mathematics Olympiad (IMO) and the International Physics Olympiad (IPhO). Except for the school level, all competitions involved some kind of travel with like-minded kids, and that was a big part of it all for me. Thanks to these competitions, I met many life-long friends (my fellow students and my future college professors) and visited wonderful places in former Yugoslavia: Postojna cave and Portoroz in Slovenia, Sarajevo in Bosnia, Decani, with its famous fourteenth-century monastery, and the Danube's Djerdap Gorge in Serbia.

Regardless of the role a student chooses, all students grow their skills faster when collaborating toward a common goal than they would on their own, because when you engage in the trade of ideas, everyone improves.

The learning environment for the game is critical to bringing out the best in the players and the rewards are *range*, *rigor*, and *resilience*.

CHAPTER 1

Rewards for Problem-Based Approach: Range, Rigor, and Resilience

Range Ignites Curiosity

As educators, we understand the importance of depth and breadth in learning. For beginning piano players, listening to a concert pianist perform can ignite curiosity and inspire them to practice more. In mathematics, there seems to be a reticence to hear the symphony for fear that it will be too much, too soon, and by limiting the range, we limit curiosity and growth.

Our AwesomeMath Enrichment programs are filled with students whose schools placed a ceiling on their mathematics education and they are seeking outside resources. Sometimes, this is because the school is concerned that if students are accelerated too quickly, they won't have the maturity to truly understand what is being taught. Other times, the school is concerned that if students move two or three levels ahead, by the time they are seniors they will have run out of classes to take. Another common reason our parents provide is that the school thinks it will be *too much* information for the student at too young an age and will spoil their performance in future classes. With a problem-based curriculum, there is no ceiling on learning and there is ample depth and breadth of subjects to keep students challenged throughout their lifetime. The real danger of not giving students adequate challenge and range to satiate their curiosity is that they will turn off on mathematics and learning altogether. Students need to hear the beauty and art that is mathematics to kindle their joy of learning.

Why Is a Collaborative Problem-Based Approach Worthwhile?

Dr. Emily Herzig: Collaboration in the classroom has many benefits. Research has demonstrated that active learning improves performance on exams, and the effect is especially large for disadvantaged students. Currently, education is not equitably accessible to all students, with students from underserved populations and first-generation college students in particular facing additional obstacles to entering, navigating, and excelling in higher education. Thus, collaborative learning in the classroom could be key to closing the achievement gap and allowing capable but underprepared students to reach greater success in math.

Furthermore, a collaborative and problem-based approach gives younger students a more accurate impression of what higher-level math entails. Students too often carry the belief that success in math is based in rote memorization and drilling problems. While those skills are certainly useful for efficiently carrying out the basic mechanics of solving problems, it is equally important that students are able to formulate and interpret more complex problems, and work with their colleagues to develop and execute problem-solving strategies. Arguably, this process is also what makes math such an enticing subject. A focus on collaborative problem solving is a great way to attract students to and prepare them for careers in math.

Even the terms used for learning piano and learning mathematics are different: Students ***play*** piano and ***work*** on math problems. There needs to be a fundamental shift in approach and exposure to a range of problems that are harder and more interesting so that students can see where math can take them. So much of math education today is about *waiting*:

- Wait until high school, and then what you've been learning in middle school will be useful.
- Wait until college, and then what you've been learning in high school will be useful.
- Wait until you learn topic x before you can see the beauty of topic y or z or beyond.
- Wait until you learn a subject, like geometry, in isolation before you have the ability to learn how it connects and contributes to other areas such as algebra, engineering, art, science, etc.

And so on. . . .

A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

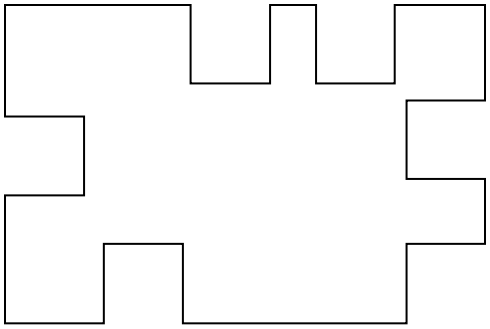
G.H. Hardy

Students need to *work together* to weave a pattern of ideas with what they know and can add more as their knowledge base grows. To fully appreciate the full tapestry of mathematics is not by adding one color of thread at a time, but weaving a picture with various thread colors within a group of learners who are just as excited by the beauty as you are.

Collaborative problem-based learning holds the key to unlocking abilities and mathematical growth. The collaboration nurtures interpersonal skills, leadership traits, and conflict resolution, while providing interesting problems gives students a common goal to work toward where they can connect and share ideas. Students can see the fun and beauty in mathematics, start to play, and see where math can take them long term.

A fun problem that works well in a group is the following:

The diagram shows a polygon made by removing six 2×2 squares from the sides of an 8×12 rectangle. Find the perimeter of this polygon.



The answer is 60. The square removed from the lower-right corner of the rectangle does not change the perimeter of the polygon, but when each of the other five squares is removed, the perimeter is increased by 4. Thus, the perimeter of the polygon is $2 \times 8 + 2 \times 12 + 5 \times 4 = 16 + 24 + 20 = 60$.[1]

Students are allowed to be transported by a musical symphony before understanding individual notes, so why not provide the symphony of mathematics and introduce students to its wonders and challenges? Parents and educators will read books beyond a student's personal reading level so that they can hear the richness of language and be exposed to more intricate sentence structure. There are so many wonderful places an enriching math education can take you.

When educating young math students, you can let them know that they are the captains of their ship, but as their navigator, you can guide them to really interesting destinations and expose them to a wider range of mathematics.

Following are some ideas for increasing range in a mathematics program:

Logic problems. Thinking through these problems primes the brain for mathematical reasoning.

Combinatorics. This discrete mathematics field involves counting and probability.

The mathematics of science. Solving, for example, real-world physics problems adds range and connectivity across academic disciplines.

Mind benders and puzzles. These are a fun way to introduce mathematical concepts, such as magic squares.

Game theory. Utilize mathematical modeling to understand the strategies employed by rational game players (decision makers).

Computational linguistics. Apply computational analysis to language and speech for linguistic phenomena.

And so much more!