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Variational Principles of Continuum Mechanics

II. Applications

With 33 Figures

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Contents - II. Applications

Contents - I. Fundamentals

Part I Fundamentals

Part III
Some Applications of Variational Methods to Development of Continuum Mechanics Models

In this part, the application of the variational approach to constructing the governing equations will be considered for several areas of continuum mechanics. The first two chapters are concerned with one of the most beautiful areas of solid mechanics – the theory of elastic shells and beams. In a sense, this is a physical theory of surfaces and curves in three-dimensional space. It is attractive by its exceptional elegance, the profound relations with geometry and the astonishing diversity and complexity of the behavior of the objects it describes. The extent of the book allows us to discuss only the derivation of the classical and refned shell theories and the classical beam theory from the three-dimensional elasticity theory, and a case when the classical shell theory does not work: theory of hard-skin plates and shells. The next chapter gives a review of stochastic variational problems. Then we turn to consideration of homogenization, one of the central problems of continuum mechanics. This is followed by several other examples of applications of variational methods to construction of continuum models: shallow water theory, theory of heterogeneous mixtures, a model of granular media and a turbulence model. The discussion of each theory is concluded by constructing the governing equations, and the issues related to the features of these equations are not discussed. The only exception is the homogenization theory where we consider the exact solutions of the cell problem which are found by means of the variational methods.

The chapters can be read independently.

Chapter 14
Theory of Elastic Plates and Shells

Consider the surface $\mathring{\Omega}$ in three-dimensional space and, at each point on the surface, erect a segment of length h directed along the normal to the surface; the centers of the segments are on $\mathring{\Omega}$. The segments cover some three-dimensional region, $\mathring{V}$ (Fig. 14.1). If h is much smaller than the minimum curvature radius of the surface $\mathring{\Omega}$, R, and the characteristic size of the surface $\mathring{\Omega}$, L,

$$\frac{h}{R} \ll 1, \quad \frac{h}{L} \ll 1,$$

then an elastic body occupying the region $\mathring{V}$ in its undeformed state is called an elastic shell. If $\mathring{\Omega}$ is a plane, i.e. $R = \infty$, then there is only one small parameter, h/L.

One can expect that the deformation of the elastic shells can be approximately described by functions which depend only on the two surface coordinates and time. The problem of constructing the shell theory consists of the proper choice for these functions, derivation of the governing equations for these functions and establishing the link between the two-dimensional characteristics and the three-dimensional stress state. These issues are addressed in this chapter. The last three sections of the chapter are concerned with the theory of laminated plates and shells, and, in

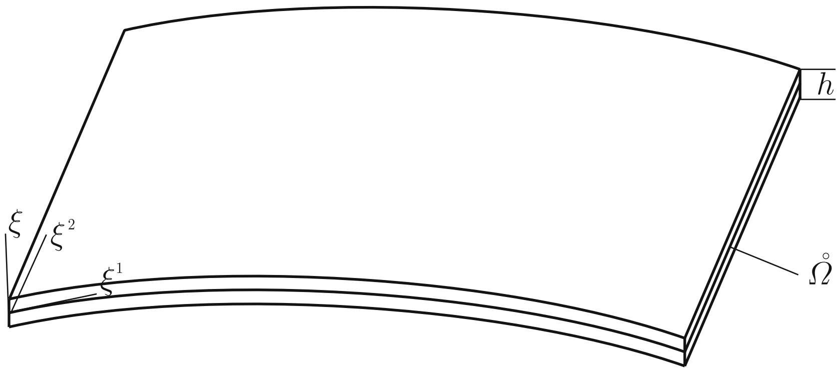

Fig. 14.1 Notation for shells

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particular, hard-skin plates and shells. In case of a hard skin an additional small parameter, the ratio of elastic moduli of the core and the skin, comes into play and changes the leading asymptotics. We begin with introduction of necessary facts from theory of surfaces.

14.1 Preliminaries from Geometry of Surfaces

The surface tensors. Consider in three-dimensional space a two-dimensional surface Ω, defined by the parametric equations

$$x^i = r^i\left(\xi^\alpha\right), \tag{14.1}$$

where ξ^α are the surface parameters and the small Greek indices $\alpha, \beta, \gamma, \ldots$ run values $1, 2$.

The parametric equations (14.1) contain more information than just the definition of the surface because they distinguish the individual points on the surface marked by the parameters ξ^α. The specific choice of the parameters on the surface is not essential, and it appears to be necessary to consider the invariance of all relationships with respect to the transformation group of the surface coordinates, $\xi^\alpha \to \xi'^\alpha$,

$$\xi'^\alpha = \xi'^\alpha\left(\xi^\beta\right). \tag{14.2}$$

The vectors and tensors with respect to this group are called the surface vectors and tensors, and the corresponding tensor indices the surface indices.

The tangent vectors and the metric tensor. The derivatives, $r^i_\alpha = \partial r^i/\partial\xi^\alpha$, are the components of two vectors in the observer's frame, r^i_1 and r^i_2. The three-dimensional vectors, r^i_1 and r^i_2, are tangential to the surface Ω. At the same time, for each fixed index i, they form a surface vector with respect to index α.

The observer's metrics, allowing one to measure distances in three-dimensional space, induces the surface intrinsic metrics on Ω, which determines the distances between the points of the surface: the squared distance, ds^2, between the points ξ^α and $\xi^\alpha + d\xi^\alpha$,

$$ds^2 = g_{ij}\left(r^i\left(\xi^\alpha + d\xi^\alpha\right) - r^i\left(\xi^\alpha\right)\right)\left(r^j\left(\xi^\alpha + d\xi^\alpha\right) - r^j\left(\xi^\alpha\right)\right) = g_{ij} r^i_\alpha d\xi^\alpha r^j_\beta d\xi^\beta,$$

can be written as

$$ds^2 = a_{\alpha\beta} d\xi^\alpha d\xi^\beta,$$

where the tensor

$$a_{\alpha\beta} = g_{ij} r^i_\alpha r^j_\beta, \tag{14.3}$$

is called the surface metric tensor or the first quadratic form of the surface.

The contravariant components of the surface metric tensor, $a^{\alpha\beta}$, are introduced as the solutions of the system of linear equations

$$a^{\alpha\beta} a_{\gamma\beta} = \delta^{\alpha}_{\gamma}.$$

According to (3.20),

$$a^{\alpha\beta} = \frac{1}{a} \frac{\partial a}{\partial a_{\alpha\beta}}, \tag{14.4}$$

where a is the determinant of the matrix $\left\| a_{\alpha\beta} \right\|$.

Using the space and the surface metrics, we can juggle the space and the surface indices; for example, for r^i_α we have

$$r_{i\alpha} \equiv g_{ij} r^j_\alpha, \quad r^\alpha_i \equiv a^{\alpha\beta} r_{i\beta}. \tag{14.5}$$

The Levi-Civita tensor. The two-dimensional Levi-Civita tensor is defined as

$$\varepsilon_{\alpha\beta} = \sqrt{a} e_{\alpha\beta}$$

with $e_{\alpha\beta}$ being the two-dimensional Levi-Civita symbol ($e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$). By definition, $e_{\alpha\beta} = e^{\alpha\beta}$.

One can check that

$$\varepsilon^{\alpha\beta} \equiv a^{\alpha\alpha'} a^{\beta\beta'} \varepsilon_{\alpha'\beta'} = \frac{1}{\sqrt{a}} e_{\alpha\beta}.$$

Note the identities

$$\varepsilon^{\alpha\beta} \varepsilon_{\gamma\beta} = \delta^{\alpha}_{\gamma}, \quad e^{\alpha\beta} e_{\gamma\beta} = \delta^{\alpha}_{\gamma}. \tag{14.6}$$

The normal vector. Consider a vector with the components

$$n_i = \frac{1}{\sqrt{a}} \varepsilon_{ijk} r^j_1 r^k_2 = \frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon_{ijk} r^j_\alpha r^k_\beta. \tag{14.7}$$

Here ε_{ijk} is the three-dimensional Levi-Civita tensor (see Sect. 3.1).

The vector, n_i, is orthogonal to the surface, since, due to (14.7),

$$n_i r^i_\alpha = 0. \tag{14.8}$$

Let us show that the vector n_i has the unit length

$$g^{ij} n_i n_j = 1. \tag{14.9}$$

From (14.7) and (3.19),

$$g^{ij}n_in_j = \frac{1}{a}g^{ij}\varepsilon_{ikl}r_1^k r_2^l \varepsilon_{jmn}r_1^m r_2^n =$$
$$= \frac{1}{a}\left(g_{km}g_{ln} - g_{kn}g_{lm}\right)r_1^k r_2^l r_1^m r_2^n = \frac{1}{a}\left(a_{11}a_{22} - a_{12}^2\right) = 1.$$

So, n_i are the components of the unit vector normal to the surface Ω indeed. It follows from (14.7) and (3.19) that

$$n_i\varepsilon^{ijk} = \varepsilon^{\alpha\beta}r_\alpha^j r_\beta^k. \tag{14.10}$$

Note also the relation

$$\varepsilon_{ijk}r_\alpha^j r_\beta^k = n_i\varepsilon_{\alpha\beta}. \tag{14.11}$$

It can be obtained from the following reasoning. The scalar product of $\varepsilon_{ijk}r_\alpha^j r_\beta^k$ with r_1^i and r_2^i is zero. Hence, for each fixed α and β, $\varepsilon_{ijk}r_\alpha^j r_\beta^k$ is proportional to n_i, and one can write

$$\varepsilon_{ijk}r_\alpha^j r_\beta^k = c_{\alpha\beta}n_i. \tag{14.12}$$

Tensor $c_{\alpha\beta}$ must be antisymmetric. Therefore, $c_{\alpha\beta} = c\varepsilon_{\alpha\beta}$. Contracting (14.12) with $n_i\varepsilon^{\alpha\beta}$ and using (14.7), we obtain the value of $c : c = 1$.

The area element. The area element, $d\omega$, of the surface, Ω, is the area of the infinitesimally small parallelogram with the sides, $r_1^i d\xi^1$ and $r_2^i d\xi^2$. It is equal to the length of the vector product of these two vectors, $\varepsilon_{ijk}r_1^j d\xi^1 r_2^k d\xi^2$. The vector product, according to (14.7), is equal to $n_i\sqrt{a}d\xi^1 d\xi^2$. Since the normal vector, n_i, has the unit length,

$$d\omega = \sqrt{a}d\xi^1 d\xi^2. \tag{14.13}$$

The surface in the initial state. Let the position of the surface Ω change with time and be given by the functions $x^i = r^i(\xi^\alpha, t)$. The position of the surface Ω at the initial instant, t_0, is denoted by $\mathring{\Omega}$, and all the other quantities in the initial state will be furnished with the symbol $^\circ$. In particular,

$$r^i(\xi^\alpha, t_0) \equiv \mathring{r}^i(\xi^\alpha), \quad \mathring{r}_\alpha^i \equiv \frac{\partial \mathring{r}^i}{\partial \xi^\alpha}, \quad \mathring{a}_{\alpha\beta} = g_{ij}\mathring{r}_\alpha^i \mathring{r}_\beta^j,$$
$$\mathring{a} = \det\left\|\mathring{a}_{\alpha\beta}\right\|, \quad \mathring{a}^{\alpha\beta} = \frac{1}{\mathring{a}}\frac{\partial \mathring{a}}{\mathring{a}_{\alpha\beta}}, \quad \mathring{n}_i = \frac{1}{\sqrt{\mathring{a}}}\varepsilon_{ijk}\mathring{r}_1^j \mathring{r}_2^k,$$
$$\mathring{r}_{i\alpha} = g_{ij}\mathring{r}_\alpha^i, \quad \mathring{r}_i^\alpha = \mathring{a}^{\alpha\beta}\mathring{r}_{i\beta}. \tag{14.14}$$

The decomposition of Kronecker's delta. The following identity holds:

$$r_\alpha^i r_j^\alpha + n^i n_j = \delta_j^i. \tag{14.15}$$

In order to check that, it is sufficient to project (14.15) onto the tangent vectors and the normal vector (contract (14.15) with r^j_α and n^j) and inspect that the resulting equations are identities.[1]

The decomposition of Kronecker's delta is used in constructing the projections onto the tangent plane and the normal direction. For example, the vector with components T^i can be represented by the sum of a vector tangent to the plane and a normal vector,

$$T^i = T^i \delta^i_j = T^j r^i_\alpha r^\alpha_j + T^j n_j n^i = T^\alpha r^i_\alpha + T n^i,$$

where $T^\alpha = T^i r^\alpha_i$ is a surface vector, $T^\alpha r^i_\alpha$ is the vector tangent to the surface (i.e. $T^\alpha r^i_\alpha n_i = 0$), and $T = T^j n_j$ is a scalar. Similarly, the tensor of the second order can be written as

$$\begin{aligned} T^{ij} &= T^{kl}\delta^i_k \delta^j_l = T^{kl}\left(r^i_\alpha r^\alpha_k + n^i n_k\right)\left(r^j_\beta r^\beta_l + n^j n_l\right) = \\ &= T^{\alpha\beta} r^i_\alpha r^j_\beta + T^\alpha_1 r^i_\alpha n^i + T^\alpha_2 r^j_\alpha n^i + T n^i n^j, \end{aligned} \tag{14.16}$$

where $T^{\alpha\beta} = T^{kl} r^\alpha_k r^\beta_l$ is the surface tensor, $T^\alpha_1 = T^{kl} r^\alpha_k n_l$, $T^\alpha_2 = T^{kl} n_k r^\alpha_l$ are the surface vectors which coincide in the case of a symmetric tensor T^{ij}, and $T = T^{ij} n_i n_j$ is a scalar. The first term of the sum (14.16) "lies in the tangent plane" to the surface in the sense that it is orthogonal to the normal vector with respect to both indices, the second term is orthogonal to the normal vector with respect to index i, the third term – with respect to index j, and the fourth term is "orthogonal to the tangent plane" (its contraction with the tangent vectors is equal to zero).

The decomposition of Kronecker's delta is also used in the decomposition of the gradient along the tangent and the normal directions,

$$\frac{\partial}{\partial x^i} = \delta^j_i \frac{\partial}{\partial x^j} = \left(r^j_\alpha r^\alpha_i + n^j n_i\right)\frac{\partial}{\partial x^j} = r^\alpha_i \frac{\partial}{\partial \xi^\alpha} + n_i \frac{\partial}{\partial n}. \tag{14.17}$$

[1] The decomposition of the Kronecker's delta (14.15) uses the space metrics. Actually, such decomposition does not need metrics and can be done without the use of the metric properties. Indeed, consider on the surface Ω a vector field, r^i_3, which is not tangent to Ω at any point. This means that the determinant, r, of the matrix with the components r^i_1, r^i_2 and r^i_3 is not zero. Let us define three vector fields g^1_i, g^2_i, g^3_i by the system of linear equations

$$r^i_\alpha g^\alpha_j + r^i_3 g^3_j = \delta^i_j. \tag{14.18}$$

Since $r \neq 0$, the solution of the system of equations (14.18) exists, and it is unique. Equation (14.18) is the sought decomposition of the Kronecker's delta. Note that vector g^3_i is normal to Ω in the sense that

$$g^3_i r^i_\alpha = 0.$$

Indeed,

$$g^3_i r^i_\alpha = \left(\frac{1}{r}\frac{\partial r}{\partial r^i_3}\right) r^i_\alpha = \left(\frac{1}{r} e_{ijk} r^j_1 r^k_2\right) r^i_\alpha \equiv 0.$$

Here $\partial/\partial n \equiv n^j \partial/\partial x^j$is the derivative along the normal vector, while $\partial/\partial \xi^a \equiv r^j_\alpha \partial/\partial x^j$ is the derivative along the surface: for any function $\varphi\left(x^i\right)$ considered on the surface, $\varphi\left(x^i\right) = \varphi\left(r^i\left(\xi^\alpha\right)\right)$,

$$r^i_\alpha \frac{\partial \varphi\left(x^i\right)}{\partial x^i} = \frac{\partial r^i\left(\xi^\alpha\right)}{\partial \xi^\alpha}\frac{\partial \varphi\left(r^i\left(\xi^\alpha\right)\right)}{\partial r^i} = \frac{\partial}{\partial \xi^\alpha}\varphi\left(r^i\left(\xi^\alpha\right)\right).$$

The decomposition of the Kronecker's delta in terms of the initial state,

$$\mathring{r}^i_\alpha \mathring{r}^\alpha_j + \mathring{n}^i \mathring{n}_j = \delta^i_j,$$

yields similar relations.

Two covariant derivatives. In the same coordinates system, ξ^α, we have two metric tensors, $a_{\alpha\beta}$ and $\mathring{a}_{\alpha\beta}$. We introduce two Christoffel's symbols: for the surface metrics, $a_{\alpha\beta}$,

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}a^{\gamma\delta}\left(a_{\alpha\delta,\beta} + a_{\beta\delta,\alpha} - a_{\alpha\beta,\delta}\right), \tag{14.19}$$

and the surface metrics, $\mathring{a}_{\alpha\beta}$,

$$\mathring{\Gamma}^\gamma_{\alpha\beta} = \frac{1}{2}\mathring{a}^{\gamma\delta}\left(\mathring{a}_{\alpha\delta,\beta} + \mathring{a}_{\beta\delta,\alpha} - \mathring{a}_{\alpha\beta,\delta}\right). \tag{14.20}$$

The comma before a Greek index, α, denotes partial derivative with respect to ξ^α. The corresponding covariant derivatives are denoted by the bar and the semicolon in indices. For example, for a surface vector, T^α,

$$T^\alpha_{|\beta} = \frac{\partial T^\alpha}{\partial \xi^\beta} + \Gamma^\alpha_{\lambda\beta}T^\lambda, \quad T^\alpha_{;\beta} = \frac{\partial T^\alpha}{\partial \xi^\beta} + \mathring{\Gamma}^\alpha_{\lambda\beta}T^\lambda,$$

for a surface tensor, $T_{\alpha\beta}$,

$$T_{\alpha\beta|\gamma} = \frac{\partial T_{\alpha\beta}}{\partial \xi^\gamma} - \Gamma^\lambda_{\alpha\gamma}T_{\lambda\beta} - \Gamma^\lambda_{\beta\gamma}T_{\alpha\lambda},$$
$$T_{\alpha\beta;\gamma} = \frac{\partial T_{\alpha\beta}}{\partial \xi^\gamma} - \mathring{\Gamma}^\lambda_{\alpha\gamma}T_{\lambda\beta} - \mathring{\Gamma}^\lambda_{\beta\gamma}T_{\alpha\lambda},$$

and for a surface scalars, like $r^i\left(\xi^\alpha\right)$ or $n^i\left(\xi^\alpha\right)$,

$$r^i_\alpha = r^i_{,\alpha} = r^i_{|\alpha} = r^i_{;\alpha}, \quad n^i_{,\alpha} = n^i_{|\alpha} = n^i_{;\alpha}.$$

One can show by direct inspection, using (14.19), that the covariant derivatives of $a_{\alpha\beta}$ vanish:

$$a_{\alpha\beta|\gamma} = 0, \quad a^{\alpha\beta}_{|\gamma} = 0,$$

and, similarly,

$$\mathring{a}_{\alpha\beta;\gamma} = 0, \quad \mathring{a}^{\alpha\beta}_{;\gamma} = 0.$$

Besides,[2]

$$\varepsilon_{\alpha\beta|\gamma} = 0, \qquad \varepsilon^{\alpha\beta}_{|\gamma} = 0. \tag{14.21}$$

Note the relation

$$\Gamma^{\beta}_{\alpha\beta} = \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial \xi^{\alpha}} \tag{14.22}$$

which follows from (14.19) similarly to (4.80).

The second quadratic form of the surface. Denote by $\mathbf{e}_i$ the basic vectors of Cartesian coordinates in three-dimensional space. Consider the increment of the tangent vectors $\mathbf{t}_{\alpha} = r^i_{\alpha}\mathbf{e}_i$ when the point ξ^{α} is shifted along the surface for $d\xi^{\alpha}$:

$$d\mathbf{t}_{\alpha} = r^i_{\alpha,\beta}\mathbf{e}_i d\xi^{\beta}. \tag{14.23}$$

The coefficients, $r^i_{\alpha,\beta}$, are symmetric with respect to α, β as the second partial derivatives of the functions $r^i\left(\xi^{\alpha}\right)$,

$$r^i_{\alpha,\beta} = \frac{\partial r^i_{\alpha}}{\partial \xi^{\beta}} = \frac{\partial^2 r^i\left(\xi\right)}{\partial \xi^{\alpha}\partial \xi^{\beta}}.$$

Let us breakdown the vectors $\mathbf{e}_k$ into their tangent and normal components by means of (14.15):

$$\mathbf{e}_k = \mathbf{e}_j \delta^i_k = \mathbf{t}_{\gamma} r^{\gamma}_k + n_k \left(\mathbf{e}_i n^i\right). \tag{14.24}$$

Substituting (14.24) into (14.23), we obtain

$$d\mathbf{t}_{\alpha} = \Gamma^{\gamma}_{\alpha\beta}\mathbf{t}_{\gamma} d\xi^{\beta} + b_{\alpha\beta} d\xi^{\beta}\left(n^i \mathbf{e}_i\right). \tag{14.25}$$

Here

$$\Gamma^{\gamma}_{\alpha\beta} = r^k_{\alpha,\beta} r^{\gamma}_k, \quad b_{\alpha\beta} = r^k_{\alpha,\beta} n_k. \tag{14.26}$$

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[2] Indeed, differentiating the identity, $\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} = 2$, we have

$$\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta|\gamma} + \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta}_{|\gamma} = 0.$$

Since the covariant derivatives of the surface metric tensor are zero, $\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta|\gamma} = \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta}_{|\gamma}$. That yields $\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta|\gamma} = 0$. The tensor, $\varepsilon_{\alpha\beta|\gamma}$, is antisymmetric over α, β. Therefore, the only non-zero components have the indices, $\alpha = 1$, $\beta = 2$ and $\alpha = 2$, $\beta = 1$. Hence, $\varepsilon_{\alpha\beta|\gamma} = 0$.

The object, $\Gamma^{\gamma}_{\alpha\beta}$, introduced by $(14.26)_1$ coincides with that of (14.19). Indeed, from (14.19) and (14.3) we have

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}a^{\gamma\delta}\left(\frac{\partial\left(r^i_{\alpha}r_{i\delta}\right)}{\partial\xi^{\beta}} + \frac{\partial\left(r^i_{\beta}r_{i\delta}\right)}{\partial\xi^{\alpha}} - \frac{\partial\left(r^i_{\alpha}r_{i\beta}\right)}{\partial\xi^{\delta}}\right) =$$
$$= \frac{1}{2}a^{\gamma\delta}\left(r^i_{,\alpha\beta}r_{i\delta} + r_{i\alpha}r^i_{,\beta\delta} + r^i_{,\beta\alpha}r_{i\delta} + r_{i\beta}r^i_{,\delta\alpha} - r^i_{,\alpha\delta}r_{i\beta} - r_{i\alpha}r^i_{,\beta\delta}\right).$$

In the brackets the first and the third terms are equal while the second and last terms cancel out as well as the fourth and fifth terms, and we arrive at $(14.26)_1$.

The object $b_{\alpha\beta}$ $(14.26)_2$ can be written in terms of the covariant derivatives of the tangent vectors,

$$r^i_{\alpha|\beta} = r^i_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta}r^i_{\lambda} \quad \text{or} \quad r^i_{\alpha;\beta} = r^i_{\alpha,\beta} - \mathring{\Gamma}^{\lambda}_{\alpha\beta}r^i_{\lambda}. \tag{14.27}$$

Since, according to (14.8) and (14.27),

$$r^i_{\alpha|\beta}n_i = r^i_{\alpha;\beta}n_i = r^i_{\alpha,\beta}n_i,$$

we have

$$b_{\alpha\beta} = r^i_{\alpha,\beta}n_i = r^i_{\alpha|\beta}n_i = r^i_{\alpha;\beta}n_i. \tag{14.28}$$

Formula (14.28) shows that $b_{\alpha\beta}$ form the components of a surface tensor. They are called the components of the second quadratic form of the surface. Tensor $b_{\alpha\beta}$ is symmetric, because $r^i_{\alpha,\beta} = r^i_{,\alpha\beta} = r^i_{,\beta\alpha}$. Using (14.8), the definition of $b_{\alpha\beta}$ $(14.26)_2$ can also be written as

$$b_{\alpha\beta} = -r^i_{\alpha}n_{i,\beta}. \tag{14.29}$$

The derivatives of the tangent and normal vectors can be expressed in terms of $b_{\alpha\beta}$. To obtain such relations, we note that the three-dimensional vectors, $r^i_{\alpha|\beta}$, are directed along the normal vector to the surface: contracting (14.27) with the tangent vectors, $r_{i\gamma}$,

$$r_{i\gamma}r^i_{\alpha|\beta} = r_{i\gamma}r^i_{\alpha,\beta} - a_{\gamma\lambda}\Gamma^{\lambda}_{\alpha\beta}, \tag{14.30}$$

and using $(14.26)_1$, we see that the right hand side of (14.30) is zero. The magnitudes of vectors, $r^i_{\alpha|\beta}$, are determined by (14.28). Hence, for the derivatives of the tangent vectors, we obtain

$$r^i_{\alpha|\beta} = r^i_{\beta|\alpha} = r^i_{|\alpha\beta} = b_{\alpha\beta}n^i. \tag{14.31}$$

To find the derivatives of the normal vector, we note the relation

$$n_i n^i_{,\alpha} = 0, \tag{14.32}$$

which is obtained by differentiation of the equation, $n_i n^i = 1$, with respect to the surface coordinates. Equation (14.32) means that the three-dimensional vectors, $n^i_{,\alpha}$, are tangent to the surface, and hence can be presented as a sum of tangent vectors. The coefficients of the sum can be obtained by projecting $n^i_{,\alpha}$ on r^i_β. From (14.29) these coefficients are $-b_{\alpha\beta}$. Finally,

$$n^i_{,\alpha} = -b^\beta_\alpha r^i_\beta. \tag{14.33}$$

Equation (14.33) can also be taken as the initial definition of the second quadratic form. It shows that b^β_α are the measures of the rate of the normal vector when the point is moving over the surface.

The key role of the two quadratic forms of the surface, $a_{\alpha\beta}$ and $b_{\alpha\beta}$, in the surface geometry is explained by the following statement: each surface is determined uniquely (up to a rigid motion) by its quadratic forms, $a_{\alpha\beta}$ and $b_{\alpha\beta}$.

Curvatures. Consider the eigenvectors of the second quadratic form, i.e. the vectors t^α which are solutions of a system of linear equations,

$$b_{\alpha\beta} t^\beta = \varkappa a_{\alpha\beta} t^\beta. \tag{14.34}$$

In a generic case, there are two eigenvectors, t_1^α and t_2^α, and two corresponding eigenvalues, $\varkappa_1$ and $\varkappa_2$. The vectors t_1^α and t_2^α determine the directions of principal curvature. The corresponding eigenvalues are called the principle curvatures, and their inverse, $R_1 = 1/\varkappa_1$ and $R_2 = 1/\varkappa_2$, radii of curvature. The lines tangent to the eigenvectors are called curvature lines. There is a special coordinate system the coordinate lines of which are the curvature lines. The coordinates of this system are called principal coordinates.

The two invariants of the second quadratic form,

$$H = \frac{1}{2} b^\alpha_\alpha = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \text{ and } K = \frac{\det \left\| b_{\alpha\beta} \right\|}{\det \left\| a_{\alpha\beta} \right\|} = \frac{1}{R_1 R_2},$$

are called the mean and the Gaussian curvature, respectively.

Compatibility conditions. The tensors, $a_{\alpha\beta}$ and $b_{\alpha\beta}$, are not independent because their six components are expressed in terms of three functions, $r^i(\xi^\alpha)$. Therefore, there must be some compatibility relations linking these tensors. These relations are the Codazzi equations:

$$b_{\alpha\beta|\gamma} - b_{\alpha\gamma|\beta} = 0, \tag{14.35}$$

and the Gauss equations:

$$R_{\sigma\alpha\beta\gamma} = b_{\sigma\beta} b_{\alpha\gamma} - b_{\sigma\gamma} b_{\alpha\beta}, \tag{14.36}$$

where $R_{\sigma\alpha\beta\gamma}$ is the curvature tensor of the surface:

$$R_{\sigma\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial a_{\sigma\gamma}}{\partial\xi^\alpha \partial\xi^\beta} + \frac{\partial a_{\alpha\beta}}{\partial\xi^\sigma \partial\xi^\gamma} - \frac{\partial a_{\sigma\beta}}{\partial\xi^\alpha \partial\xi^\gamma} - \frac{\partial a_{\alpha\gamma}}{\partial\xi^\sigma \partial\xi^\beta}\right).$$

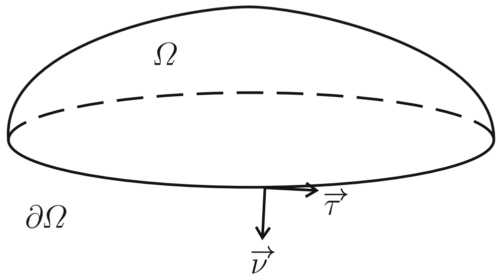

Fig. 14.2 Notation to the divergence theorem

The compatibility equations (14.35) and (14.36) are the equations containing only $a_{\alpha\beta}$, $b_{\alpha\beta}$ and their derivatives. Due to antisymmetry of (14.35) with respect to β, γ and (14.36) with respect to σ, α and β, γ, there are two independent equations (14.35) and one independent equation (14.36). So, there are three constraints for the six components of the two quadratic forms of the surface.

Divergence theorem. Let $\vec{v}$ be a vector field tangent to the surface Ω. Denote its surface components by v^α. For the surface divergence, $v^\alpha_{|\alpha}$ the following divergence theorem holds:

$$\int\limits_{\Omega} v^\alpha_{|\alpha} d\omega = \int\limits_{\partial\Omega} v^\alpha \nu_\alpha ds, \tag{14.37}$$

where $\vec{\nu}$ is the unit tangent vector to Ω which is normal to the tangent vector $\vec{\tau}$ of the curve $\partial\Omega$ (Fig. 14.2), ν_α are its surface components, s the arc length along $\partial\Omega$.

The analytical origin of (14.37) is the formula following from (14.22)

$$v^\alpha_{|\alpha} = \frac{\partial v^\alpha}{\partial \xi^\alpha} + \Gamma^\alpha_{\alpha\beta} v^\beta = \frac{1}{\sqrt{a}} \frac{\partial \left(\sqrt{a} v^\alpha\right)}{\partial \xi^\alpha}, \tag{14.38}$$

which yields

$$v^\alpha_{|\alpha} d\omega = \frac{\partial \left(\sqrt{a} v^\alpha\right)}{\partial \xi^\alpha} d\xi^1 d\xi^2.$$

Therefore, the covariant formula (14.37) is equivalent to the usual statement for an integral of divergence over a two-dimensional region.

14.2 Classical Shell Theory: Phenomenological Approach

It is natural to model the position of a thin elastic shell by a surface. Then the key kinematic characteristics of the elastic shell are the functions $x^i = r^i(\xi^\alpha, t)$, defining the position of the surface at the instant t. To obtain the dynamical equations of the shell theory we have to construct the action functional. Elastic energy must depend on the functions $r^i(\xi^\alpha, t)$ in a very special way because energy does not feel rigid motion. Therefore, we forewarn the construction of the action functional by the description of the surface deformation measures.

Strain measures. As was mentioned, any surface, Ω, up to its rigid motion, is determined by the first and second quadratic forms of the surface, $a_{\alpha\beta}$ and $b_{\alpha\beta}$. Their values in the initial state, $\mathring{\Omega}$, are denoted by $\mathring{a}_{\alpha\beta}$ and $\mathring{b}_{\alpha\beta}$. Recall that

$$\mathring{b}_{\alpha\beta} = \mathring{r}^k_{\alpha;\beta}\mathring{n}_k, \quad \mathring{n}^i_{,\alpha} = -\mathring{b}^\beta_\alpha \mathring{r}^i_\beta. \tag{14.39}$$

The juggling of indices of $\mathring{b}_{\alpha\beta}$ and other tensors in the initial state is done by means of the metric tensor of the initial state, $\mathring{a}_{\alpha\beta}$. Juggling of indices in the deformed state, if not otherwise stated, is done by means of the current metric tensor, $a_{\alpha\beta}$.

The tensors

$$A_{\alpha\beta} = \frac{1}{2}\left(a_{\alpha\beta} - \mathring{a}_{\alpha\beta}\right) \text{ and } B_{\alpha\beta} = b_{\alpha\beta} - \mathring{b}_{\alpha\beta} \tag{14.40}$$

characterize the surface deformation and can serve as the strain measures of the surface.

The tensor $A_{\alpha\beta}$ is a measure of elongations of the surface; if $A_{\alpha\beta} = 0$, the distances between any two points of the surface measured along the surface do not change. The tensor $B_{\alpha\beta}$ is a measure of bending.

To appreciate better the role of $B_{\alpha\beta}$ as a bending measure, note that a plane can be deformed in a cylindrical surface without change of the lengths of any line on the surface. For such deformation, $A_{\alpha\beta} = 0$. The only indicator of the deformation occurred is the tensor $B_{\alpha\beta}$. Deformations for which $A_{\alpha\beta} = 0$ are called pure bending. Juggling the indices in $A_{\alpha\beta}$ and $B_{\alpha\beta}$ and other strain measures, encountered further, is done by means of the metric tensor of the initial state, $\mathring{a}_{\alpha\beta}$.

Energy. It is natural to assume that kinetic and free energies of the shell possess the surface densities, i.e. they can be written as the surface integrals

$$\mathcal{K} = \int_{\mathring{\Omega}} K d\mathring{\omega}, \quad \mathcal{F} = \int_{\mathring{\Omega}} \Phi d\mathring{\omega},$$

$d\mathring{\omega}$ being an area element of $\mathring{\Omega}$.

In classical shell theory kinetic and free energies are considered as functionals of the position vector of the surface, $r^i(\xi^\alpha, t)$. One assumes that

$$K = \frac{1}{2}\bar{\rho} r^i_{,t} r_{i,t},$$

$\bar{\rho}$ being the surface mass density, while Φ is a function of the strain and bending measures:

$$\Phi = \Phi\left(A_{\alpha\beta}, B_{\alpha\beta}\right).$$

Variational principle. First let no external forces act on the shell. Then the action functional for an elastic shell is

$$I\left(r^i\left(\xi^\alpha, t\right)\right) = \int_{t_0}^{t_1}\int_{\mathring{\Omega}} \left(\frac{1}{2}\bar{\rho} r^i_{,t} r_{i,t} - \Phi\left(A_{\alpha\beta}, B_{\alpha\beta}\right)\right) d\mathring{\omega} dt. \tag{14.41}$$

The true motion of the shell is a stationary point of the functional (14.41) on a set of all functions $r^i\left(\xi^\alpha, t\right)$ with given initial and final values,

$$r^i(\xi^\alpha, t_0) = \mathring{r}^i\left(\xi^\alpha\right), \quad r^i\left(\xi^\alpha, t_1\right) = \overset{1}{r}{}^i\left(\xi^\alpha\right),$$

and, possibly, some boundary values. Consider a typical setting of the kinematic boundary constraints.

Kinematic boundary conditions. If the dependence of Φ on $A_{\alpha\beta}$ and $B_{\alpha\beta}$ is not degenerated, one can show that free energy, $\mathcal{F}$, "feels" (see Sect. 5.5) the change of the values of r^i and n^i at the boundary. Therefore, the kinematic boundary conditions include an assigning of r^i and n^i on a part of the boundary, $\mathring{\Gamma}_u$, of the surface $\mathring{\Omega}$:

$$r^i = r^i_{(b)}, \quad n^i = n^i_{(b)} \quad \text{on } \mathring{\Gamma}_u. \tag{14.42}$$

The index u in $\mathring{\Gamma}_u$ emphasizes that on $\mathring{\Gamma}_u$ the displacements of the shell,

$$u^i \equiv r^i(\xi^\alpha, t) - r^i(\xi^\alpha, t_0),$$

are known.

Among the six boundary conditions (14.42), only four conditions are independent. Indeed, let σ be a parameter on the curve $\mathring{\Gamma}_u$, and $\xi^\alpha = \xi^\alpha(\sigma)$ are the parametric equations of $\mathring{\Gamma}_u$. Then equation $(14.42)_1$ can be written as

$$r^i\left(t, \xi^\alpha(\sigma)\right) = r^i_{(b)}(t, \sigma).$$

It determines a space curve, Γ_u, given by the parametric equation

$$x^i = r^i_{(b)}(t, \sigma).$$

The normal vector to the surface, n^i, must be orthogonal to Γ_u, and therefore the prescribed boundary values of the normal vector, $n^i_{(b)}(t, \sigma)$, must obey the two equations

$$n_{i(b)}\frac{\partial r^i_{(b)}(t,\sigma)}{\partial\sigma}=0,\qquad n^i_{(b)}n_{i(b)}=1,$$

which leave only one independent constraint $(14.42)_2$.

Variations of characteristics of the surface. To derive the governing equations of the shell theory we first need to find the variations of the geometrical characteristics of the deformed surface. Let functions $r^i(\xi^\alpha,t)$ acquire infinitesimally small increments, δr^i. Varying (14.3), we get[3]

$$\delta a_{\alpha\beta}=r_{i\alpha}\left(\delta r^i\right)_{,\beta}+(\alpha\leftrightarrow\beta).\tag{14.43}$$

It is taken into account that $\delta r^i_\alpha=\left(\delta r^i\right)_{,\alpha}$ due to the permutability of the operators δ and $\partial/\partial\xi^\alpha$.

If the projections of the vector, δr^i, on the normal and the tangent vectors are used,

$$\delta r^i=\delta r^\alpha r^i_\alpha+\delta r\, n^i,$$

then (14.43) can be written as[4]

$$\begin{aligned}\delta a_{\alpha\beta}&=r_{i\alpha}\left(\delta r^\gamma r^i_\gamma+\delta r n^i\right)_{,\beta}+(\alpha\leftrightarrow\beta)=\\&=\left(r_{i\alpha}\left(\delta r^\gamma\right)_{|\beta}r^i_\gamma+\delta r^\gamma r_{i\alpha}r^i_{\gamma|\beta}+\delta r_{,\beta}r_{i\alpha}n^i+\delta r r_{i\alpha}n^i_{,\beta}\right)+(\alpha\leftrightarrow\beta).\end{aligned}\tag{14.44}$$

In the brackets, the first term is equal to $a_{\alpha\gamma}\delta r^\gamma_{|\beta}$ due to (14.3); since the covariant derivatives of the metric tensor are zeros, it can also be written equal to $\delta r_{\alpha|\beta}$ where

$$\delta r_\alpha=a_{\alpha\gamma}\delta r^\gamma=a_{\alpha\gamma}r^\gamma_i\delta r^i=r_{i\alpha}\delta r^i.$$

The second and the third terms are zero in accordance with (14.31) and (14.8). The last term is equal to $-b_{\alpha\beta}\delta r$ due to (14.33). Finally,

$$\delta a_{\alpha\beta}=\delta r_{\alpha|\beta}+\delta r_{\beta|\alpha}-2b_{\alpha\beta}\delta r.\tag{14.45}$$

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[3] Recall that the notation $(\alpha\leftrightarrow\beta)$ means the previous term in the equation with the indices α and β replaced by β and α, respectively.

[4] Since $\left(\delta r^\gamma r^i_\gamma\right)_{,\beta}$, for each i, is a surface scalar, it can be written in various ways:

$$\begin{aligned}\left(\delta r^\gamma r^i_\gamma\right)_{,\beta}&=\left(\delta r^\gamma\right)_{,\beta}r^i_\gamma+\delta r^\gamma\left(r^i_\gamma\right)_{,\beta}=\\&=\delta r^\gamma_{|\beta}r^i_\gamma+\delta r^\gamma r^i_{\gamma|\beta}=\delta r^\gamma_{;\beta}r^i_\gamma+\delta r^\gamma r^i_{\gamma;\beta}.\end{aligned}$$

We use the second option, the covariant form of this equation, for the deformed state.

The variation of the normal vector can be found by varying the equations

$$n_i r^i_\alpha = 0, \quad n_i n^i = 1.$$

We have

$$r^i_\alpha \delta n_i = -n_i \delta r^i_\alpha, \quad n_i \delta n^i = 0. \tag{14.46}$$

Equations (14.46) define the projections of the vector δn^i on the tangent vectors and the normal vector. The projection on the normal vector is zero. Consequently,

$$\delta n_i = -r^\alpha_i n_k \left(\delta r^k\right)_{,\alpha}. \tag{14.47}$$

Let us find $\delta b_{\alpha\beta}$. Since in the Cartesian observer's frame, $b_{\alpha\beta} \equiv n_i r^i_{,\alpha\beta}$, we have

$$\delta b_{\alpha\beta} = n_i \left(\delta r^i\right)_{,\alpha\beta} + r^i_{,\alpha\beta} \delta n_i.$$

Taking into account (14.47) and $(14.26)_1$, we get

$$\begin{aligned}\delta b_{\alpha\beta} &= n_i \delta r^i_{\alpha,\beta} - r^i_{\alpha,\beta} r^\gamma_i n_k \delta r^k_\gamma \\ &= n_i \left(\delta r^i_{\alpha,\beta} - \Gamma^\gamma_{\alpha\beta} \delta r^i_\gamma\right) = n_i \delta r^i_{\alpha|\beta} = n_i \delta r^i_{|\alpha\beta}.\end{aligned} \tag{14.48}$$

From (14.48) and (14.43), the variations of the two deformation measures of the surface are[5]

$$\begin{aligned}\delta A_{\alpha\beta} &= r_{i(\alpha} \delta r^i_{,\beta)}, \\ \delta B_{\alpha\beta} &= n_i \delta r^i_{|\alpha\beta}.\end{aligned} \tag{14.49}$$

The system of equations. Everything is prepared now to proceed to the derivation of the governing equations of shell dynamics.

In accordance with (14.49), the variation of the free energy $\mathcal{F}$ is

$$\delta \mathcal{F} = \int\limits_{\mathring{\Omega}} \left(\frac{\partial \Phi}{\partial A_{\alpha\beta}} r_{i\alpha} \delta r^i_{,\beta} + \frac{\partial \Phi}{\partial B_{\alpha\beta}} n_i \delta r^i_{|\alpha\beta} \right) d\mathring{\omega}. \tag{14.50}$$

Since we have to integrate by parts, and the second term contains the covariant derivatives in the deformed state, it is convenient to transform the integral (14.50) to the integral over the deformed surface Ω. We note that the area elements of $\mathring{\Omega}$ and Ω are linked by a factor θ:

$$d\mathring{\omega} = \theta d\omega, \quad \theta = \sqrt{\mathring{a}}/\sqrt{a} \tag{14.51}$$

[5] Recall that the parenthesis in indices mean symmetrization, i.e.

$$r_{i(\alpha} \delta r^i_{,\beta)} = \frac{1}{2} \left(r_{i\alpha} \delta r^i_{,\beta} + r_{i\beta} \delta r^i_{,\alpha} \right).$$

This factor may be interpreted as the ratio of the surface mass densities in the deformed and undeformed states.

Introducing the notations

$$S^{\alpha\beta} = \theta \frac{\partial \Phi(A, B)}{\partial A_{\alpha\beta}}, \quad M^{\alpha\beta} = -\theta \frac{\partial \Phi(A, B)}{\partial B_{\alpha\beta}}, \tag{14.52}$$

we have

$$\delta \mathcal{F} = \int_{\Omega} \left(S^{\alpha\beta} r^i_{\alpha} \delta r_{i,\beta} - M^{\alpha\beta} n_i \delta r^i_{|\alpha\beta} \right) d\omega. \tag{14.53}$$

The tensors $S^{\alpha\beta}$ and $M^{\alpha\beta}$ are symmetric. The tensor $S^{\alpha\beta}$ "works" on the surface strains, while $M^{\alpha\beta}$ "works" on the surface bending; therefore, they are called the stress resultants and the stress moments, respectively.

For the variation of kinetic and free energies we have[6]

$$\delta \int_{t_0}^{t_1} \mathcal{K} dt = \int_{t_0}^{t_1} \int_{\mathring{\Omega}} \bar{\rho} r_{i,t} \frac{\partial \delta r^i}{\partial t} d\mathring{\omega} dt = -\int_{t_0}^{t_1} \int_{\mathring{\Omega}} \bar{\rho} r_{i,tt} \delta r^i d\mathring{\omega} dt =$$

$$= -\int_{t_0}^{t_1} \int_{\Omega} \bar{\rho} \theta r_{i,tt} \delta r^i d\omega dt,$$

$$\delta \int_{t_0}^{t_1} \mathcal{F} dt = \int_{t_0}^{t_1} \int_{\Omega} \left(S^{\alpha\beta} r^i_{\alpha} \delta r^i_{,\beta} - M^{\alpha\beta} n_i \delta r^i_{|\alpha\beta} \right) d\omega dt =$$

$$= \int_{t_0}^{t_1} \int_{\Omega} \left(S^{\alpha\beta} r^i_{\beta} + \left(M^{\alpha\beta} n^i \right)_{|\beta} \right) \delta r_{i,\alpha} d\omega dt -$$

$$- \int_{t_0}^{t_1} \int_{\partial\Omega} M^{\alpha\beta} n_i \nu_{\beta} \delta r^i_{|\alpha} ds dt$$

$$= -\int_{t_0}^{t_1} \int_{\Omega} \left(S^{\alpha\beta} r^i_{\beta} + \left(M^{\alpha\beta} n^i \right)_{|\beta} \right)_{|\alpha} \delta r_i d\omega dt +$$

$$+ \int_{t_0}^{t_1} \int_{\partial\Omega} \left(\left(S^{\alpha\beta} r^i_{\beta} + \left(M^{\alpha\beta} n^i \right)_{|\beta} \right) \nu_{\alpha} \delta r_i - M^{\alpha\beta} n_i \nu_{\beta} \delta r^i_{|\alpha} \right) ds dt.$$

[6] Here we use a covariant form of the divergence theorem (14.37) for a surface Ω.

Taking first the variations δr^i and $\delta r^i_{,\alpha}$ equal to zero at the boundary, we obtain from the condition, $\delta I = 0$, Euler equations,

$$\bar{\rho}\theta\frac{\partial^2 r^i(t,\xi^\alpha)}{\partial t^2} = \left(S^{\alpha\beta}r^i_\beta + \left(M^{\alpha\beta}n^i\right)_{|\beta}\right)_{|\alpha}. \tag{14.54}$$

Equations (14.54), along with the constitutive equations for $S^{\alpha\beta}$ and $M^{\alpha\beta}$ (14.52) and the kinematical relations (14.3), (14.7), (14.29) and (14.40), form a closed system of equations for three functions, $r^i(t,\xi^\alpha)$.

Boundary conditions. In the case of non-zero variations on $\partial\Omega$ the following equality is to be satisfied at any instant:

$$\int\limits_{\partial\Omega}\left[\left(S^{\alpha\beta}r^i_\alpha + \left(M^{\alpha\beta}n^i\right)_{|\alpha}\right)\delta r_i - M^{\alpha\beta}n^i\delta r_{i,\alpha}\right]\nu_\beta ds = 0. \tag{14.55}$$

The integral (14.55) contains the variations δr^i and their derivatives $\delta r^i_{,\alpha}$. They are not independent: the derivative of δr^i along the contour is determined completely by the values of δr^i on this contour. To obtain the boundary conditions from (14.55), first we have to rewrite (14.55) in the form containing only independent variations. To this end we break down the two-dimensional Kronecker's delta, δ^β_α, in terms of the tangent vector, τ^α, and the normal vector, ν^α,

$$\delta^\beta_\alpha = \tau_\alpha\tau^\beta + \nu_\alpha\nu^\beta,$$

and make the corresponding decomposition of the derivatives,

$$\delta r^i_{,\alpha} = \delta r^i_{,\beta}\delta^\beta_\alpha = \delta r^i_{,\beta}\tau^\beta\tau_\alpha + \delta r^i_{,\beta}\nu^\beta\nu_\alpha \tag{14.56}$$

If $\xi^\beta = \xi^\beta(s)$ are the parametric equations of the contour Γ, s being the arc length of Γ, then

$$\tau^\beta = \frac{d\xi^\beta(s)}{ds},$$

and the first term of (14.56) can be written as

$$\delta r^i_{,\beta}\tau^\beta\tau_\alpha = \tau_\alpha\frac{d\delta r^i}{ds}.$$

The second term in (14.56) contains the normal derivative of δr^i which we denote by $d\delta r^i/d\nu$:

$$\frac{d\delta r^i}{d\nu} \equiv \delta r^i_{,\beta}\nu^\beta.$$

The integral (14.55) takes the form

$$\int_{\partial\Omega}\left[\left(S^{\alpha\beta}r^i_\alpha+\left(M^{\alpha\beta}n^i\right)_{|\alpha}\right)\delta r_i-M^{\alpha\beta}\tau_\alpha n_i\frac{d\delta r^i}{ds}-M^{\alpha\beta}\nu_\alpha n_i\frac{d\delta r^i}{d\nu}\right]\nu_\beta ds=0.$$

Obviously, δr^i and $n_i d\delta r^i/d\nu$ are independent. Integrating the second term by parts we obtain the equation containing only independent variations:

$$\int_{\partial\Omega}\left[\left(\left(S^{\alpha\beta}r^i_\alpha+\left(M^{\alpha\beta}n^i\right)_{|\alpha}\right)\nu_\beta+\frac{d}{ds}\left(M^{\alpha\beta}\tau_\alpha\nu_\beta n_i\right)\right)\delta r^i-M^{\alpha\beta}\nu_\alpha\nu_\beta n_i\frac{d\delta r^i}{d\nu}\right]ds=0. \tag{14.57}$$

Variations δr^i and $n_i d\delta r^i/d\nu$ are zero at $\mathring{\Gamma}_u$ due to (14.42). On the other part of the boundary, $\mathring{\Gamma}_f=\mathring{\Gamma}-\mathring{\Gamma}_u$, the arbitrariness of δr_i and $n_i d\delta r^i/d\nu$, yields the boundary conditions on $\mathring{\Gamma}_f$:

$$\left(S^{\alpha\beta}r^i_\alpha+\left(M^{\alpha\beta}n^i\right)_{|\alpha}\right)\nu_\beta+\frac{d}{ds}\left(M^{\alpha\beta}\tau_\alpha\nu_\beta n^i\right)=0,\quad M^{\alpha\beta}\nu_a\nu_\beta=0. \tag{14.58}$$

These boundary conditions are quite unusual: in contrast to other problems we have dealt with in continuum mechanics, they contain the derivatives of normal vector to the boundary, and, thus, curvatures of the boundary. This is caused by the presence of the second derivatives of position vector in energy density and occurs for both plates and shells. The issue of the proper boundary conditions for plates was a long-standing problem in the nineteenth century. It was solved by Kirchhoff. He was the first to apply the energy method to the derivation of the equations and the boundary conditions of plate theory and make the transformation from (14.55), (14.56) and (14.57) in case of plates. Interestingly, it took more than half a century to understand how to get Kirchhoff's boundary conditions from the differential equations of linear elasticity.

External forces. Let the external forces now be non-zero. Denote by Q_i and R_i the forces working on the variations of the positions of the points of Ω and $\partial\Omega$, respectively, and by M the "generalized force" working on the rotations of the fibers normal to $\partial\Omega$. Then the variation of the action functional (14.41) must be equated to the negative work of the external forces on the shell displacements,

$$-\int_{t_0}^{t_1}\left[\int_\Omega Q_i\delta r^i d\omega+\int_{\partial\Omega}\left(R_i\delta r^i+Mn_i\frac{d\delta r^i}{d\nu}\right)ds\right]dt. \tag{14.59}$$

This yields the corresponding contributions to the momentum equations

$$\bar\rho\theta\frac{\partial^2 r^i\left(t,\xi^\alpha\right)}{\partial t^2}=\left(S^{\alpha\beta}r^i_\alpha+\left(M^{\alpha\beta}n^i\right)_{|\beta}\right)+Q^i, \tag{14.60}$$

and the boundary conditions

$$\left(S^{\alpha\beta}r^i_\alpha + \left(M^{\alpha\beta}n^i\right)_{|\alpha}\right)\nu_\beta + \frac{d}{ds}\left(M^{\alpha\beta}\tau_\alpha n_\beta n^i\right) = R^i,$$
$$M^{\alpha\beta}\nu_\alpha\nu_\beta = M. \tag{14.61}$$

The governing system of equations projected to the tangent plane and the normal vector. It is often convenient to write down the "intrinsic" system of equations projected to the tangent directions and the normal vector. According to (14.33),

$$S^{\alpha\beta}r^i_\alpha + \left(M^{\alpha\beta}n^i\right)_{|\alpha} = \left(S^{\alpha\beta} - b^\alpha_\lambda M^{\lambda\beta}\right)r^i_\alpha + M^{\alpha\beta}_{|\alpha}n^i.$$

It is convenient to introduce a non-symmetric tensor, $T^{\alpha\beta}$, as

$$T^{\alpha\beta} = S^{\alpha\beta} - b^\alpha_\lambda M^{\lambda\beta}. \tag{14.62}$$

Projecting equations (14.60) on the tangent vectors and the normal vector, we have

$$T^{\alpha\beta}_{|\beta} - b^\alpha_\lambda M^{\lambda\beta}_{|\beta} + Q^\alpha = \bar{\rho}\theta r^\alpha_i\frac{\partial^2 r^i}{\partial t^2},$$
$$M^{\alpha\beta}_{|\alpha\beta} + b_{\alpha\beta}T^{\alpha\beta} + Q = \bar{\rho}\theta n_i\frac{\partial^2 r^i}{\partial t^2}. \tag{14.63}$$

Here the projections of the external forces are denoted by

$$Q^\alpha = r^\alpha_i Q^i, \quad Q = n_i Q^i, \quad R^\alpha = R^i r^\alpha_i, \quad R = R^i n_i.$$

The projections of the boundary conditions are

$$T^{\alpha\beta}\nu_\beta - M^{\gamma\beta}\tau_\gamma\nu_\beta\tau^\lambda b^\alpha_\lambda = R^\alpha,$$
$$\nu_\beta M^{\alpha\beta}_{|\alpha} + \frac{d}{ds}\left(M^{\alpha\beta}\tau_\alpha\nu_\beta\right) = R,$$
$$M^{\alpha\beta}\nu_a\nu_\beta = M. \tag{14.64}$$

If the function Φ is known, (14.63) and (14.64) augmented by the constitutive equations (14.52), and the initial conditions, form a closed system of equations of shell dynamics.

Physically linear theory. As in elasticity theory, by physically linear one means a simplification of general theory based on smallness of strains. The strains in shell theory are characterized by two dimensionless parameters:

$$\varepsilon_A = \max_\Omega\left(A_{\alpha\beta}A^{\alpha\beta}\right)^{1/2} \quad \text{and} \quad \varepsilon_B = h\max_\Omega\left(B_{\alpha\beta}B^{\alpha\beta}\right)^{1/2}. \tag{14.65}$$

In physically linear shell theory one neglects the contributions on the order of ε_A and ε_B with respect to unity. This considerably simplifies the system of equations. First of all, one can replace the covariant differentiation over Ω by the covariant differentiation over $\overset{\circ}{\Omega}$. Indeed, there is a simply verified identity:

$$\Gamma^{\gamma}_{\alpha\beta} - \overset{\circ}{\Gamma}{}^{\gamma}_{\alpha\beta} = a^{\gamma\sigma}\left(A_{\alpha\sigma;\beta} + A_{\beta\sigma;\alpha} - A_{\alpha\beta;\sigma}\right). \tag{14.66}$$

Therefore, $\Gamma^{\gamma}_{\alpha\beta} - \overset{\circ}{\Gamma}{}^{\gamma}_{\alpha\beta} \sim \varepsilon_A/l$, where l is the characteristic length of the stress state on Ω,[7] and replacement of $\Gamma^{\gamma}_{\alpha\beta}$ by $\overset{\circ}{\Gamma}{}^{\gamma}_{\alpha\beta}$ corresponds to neglecting the terms on the order of ε_A in comparison with unity. Second, θ can be replaced by unity. Third, $b_{\alpha\beta}$ can be replaced by $\overset{\circ}{b}_{\alpha\beta}$ when they enter in the products like $b^{\alpha}_{\lambda}M^{\lambda\beta}$ or $T^{\alpha\beta}b_{\alpha\beta}$. Finally, the governing equations take the form

$$\begin{aligned} T^{\alpha\beta}_{\ ;\beta} - \overset{\circ}{b}{}^{\alpha}_{\lambda} M^{\lambda\beta}_{\ ;\beta} + Q^{\alpha} &= \bar{\rho} r^{\alpha}_{i} \frac{\partial^2 r^i}{\partial t^2}, \\ M^{\alpha\beta}_{\ ;\alpha\beta} + \overset{\circ}{b}_{\alpha\beta} T^{\alpha\beta} + Q &= \bar{\rho} n_i \frac{\partial^2 r^i}{\partial t^2}, \\ T^{\alpha\beta} &= S^{\alpha\beta} - \overset{\circ}{b}{}^{\alpha}_{\lambda} M^{\lambda\beta}. \end{aligned} \tag{14.67}$$

In the boundary conditions vectors τ^{α} and ν^{α} may be replaced by $\overset{\circ}{\tau}{}^{\alpha}$ and $\overset{\circ}{\nu}{}^{\alpha}$, respectively.[8] Indeed, let $\xi^{\alpha} = \xi^{\alpha}(\overset{\circ}{s})$ be the equation for the contour $\overset{\circ}{\Gamma}$, $\overset{\circ}{s}$ being the arc lengths on $\overset{\circ}{\Gamma}$. Then $\overset{\circ}{\tau}{}^{\alpha} = d\xi^{\alpha}/d\overset{\circ}{s}$. Since the contour $\overset{\circ}{\Gamma}$ does not move over particles, the vector τ^{α} is proportional to $d\xi^{\alpha}/d\overset{\circ}{s}$. The proportionality coefficient, $ds/d\overset{\circ}{s}$, differs from unity by a small term on the order of ε_A. Consequently, $\tau^{\alpha} = \overset{\circ}{\tau}{}^{a} + O(\varepsilon_A)$. The vectors ν^{α} and $\overset{\circ}{\nu}{}^{a}$ are defined by the equations $a_{\alpha\beta}\nu^{\alpha}\nu^{\beta} = 1$, $a_{\alpha\beta}\nu^{\alpha}\tau^{\beta} = 0$, and $\overset{\circ}{a}_{\alpha\beta}\overset{\circ}{\nu}{}^{\alpha} = 1$, $\overset{\circ}{a}_{\alpha\beta}\overset{\circ}{\nu}{}^{\alpha}\overset{\circ}{\tau}{}^{\beta} = 0$. Therefore, $\nu^{\alpha} = \overset{\circ}{\nu}{}^{\alpha} + 0(\varepsilon_A)$.

So the boundary conditions are

$$\begin{aligned} T^{\alpha\beta}\overset{\circ}{\nu}_{\beta} - M^{\gamma\beta}\overset{\circ}{\tau}_{\gamma}\overset{\circ}{\nu}_{\beta}\overset{\circ}{\tau}{}^{\lambda}\overset{\circ}{b}{}^{\alpha}_{\lambda} &= R^{\alpha}, \\ \overset{\circ}{\nu}_{\beta} M^{\alpha\beta}_{\ ;\alpha} - \frac{d}{d\overset{\circ}{s}}\left(M^{\alpha\beta}\overset{\circ}{\tau}_{\alpha}\overset{\circ}{\nu}_{\beta}\right) &= R, \\ M^{\alpha\beta}\overset{\circ}{\nu}_{\alpha}\overset{\circ}{\nu}_{\beta} &= M. \end{aligned} \tag{14.68}$$

In physically linear theory energy is a quadratic form of $A_{\alpha\beta}$ and $B_{\alpha\beta}$. Accordingly, $S^{\alpha\beta}$ and $M^{\alpha\beta}$ are linear functions of $A_{\alpha\beta}$ and $B_{\alpha\beta}$. We consider these relations further.

Geometrically linear theory. Another case where considerable simplifications are possible is the case of small displacements, $u^i = r^i - \overset{\circ}{r}{}^i$. Then, setting $\delta r^i = u^i$ in (14.49), we get

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[7] About the characteristic length (see Sect. 5.11); a complete definition will be given below.

[8] Note that the three-dimensional vectors $r^i_{\alpha}\tau^{\alpha}$ and $r^i_{\alpha}\nu^{\alpha}$ may differ considerably from $\overset{\circ}{r}{}^i_{\alpha}\overset{\circ}{\tau}{}^{\alpha}$ and $\overset{\circ}{r}{}^i_{\alpha}\overset{\circ}{\nu}{}^{\alpha}$ because $r^i_{\alpha} - \overset{\circ}{r}{}^i_{\alpha}$ may be not small even for small strains.

$$A_{\alpha\beta} = \mathring{r}^i_{(\alpha} u_{i,\beta)}, \tag{14.69}$$
$$B_{\alpha\beta} = \mathring{n}_i u^i_{;\alpha\beta}.$$

As in three-dimensional elasticity, even for small displacements, theory can be, in principle, physically nonlinear, i.e. the dependence of $S^{\alpha\beta}$ and $M^{\alpha\beta}$ on the strain measures is nonlinear.

Equations (14.63) and (14.64) can be simplified due to smallness of displacements: the acceleration terms $r^\alpha_i \partial^2 r^i/\partial t^2$ and $n_i \partial^2 r^i/\partial t^2$ can be replaced by $\partial^2 u^\alpha/\partial t^2$ and $\partial^2 u/\partial t^2$, respectively, with u^α and u being the projection of the displacements to the tangent plane and the normal vector of the initial state, $u^\alpha \equiv \mathring{r}^\alpha_i u^i$, $u \equiv \mathring{n}_i u^i$, and covariant derivatives over Ω may be replaced by covariant derivatives over $\mathring{\Omega}$. We obtain the governing equations

$$\begin{aligned}
\bar{\rho}\frac{\partial^2 u^\alpha}{\partial t^2} &= T^{\alpha\beta}_{\ ;\beta} - \mathring{b}^\alpha_\lambda M^{\lambda\beta}_{\ ;\beta} + Q^\alpha, \\
\bar{\rho}\frac{\partial^2 u^\alpha}{\partial t^2} &= M^{\alpha\beta}_{\ ;\alpha\beta} - \mathring{b}_{\alpha\beta} T^{\alpha\beta} + Q, \\
T^{\alpha\beta} &= S^{\alpha\beta} - \mathring{b}^\alpha_\lambda M^{\lambda\beta},
\end{aligned} \tag{14.70}$$

and the boundary conditions

$$\begin{aligned}
T^{\alpha\beta}\mathring{\nu}_\beta - M^{\gamma\beta}\mathring{\tau}_\gamma \mathring{\nu}_\beta \mathring{\tau}^\lambda \mathring{b}^\alpha_\lambda &= R^\alpha, \\
\mathring{\nu}_\beta M^{\alpha\beta}_{\ ;\alpha} - \frac{d}{d\xi}\left(M^{\alpha\beta}\mathring{\tau}_\alpha \mathring{\nu}_\beta\right) &= R, \\
M^{\alpha\beta}\mathring{\nu}_\alpha \mathring{\nu}_\beta &= M.
\end{aligned} \tag{14.71}$$

In linear shell theory equations (14.70) and (14.71) are closed by the linear relations between $S^{\alpha\beta}$, $M^{\alpha\beta}$ and $A_{\alpha\beta}$, $B_{\alpha\beta}$ and the linear relations between the deformation measures and displacements (14.69).

Various bending measures. Instead of the bending measure $B_{\alpha\beta}$ one can use another bending measure $\rho_{\alpha\beta}$ which is a function of $B_{\alpha\beta}$ and $A_{\alpha\beta}$ as long as the couple $(A_{\alpha\beta}, B_{\alpha\beta})$ is in one-to-one correspondence with the couple $(A_{\alpha\beta}, \rho_{\alpha\beta})$. If the energy density Φ is given as a function of $A_{\alpha\beta}$ and $B_{\alpha\beta}$, it can be computed in terms of $A_{\alpha\beta}$ and $\rho_{\alpha\beta}$, and we get another form of the theory. In linear shell theory we have an additional opportunity to drop small terms of order ε_A and ε_B in comparison to unity and, thus, get a different set of equations which still have the same accuracy. For example, let Φ be a quadratic function, which we write in a symbolic form:

$$\frac{1}{\mu h}\Phi = A^2 + h^2 B^2,$$

with μ being the characteristic value of the shear modulus. If we take

$$\rho_{\alpha\beta} = B_{\alpha\beta} - \mathring{b}^\lambda_{(\alpha} A_{\lambda\beta)},$$