Applied and Numerical Harmonic Analysis

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma}\, dx$$

Stephen D. Casey
Kasso A. Okoudjou
Michael Robinson
Brian M. Sadler
Editors

Sampling: Theory and Applications

A Centennial Celebration of Claude Shannon

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Applied and Numerical Harmonic Analysis

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time-frequency and*
Numerical partial differential equations	*time-scale analysis*
	Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

University of Maryland
College Park

John J. Benedetto
Series Editor

The figure is designed to reflect the rich harmonic analysis/signal processing community in the greater DC metropolitan region. Like DC itself, it radiates outward symmetrically from the Capitol. First, the function

$$\frac{\sin(\pi|z|)}{\pi|z|}$$

was plotted in MATLAB in the region $\mathbb{C} \setminus \mathbb{D}$, where $\mathbb{C}$ represents the complex plane and $\mathbb{D}$ represents the unit disk in the plane. Then, the U.S. Capitol Dome was "lathed" onto the image, joining the radial sinc function at the unit circle $\partial\mathbb{D}$. The design was created by Stephen Casey and Randy Mays, with Mays providing the Capitol Dome framework, filling it in with MATLAB coloring.

Preface

Seminal ideas are triggered by amazing insights, often derived from straightforward ideas. Many lead to important developments far beyond what was envisioned. The ideas of Claude E. Shannon on sampling and signal processing, entropy, information theory, and cryptography are examples of such seminal ideas. He placed what we commonly refer to as the Classical (a.k.a., Whittaker–Kotel'nikov–Shannon (WKS)) Sampling Theorem as a cornerstone of information theory in his paper *Communications in the Presence of Noise*.[1] The harmonic analysis and signal processing communities have adopted Shannon's point of view, as is evidenced in the papers in this monograph. Our "Age of Information" presents us with new challenges, and, in particular, with respect to sampling. We are in the time of miniature and handheld devices for communications, robotics, and micro aerial vehicles, cognitive radio, radar, etc. We are also presented with the challenge of powering these and other systems.

We marvel at the rich extent of mathematical research and significant applications that Shannon's ideas have inspired, from the mathematics hiding in compact discs and videos to modern advances in signal and image processing, compressed sensing, deep learning, real and complex analysis, applied and computational harmonic analysis, geosciences, inverse problems, optics, computational neuroscience, etc. We have witnessed many breakthroughs in some of these areas in the past two decades, and more is yet to come. The Applied and Numerical Harmonic Analysis (ANHA) series publishes works in harmonic analysis and its myriad of incredible applications. It has and will continue to report on the rich tapestry of these disciplines as they continue to grow.

The papers in this volume are outgrowths of the SampTA conferences. The 13th SampTA occurred in July 2019 at Université de Bordeaux.

[1] Shannon himself was careful to note that the Sampling Theorem did not originate with him. The history of the theorem is rich and includes the names Cauchy, Raabe, and Ogura, among others.

Overview of SAMPTA

SAMPTA *(Sampling Theory and Applications)* is a biennial interdisciplinary international conference for mathematicians, engineers, and applied scientists. The main purpose of SAMPTA is to exchange recent advances in sampling theory and to explore new trends and directions in the related areas of application.

SAMPTA has traditionally focused on such fields as signal processing and image processing, coding theory, control theory, real and complex analysis, harmonic analysis, and the theory of differential equations. The conference has always featured plenary talks by prominent speakers, special sessions on selected topics reflecting the current trends in sampling theory and its applications to the engineering sciences, as well as regular sessions about traditional topics in sampling theory, and poster sessions. The conferences are a bridge between the mathematical and engineering signal processing communities. The mix between mathematicians and engineers is unique and leads to extremely useful and constructive dialogs. SAMPTA has had sessions on theory—compressed sensing, frames, geometry, wavelets, nonuniform and weighted sampling, finite rate of innovation, universal sampling, time-frequency analysis, deep learning, operator theory, and applications—A-to-D conversion, computational neuroscience, mobile sampling issues, and biomedical imaging.

SAMPTA Meetings

- SAMPTA 2019: Université de Bordeaux, Bordeaux, France, July 8–12, 2019
- SAMPTA 2017: Tallinn University, Tallinn, Estonia, July 3–7, 2017
- SAMPTA 2015: American University, Washington DC, USA, May 25–29, 2015
- SAMPTA 2013: Jacobs University, Bremen, Germany, July 1–5, 2013
- SAMPTA 2011: Nanyang Technical University, Singapore, May 2–6, 2011
- SAMPTA 2009: CIRM, Marseilles, France, May 18–22, 2009
- SAMPTA 2007: Aristotle University, Thessaloniki, Greece, June 1–5, 2007
- SAMPTA 2005: Samsun, Turkey, July 10–15, 2005
- SAMPTA 2003: Strobl, Austria, May 26–30, 2003
- SAMPTA 2001: UCF, Orlando, Florida, USA, May 13–17, 2001
- SAMPTA 1999: Loen, Norway, August 11–14, 1999
- SAMPTA 1997: University of Aveiro, Aveiro, Portugal, July 16–19, 1997
- SAMPTA 1995: Riga, Latvia, September 20–22, 1995

The SAMPTA 2015 conference had 203 attendees and achieved some rather notable milestones. The meeting was endorsed by the *Institute of Electrical and Electronics Engineers (IEEE)* and the *Society for Industrial and Applied Mathematics (SIAM)*. The conference papers were published in *IEEE Xplore*. SAMPTA 2015 also received grant support from the *Air Force Office of Scientific Research (AFOSR)* (BAA-AFOSR-2014-0001) and the *Army Research Office (ARO)* (BAA W911NF-12-R-0012-02). The conference was also supported by the efforts of the

College of Arts and Sciences and the Department of Mathematics and Statistics at American University and the Norbert Wiener Center at the University of Maryland.

The interaction at SAMPTA has pushed the envelopes forward in both the mathematics and engineering communities. The SAMPTA conferences have and will continue to serve as a meeting ground for harmonic analysts and electrical engineers and give graduate students and junior investigators a chance to learn about the developments of the subjects. The conference provides the community an opportunity to interact with some of the leaders in the field in a relaxed and yet very constructive environment.

We thank all the authors, editors, and referees for their contributions to this volume and their ongoing efforts to support the Sampling/ANHA/SAMPTA community.

Book Chapters

The volume opens with the delightful essay *Claude Shannon: American Genius* by Professor John Rowland Higgins (Anglia Polytech), our historian on sampling theory and expositor par excellence. The essay is a salute to Shannon's genius and tells us the tales of not only his seminal mathematics but also of several of his extremely clever inventions. The subsequent chapters report on major advances in their areas; collectively, they explore impressive frontiers of modern sampling theory.

The first book chapter—*Reconstruction of Signals: Uniqueness and Stable Sampling*—is by Alexander Olevskii (Tel Aviv) and Alexander Ulanovskii (Stavanger). It addresses sets of uniqueness and sampling, discussing uniform densities, universal sampling, and universal completeness. Recall that the classical *sampling problem* is to reconstruct, in a unique and stable way, a continuous signal with a given spectrum S from its samples on a discrete set Λ. Through the Fourier transform, the problems ask for which sets of frequencies Λ are the exponential system $\{e^{i\lambda t}, \lambda \in \Lambda\}$ *complete* or constitute a *frame* or a *Riesz sequence* in the space L^2 on a given set S of finite measure? When S is a single interval, these problems were essentially solved in deep papers by A. Beurling and P. Malliavin in terms of appropriate densities of the discrete set Λ. The main tool in these papers was the theory of entire functions. However, this tool becomes considerably less effective in the case of disconnected spectra S. H. Landau extended the necessity of the density conditions in these results to the general bounded spectra. However, when S is a disconnected set, no sharp sufficient condition for sampling and completeness can be expressed in terms of the density of the set Λ. Both the size and arithmetic structure of Λ come into the play. This chapter gives a short introduction into the subject of sampling and related problems. It presents both classical and recent results on universal sampling.

The chapter by M. Maurice Dodson (York) and J. Rowland Higgins (Anglia Polytech)—*Sampling Theory in a Fourier Algebra Setting*—gives a way of developing sampling in terms of the Approximate Sampling Theorem. This structure, a Fourier algebra setting, is not as well-known as it might be, possibly because the Approximate Sampling Theorem, central to their discussion, had what could be called a mysterious birth and a confused adolescence. They also discuss functions of the familiar bandlimited and bandpass types, showing that these too have a place in this viewpoint. The chapter combines an expository and historical treatment of the origins of exact and approximate sampling, including bandpass sampling, all from the Fourier algebra viewpoint. It has two objectives. The first is to provide an accessible and rigorous account of this sampling theory. The various cases mentioned above are each discussed in order to show that the Fourier algebra, a Banach space, is a broad and natural setting for the theory. The second objective is to clarify the early development of approximate sampling in the Fourier algebra and to unravel its origins.

This is followed by *Sampling Series, Refinable Sampling Kernels, and Frequency Band Limited Functions* by Wolodymyr R. Madych (Connecticut). It studies sampling series and their relationship to frequency bandlimited functions. Motivated by the theories of multiresolution analysis and subdivision, particular attention is paid to sampling series whose kernels are refinable. After developing the basic properties of the sampling kernels understudy, the chapter focuses on three families of such kernels:

1. damped cardinal sines,
2. the fundamental functions for cardinal spline interpolation, and
3. a family of compactly supported kernels defined in terms of their masks.

The limiting kernel of each family is the cardinal sine. In the cases (i) and (ii), it presents results concerning the limiting behavior of the corresponding sampling series when the data samples $\{c_n\}$ have polynomial growth as $n \longrightarrow \pm\infty$.

The chapter *Prolate Shift Frames and Sampling of Bandlimited Functions* by Jeffrey Hogan (Newcastle) and Joseph Lakey (New Mexico State) reviews recent developments involving frames of time- and bandlimited functions for Paley–Wiener spaces. The classic Shannon sampling theorem can be viewed as a special case of (generalized) sampling reconstructions for bandlimited signals in which the signal is expressed as a superposition of shifts of finitely many bandlimited generators. The coefficients of these expansions can be regarded as generalized samples taken at a Nyquist rate determined by the number of generators and basic shift rate parameters. When the shifts of the generators form a frame for the Paley–Wiener space, the coefficients are inner products with dual frame elements. There is a trade-off between time localization of the generators and localization of dual generators. The Shannon sampling theorem is an extreme manifestation in which the coefficients are point values but the generating sinc function is poorly localized in time. This work reviews and extends some recent related work of the authors regarding frames for the Paley–Wiener space generated by shifts of prolate spheroidal wave functions and considers the question of trade-off between localization of the generators and of the dual frames.

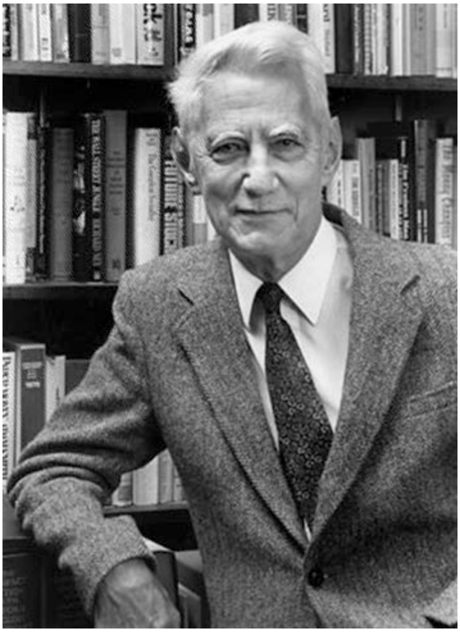
The volume finishes with *A Survey on the Unconditional Convergence and the Invertibility of Frame Multipliers with Implementation* by Peter Balazs and Diana Stoeva (Acoustics Research Institute, Austrian Academy of Sciences). Multipliers are operators which consist of an analysis stage, a multiplication, and then a synthesis stage. This is a very natural concept, and it occurs in a lot of scientific questions in mathematics, physics, and engineering. In physics, multipliers are the link between classical and quantum mechanics, the so-called quantization operators. Multipliers link sequences (or functions) m_k corresponding to the measurable variables in classical physics to operators $M_{m,\Phi,\Psi}$, which are the measurables in quantum mechanics. In signal processing, multipliers are a particular way to implement time-variant filters. One of the goals in signal processing is to find a transform which allows certain properties of the signal to be easily discovered. Via such a transform, one can focus on those properties of the signal one is interested in or would like to change. The coefficients can be manipulated directly in the transform domain, thus allowing certain signal features to be amplified or attenuated. This is, for example, the case of what a sound engineer operating an equalizer does during a concert, i.e., changing the amplification of certain frequency bands in real time. Convolution operators correspond to a multiplication in the Fourier domain, and therefore to a time-invariant change of frequency content.

The chapter presents a survey of frame multipliers and related concepts. In particular, it includes a short motivation of why multipliers are of interest and a review and extensions of recent results. These include the unconditional convergence of multipliers, sufficient and/or necessary conditions for the invertibility of multipliers, and representation of the inverse via Neumann-like series and via multipliers with particular parameters. Multipliers for frames with specific structure, namely Gabor multipliers, are also considered. Results for the representation of the inverse multiplier are implemented in MATLAB code, and the algorithms are described.

Crossroads

Our wonderful discipline sits at a crossroad. Just as mathematics and engineering used to be driven by mathematical physics, our mathematics and engineering will be motivated by the science of information. The Sampling/ANHA/SampTA community is in a unique position to contribute to these efforts. Our work provides the tools for advances in a very wide spectrum of areas, from communications, data, and information, to advances in signal and imaging processing, to breakthroughs in the physical, medical, social, and political sciences. These efforts are ongoing, as witnessed by the SampTA conferences and the growing body of literature generated by the Sampling/ANHA/SampTA community.

Washington, DC, USA — Stephen D. Casey
College Park, MD, USA — Kasso A. Okoudjou
Washington, DC, USA — Michael Robinson
Adelphi, MD, USA — Brian M. Sadler



Claude E. Shannon

The Classical (WKS) Sampling Theorem

Let $f \in \mathbb{PW}_\Omega$, $\operatorname{sinc}_T(t) = \dfrac{\sin(\frac{\pi}{T}t)}{\pi t}$, and $\delta_{nT}(t) = \delta(t - nT)$.

1. If $T \le 1/2\Omega$, then for all $t \in \mathbb{R}$,

$$f(t) = T\sum_{n\in\mathbb{Z}} f(nT)\frac{\sin(\frac{\pi}{T}(t - nT))}{\pi(t - nT)} = T\left(\left[\sum_{n\in\mathbb{Z}}\delta_{nT}\right]f\right) * \operatorname*{sinc}_T(t)\,.$$

2. If $T \le 1/2\Omega$ and $f(nT) = 0$ for all $n \in \mathbb{Z}$, then $f \equiv 0$.

Dedication—John Rowland Higgins (17 August 1935–24 February 2020)

The Sampling/ANHA/SampTA community is shocked and saddened by Rowland Higgins' unexpected death during the final stages of publication of this volume. Just a few weeks earlier, Rowland and his wife Nan had returned from the south of France, where they had lived since retirement, to Cambridge, where they used to live, to be with their children and grandchildren. Despite health problems over recent years, Rowland had continued to think about sampling and having completed his joint chapter in this volume, was planning to write more papers. Alas, it was not to be.

Rowland was a mainstay of our Sampling/ANHA/SampTA community from the start. His papers, books, and lectures were key elements in the growing body of knowledge of the community. He served on the original Steering Committee of the SampTA meetings and then went on to fulfill his duties as a Founding Member.

Rowland was a gift to our community. Through his deep interest in the rich mathematics of sampling theory and his wide knowledge of its fascinating history, he provided for all of us an informed perspective and a sense of the history of the subject. We were able to see the deep roots and interconnections of sampling and appreciate the tremendous potential of this theory, as it branched out to exciting new areas, even beyond the genius of Shannon.

However, even more brilliant than the excellence of Rowland as a mathematician and lecturer was his role to our community as a friend and mentor. With an encyclopediac knowledge of sampling which he wore lightly, he was in a real sense a scholar and a gentleman. His kindness, generosity, and courtesy brought out, in many ways, our best. And for that gift, we will always be grateful. He will be deeply missed by all of us and by his family.

In honor of his enormous contribution to the Sampling/ANHA/SampTA community, the editors are dedicating this book to Rowland Higgins.

Washington, DC, USA — Stephen D. Casey
York, UK — M. Maurice Dodson

Contents

Claude Shannon: American Genius

J. Rowland Higgins

Abstract Claude Shannon was born one hundred years ago. This essay is a salute to his genius.

Claude Shannon was born one hundred years ago. This essay is a salute to his genius.

What is genius? Opinions will differ of course, but in general terms we would not be far wrong in allowing that genius goes beyond mere talent. The philosopher Schopenhauer said[1] "Talent is like the marksman who hits a target which others cannot reach; genius is like the marksman who hits a target, as far as which others cannot even see."

Some of the more explicit qualities often associated with genius are: an ability to initiate major advances in a subject, sometimes even to create a new subject, a prodigious memory, and an unflagging curiosity. Astounding examples are not hard to come by, and Claude Shannon sits well in their company.

As for major advances, Shannon's three most important advances in the sciences of his time were, undoubtedly, his creation of information theory, his contributions to switching theory and his contributions to cryptography.

Next, memory. One thinks of Euler's extraordinary memory, for example; he knew Homer's Iliad by heart, and given any page number in his copy he could remember the first and last lines on that page. Shannon's memory is evident in, for example, his practice of dictating his articles for publication entirely from memory and without the need for correction. Incidentally, his collected works list 127 publications.

[1]In his book: *Die Welt als Wille und Vorstellung*. (The World as Will and Representation, Tr. R.B. Haldane).

J. R. Higgins (deceased)
Cambridge, UK

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Now to curiosity. The art historian Kenneth Clark recognizes[2] Leonardo da Vinci as one of the greatest geniuses, and writes that “all his gifts were dominated by one ruling passion—curiosity. He was the most relentlessly curious man in history. Everything he saw made him ask why and how.”

It is clear that Shannon too was strongly motivated by curiosity. In his own words[3]

> I don’t think I was ever motivated by the notion of wining prizes, ... I was more motivated by curiosity. Never by the desire for financial gain. ... There are many things I have done and never written up at all.

Much scholarly attention has been given to Claude Shannon and his work in mathematics and the theory of communications; throughout all of this attention his genius is in full view. But, strange as it may seem, among the several excellent commentaries on Shannon to be found in the literature, there are some that make little or no acknowledgment of his usage of sampling theory. This may be due to the fact that by his time basic sampling theorems were already considered part of the general background[4]; however, it was Shannon who recognized the theorem as having a central place in information theory. We shall return to these points later, with particular emphasis on sampling methods and some of their many generalizations and extensions; meanwhile, let us expand a little on Shannon and the three major advances mentioned above, and on Shannon the inventor of strange devices.

First, in the following excerpt from the paper by Gallager[5] we begin to get an idea of Shannon’s creation. Much more is to be found in this source.

> By 1948, all the pieces of “A mathematical theory of communication”[6] had come together in Shannon’s head. He had been working on this project, on and off, for eight years. There were no drafts or partial manuscripts|remarkably, he was able to keep the entire creation in his head .... His theory was about the entire process of telecommunications, from source to data compression to channel coding to modulation to channel noise to detection to error correction. .... Shannon employed the provocative term “information” for what was being communicated .... This new notion of information reopened many age-old debates about the difference between knowledge, information, data, and so forth ... the paper (that of footnote 6) totally changed both the understanding and the design of telecommunication systems ....

[2] *Civilization, a personal view*, Penguin Books, 1969, p. 105.

[3] The quotation is from an interview: Anthony Liversidge, “Profile of Claude Shannon,” *Claude Elwood Shannon Collected Papers*, edited by N. J. A. Sloan, Aaron D. Wyner (IEEE, 1993).

[4] His source is found in the work of J. M. Whittaker from the middle 1930s.

[5] “Claude E. Shannon: A retrospective on his life, work and impact.” IEEE Trans. Info. Th., 47, 2001.

[6] Bell System Tech J., 27, pp. 379–423, 623–656, 1948.

It is important to realize how surprising the paper "A mathematical theory of communication" really was, even to Shannon's close colleagues. The following quotation[7] underlines this.

> "We had an internal distribution system" recalls a Bell Labs mathematician, Brock McMillan, who worked in the office next to Shannon's at Murray Hill at the time, "so people in the math department could read things before publication. Shannon was not particularly talkative. He talked to us about his theories maybe a little bit. But the 1948 paper caught me almost stone cold." Most of the Bell researchers, with the exception of Shannon's friend Barney Oliver, experienced a similar sense of shock. John Pierce likened the surprise he felt upon encountering his friend's ideas to the dropping of a powerful explosive.

Second, in 1937 Shannon wrote a master's thesis at MIT entitled "A symbolic analysis of relay and switching circuits." Shannon's insight was decisive; the symbolic logic of Boole's *Laws of thought* (1854) provided a mathematical model for switching theory. Achieved at the age of 21, this thesis is one of his main claims to fame and was received with abundant and universal praise. A detailed study can be found in the book by Nahin.[8]

Third, Shannon became interested in cryptography during WWII. In his only published paper in this area,[9] he established a mathematical theory for secrecy systems and showed that the encryption system known as the "one-time pad" achieves perfect secrecy. This paper was enormously influential. Kotel'nikov also proved this in a report of 1941[10] which is apparently still classified!

Shannon's childhood and undergraduate days were spent in Michigan. It was here that he started inventing gadgets and strange devices, including model airplanes and a telegraph device to a neighbor's house using a wire fence. As time went by we find him inventing a calculator that worked in Roman numerals, a motorized pogo stick and a juggling machine that could actually juggle three balls and was decorated to look like the entertainer W.C. Fields. These were just a few of Shannon's devices. During his time at Bell Telephone Laboratories he could be seen in the corridors riding a unicycle and juggling four balls at the same time. Later, Shannon gave the unicycle's wheel an eccentric mounting whereupon he was seen to bob up and down "like a duck."

We may ask why Shannon gave time and effort to what may appear at first to be mere frivolity. Well, surely it was not a waste of time. I think he did it, at first anyway, out of pure *joi-de-vivre*; he could do these unusual things and doing them must have given him enormous pleasure. But a more important consideration is the attunement of the eager mind, the habituation to a way of thought as the purposes inevitably became more sophisticated and more serious.

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[7] Found in a footnote to "The idea factory: the Bell Labs and the great age of American innovation," The Penguin Press, 2012, by Jon Gertner.

[8] *The Logician and the Engineer How George Boole and Claude Shannon created the information age*. Princeton University Press, 2013.

[9] "Communication theory of secrecy systems," BSTJ, 1949.

[10] See Sergei N. Molotkov, "Quantum cryptography" and V.A.'s one-time key and sampling theorems.

One example of this emerging maturity was a mechanical mouse; another, the programming of computers for playing chess.

As for the mouse, in about 1950 Shannon introduced a mechanical mouse that could negotiate a maze successfully.[11] It should be borne in mind that this mouse belonged to the pre-electronic era; in fact, he was controlled using relay circuits and a magnet underneath the maze, and proceeded via an exhaustive search algorithm. Although it is an interesting and early example of such an algorithm, current opinion does not allow Shannon's mouse the status of an artificial learning device, or indeed to have any intelligence at all. Although striking at the time the algorithm is now considered elementary.

Shannon began the formalization of chess via computer programs in a paper published in The Philosophical Magazine[12] of 1950: "Programming a computer for playing chess."[13]

Let Shannon himself be our guide as we contemplate, briefly, the theme of his paper. Following on directly from the title Shannon tells us:

> Although perhaps of no practical importance, the question is of theoretical interest, and it is hoped that a satisfactory solution of this problem will act as a wedge in attacking other problems of a similar nature and greater significance.

Shannon goes on to list several possibilities. Among these are: machines for performing symbolic mathematical operations, for translation of one language into another, for making strategic decisions and for orchestrating a melody.

Shannon goes on:

> It is believed that all of these and many other devices of a similar nature are possible developments in the immediate future.

He describes how his chess-playing programs were designed, and emphasizes that the computers of his time were general enough and flexible enough to work symbolically with elements representing words, propositions, or other conceptual entities and speculates on whether computers can be said to "think." Thus, Shannon's interest was not so much in the game of chess itself as in the evolution of software for more important purposes. Before that could happen chess would provide a move in the right direction.

Shannon had the insight to realize[14] (from now on let us call the paper in footnote 14 just (1949)) that sampling theory is an essential part of information theory since it shows, in particular, that band limited analog signals, that is, continuous

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[11] The mouse was called Theseus, a name borrowed appropriately from Greek mythology.

[12] One of the earliest scientific journals, founded by Alexander Tulloch in 1797.

[13] In 1965 Shannon met the then world chess champion M.M. Botvinnik at a conference in Moscow. Botvinnik had been three times world-champion and was a pioneer of computerized chess-playing programs. Information about this meeting seems hard to come by, but it is known that they discussed chess and its computerization. More can be found it Botvinnik's book: *Computers, chess and long-range planning*, Heidelberg Science Library, 1970, Springer, Translator's preface.

[14] "Communication in the presence of noise," Proc. IRE, 37, 1949, pp. 10–21.