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Boundary Conditions in Electromagnetics



*Ismo V. Lindell
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Preface

Along with the Maxwell equations, boundary conditions form an essential part of defining electromagnetic problems. Mathematically, they ensure uniqueness of solution for the problem. In working with engineering problems, they offer analytic help in mathematical handling of given electromagnetic structures and synthetic help for designing new electromagnetic structures. In the former case, a given physical structure can often be approximated by boundary conditions enabling the use of simpler analytical computation schemes to find the electromagnetic response of the structure. In the latter case, certain exact boundary conditions can be used as a starting point to design (synthesize) a structure with desired electromagnetic response. This aspect is adopted throughout in the text. With the synthetic approach, the problem becomes to realize the structure in terms of suitable materials. The main topic of this book is the most general linear and local boundary conditions and the basic problem, plane-wave reflection from a planar boundary in a simple-isotropic medium.

The general boundary is approached by starting, in Chapters 1 and 2, from the simplest special cases, the Perfect Electric Conductor (PEC), the Perfect Magnetic Conductor (PMC) and their generalization, the Perfect ElectroMagnetic Conductor (PEMC). It is shown that, by applying suitable duality transformations, these boundaries can be transformed to one another. As an application, scattering of small PEMC particles is analyzed to find effective parameters of a mixture of PEMC inclusions.

In Chapter 3, the Impedance Boundary Condition (IBC) is seen to generalize the previous cases. Other special cases are obtained by splitting the impedance dyadic $\bar{\bar{Z}}_t$ in its natural components. A plane wave satisfying the boundary conditions identically is called a wave matched to the boundary, an example of which is the surface wave propagating along certain impedance surfaces. In addition to finding the plane-wave reflection dyadic for the general IBC surface, and the special cases, the Soft-and-Hard (SH) boundary with its generalization (GSH boundary), the simple-isotropic boundary problem is solved for dipole excitation in terms of its image source.

In Chapter 4, boundaries defined by conditions involving components of fields normal to the boundary surface, are considered. Main attention is given to the DB boundary, at which the fields $\mathbf{D}$ and $\mathbf{B}$ have no normal components. Along with it, the one called D'B' boundary, defined by zero normal derivatives

of the normal components of $\mathbf{D}$ and $\mathbf{B}$, is analysed. Two applications, spherical DB resonator and cylindrical DB waveguide are solved for their basic modes. Finally, attention is given to the more general case of the mixed-impedance (or DB/D'B') boundary.

In Chapter 5, the most general linear and local impedance boundary (GBC boundary) is introduced as a generalization of all of the above cases except the D'B' and DB/D'B' boundaries which are not local ones. It is shown that any plane wave can be decomposed in two parts defined by polarizations depending on the parameters of the boundary and the $\mathbf{k}$ vector of the wave. It is shown that such a polarization property is retained in reflection, whence the decomposed waves do not couple at the boundary. This allows one to express the reflection dyadic as a sum of two simple parts. Conditions for waves matched to the GBC boundary are obtained from zero reflection coefficients. Special cases, like the SHDB boundary and the GSHDB boundary, not falling in the topics of the previous Chapters, are discussed with examples on matched waves in various cases. Reciprocity of the GBC boundary is discussed by analysing different special cases, and realization of the general GBC boundary in terms of the interface of a suitable bi-anisotropic medium is suggested.

As a topic different from the previous ones, based on linear boundary conditions, the class of SQL boundaries, based on sesquilinear boundary conditions, is introduced in Chapter 6. For the basic problem of plane wave incident to an isotropic SQL boundary, it is shown that there are two reflected waves, one of which is forward-reflected and, the other one, is retro-reflected. For certain parameters of the medium, only the retro-reflected wave survives, whence the SQL boundary resembles one made of corner reflectors.

Chapter 7 deals with the numerical analysis of scattering from objects defined by boundary conditions at their surfaces. The manner how electromagnetic waves are affected by such scatterers depends on their size, geometrical shape, and in particular, the character of the boundary condition of their surface. The analysis in this chapter focuses mostly on the canonical spherical geometry. Scatterers with PEC, PMC, PEMC, DB, D'B', and isotropic impedance boundaries are analysed using the full-wave Mie scattering principles. Interesting physical responses of such scatterers are discussed, such as zero-backscattering objects, unexpected front-to-back asymmetries, and sharp resonances for sub-wavelength impedance-boundary scatterers.

The main scope of this book is to introduce the set of most general linear and local boundary conditions and analyse the basic problems of plane-wave reflection and matching associated to a planar boundary in a simple-isotropic medium. Most of the material in this book has been first presented by these authors in various journals listed at the end of the book, and is here given a unified representation. Some of these less common boundary conditions discussed here have already found practical engineering realizations by scientists working on metamaterials and metasurfaces. They are referred to in a proper context in this book.

While the analysis applies Gibbsian 3D vector formalism, familiar to all electrical engineers, the general boundary conditions can be given a simple and

natural representation in terms of 4D multiforms and dyadics, which leads to the impression of their basic physical nature. The formalism is briefly introduced in Appendix C. In Appendices A and B some useful formulas from electromagnetic analysis and Gibbsian 3D dyadics are added for convenience. Many details of the analysis are skipped and left as exercises, solutions of which are outlined in the Appendix D. The aim is as well to compact the text as to leave room for activity in self-learning.

The book is directed to anyone interested in electromagnetic theory in general. For example, those working in the fashionable engineering field of metamaterials and metasurfaces, may find the concepts discussed here useful.

Ismo V. Lindell and Ari Sihvola

Chapter 1

Introduction

1.1 Basic Equations

Maxwell Equations

Electromagnetic problems involving fields and sources are governed by the Maxwell equations, medium equations and boundary conditions. For time-harmonic fields and sources, with time dependence $\exp(j\omega t)$, the Maxwell equations, in the form of Heaviside [19] and the formalism of Gibbs [13], can be expressed as

$$\nabla \times \mathbf{E} + j\omega \mathbf{B} = -\mathbf{J}_m \quad (1.1)$$

$$\nabla \times \mathbf{H} - j\omega \mathbf{D} = \mathbf{J}_e. \quad (1.2)$$

Here, $\mathbf{J}_e$ and $\mathbf{J}_m$ denote the respective electric and magnetic current density vectors while $\mathbf{E}, \mathbf{H}, \mathbf{D}$ and $\mathbf{B}$ represent the respective electric field, magnetic field, electric flux-density and magnetic flux-density vectors. All of these vectors may have complex components.

Medium Conditions

Conditions between the field vectors in a bianisotropic medium can be expressed in the form [29, 35]

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \bar{\bar{\epsilon}} & \bar{\bar{\xi}} \\ \bar{\bar{\zeta}} & \bar{\bar{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad (1.3)$$

where $\bar{\bar{\epsilon}}, \bar{\bar{\xi}}, \bar{\bar{\zeta}}$ and $\bar{\bar{\mu}}$ are medium dyadics. In terms of a given vector basis, any dyadic can be represented in terms of a 3×3 matrix involving nine scalar components (for rules of dyadic algebra, see Appendix C).

Here we mainly consider problems involving the simple-isotropic medium ('free space, vacuum'), for which the medium conditions are reduced to

$$\mathbf{D} = \epsilon_o \mathbf{E}, \quad \mathbf{B} = \mu_o \mathbf{H}. \quad (1.4)$$

The wave impedance η_o , expressing the ratio of the electric and magnetic field magnitudes in the simple-isotropic medium, is defined by

$$\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}}. \quad (1.5)$$

Boundary Conditions

Boundary conditions form an essential part of formulating electromagnetic problems. By a boundary we mean a surface on which secondary electromagnetic sources are induced by the primary fields so that the fields beyond the surface vanish. A boundary surface is different from an interface of two media because the fields beyond the interface are not necessarily zero. From the mathematical point of view, boundary conditions are required to ensure existence and uniqueness of solutions for a particular problem. From the engineering point of view, two aspects of boundary conditions can be separated which may be called analytic and synthetic.

An *analytic* aspect of boundary conditions is encountered when a given physical problem requires mathematical analysis. Due to natural complications, a given structure must often be approximated by certain boundary conditions to find a numerical solution [21, 88]. As an example, solving radio wave propagation over ground requires that the ground be approximated by an impedance boundary or, if well-enough conducting, by the perfect electric conductor (PEC). Thus, in such a case, to analyze the effect of a given structure, we replace it by some approximate boundary conditions.

In contrast, a *synthetic* aspect emerges when we wish to realize given boundary conditions by some physical structure. As an example, when designing mobile phones with submerged antennas, the concept of perfect magnetic conductor (PMC) boundary has been suggested to solve the problem of efficient radiation [92]. In this case, the problem becomes a synthesis, how to find a physical structure to realize the PMC conditions.

In this book we are concerned about the synthetic aspect by studying different types of boundary conditions, their effect on electromagnetic fields, and possible realizations by interfaces of media defined by medium parameters. More practical physical realizations by (meta)materials are, however, beyond the topic of the book. Let us start with three most basic boundary conditions, properties of which will be considered in due course.

- Perfect Electric Conductor (PEC)

$$\mathbf{n} \times \mathbf{E} = 0 \quad (1.6)$$

- Perfect Magnetic Conductor (PMC)

$$\mathbf{n} \times \mathbf{H} = 0, \quad (1.7)$$

- Perfect Electromagnetic Conductor (PEMC) [45]

$$\mathbf{n} \times (\mathbf{H} + M\mathbf{E}) = 0 \quad (1.8)$$

The PEMC, involving a parameter M (the PEMC admittance), is a generalization of both the PEC and the PMC. For $M = 0$, the PEMC reduces to the PMC, and for $1/M = 0$, it reduces to the PEC.

1.2 Duality Transformation

A given electromagnetic problem can be transformed to another one in terms of duality transformation which does not change the geometry of the problem, but the sources, fields, medium and boundary parameters are transformed to have other values, in general. The concept of duality was evidently unknown to Maxwell, because he presented his equations in a very nonsymmetric form, in terms of 20 scalar field and potential quantities [79]. The concept was introduced by Heaviside in 1886 [19].

Duality transformation is based on the apparent symmetry of the Maxwell equations (1.1), (1.2), written more compactly as

$$\nabla \times \begin{pmatrix} \mathbf{E} \\ -\mathbf{H} \end{pmatrix} + j\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = - \begin{pmatrix} \mathbf{J}_m \\ \mathbf{J}_e \end{pmatrix}. \quad (1.9)$$

In fact, for the simple change of symbols $\mathbf{E} \leftrightarrow -\mathbf{H}$, $\mathbf{B} \leftrightarrow \mathbf{D}$ and $\mathbf{J}_e \leftrightarrow \mathbf{J}_m$, the pair of equations (1.9) is invariant. More generally, the same property can be expressed in terms of the duality transformation $(\mathbf{E}, \mathbf{H}) \rightarrow (\mathbf{E}_d, \mathbf{H}_d)$, defined by [35]

$$\begin{pmatrix} \mathbf{E}_d \\ \eta_o \mathbf{H}_d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \eta_o \mathbf{H} \end{pmatrix}. \quad (1.10)$$

The wave impedance η_o has been included to obtain dimensionless transformation parameters $A \cdots D$. They are assumed to satisfy

$$AD - BC = 1, \quad (1.11)$$

whence the inverse transformation exists and has the form

$$\begin{pmatrix} \mathbf{E} \\ \eta_o \mathbf{H} \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \begin{pmatrix} \mathbf{E}_d \\ \eta_o \mathbf{H}_d \end{pmatrix}. \quad (1.12)$$

Applying (1.10) to (1.1) and (1.2), the associated transformation rules can be expressed as

$$\begin{pmatrix} \eta_o \mathbf{D}_d \\ \mathbf{B}_d \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \eta_o \mathbf{D} \\ \mathbf{B} \end{pmatrix} \quad (1.13)$$

$$\begin{pmatrix} \eta_o \mathbf{J}_{ed} \\ \mathbf{J}_{md} \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \eta_o \mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix} \quad (1.14)$$

One can show that the PEMC boundary conditions (1.8) are transformed to

$$\mathbf{n} \times (\mathbf{H}_d + M_d \mathbf{E}_d) = 0, \quad (1.15)$$

with the transformed PEMC admittance satisfying

$$M_d \eta_o = -\frac{C - DM\eta_o}{A - BM\eta_o}. \quad (1.16)$$

From this it follows that both PEC and PMC boundaries are transformed to PEMC boundaries with $M_d \eta_o = -D/B$ and $M_d \eta_o = -C/A$, respectively. Also, any given PEMC boundary can be transformed to PEC and PMC boundaries when the transformation parameters are chosen to satisfy the respective restrictions $A/B = M\eta_o$ and $C/D = M\eta_o$.

In the general case, the dyadic parameters of the electromagnetic medium will be changed when the fields are subject to the duality transformation (1.10). For a bianisotropic medium defined by conditions of the form (1.3), the transformed medium dyadics can be shown to obey the relations [42]

$$\begin{pmatrix} \bar{\bar{\epsilon}}_d \\ \bar{\bar{\xi}}_d \\ \bar{\bar{\zeta}}_d \\ \bar{\bar{\mu}}_d \end{pmatrix} = \begin{pmatrix} D^2 & -CD/\eta_o & -CD/\eta_o & C^2/\eta_o^2 \\ -BD\eta_o & AD & BC & -AC/\eta_o \\ -BD\eta_o & BC & AD & -AC/\eta_o \\ B^2\eta_o^2 & -AB\eta_o & -AB\eta_o & A^2 \end{pmatrix} \begin{pmatrix} \bar{\bar{\epsilon}} \\ \bar{\bar{\xi}} \\ \bar{\bar{\zeta}} \\ \bar{\bar{\mu}} \end{pmatrix}. \quad (1.17)$$

Requiring a transformation in which the simple isotropic medium (μ_o, ϵ_o) is invariant leads to a choice of the form

$$A = D = \cos \varphi, \quad B = -C = \sin \varphi, \quad (1.18)$$

where φ is a free transformation parameter. The resulting transformation matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \quad (1.19)$$

can be recognized as a 2D rotation matrix. The corresponding transformation rule for the medium dyadics (1.17) becomes in this case

$$\begin{pmatrix} \bar{\bar{\epsilon}}_d/\epsilon_o \\ \bar{\bar{\xi}}_d/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\zeta}}_d/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\mu}}_d/\mu_o \end{pmatrix} = \mathcal{Q}(\varphi) \begin{pmatrix} \bar{\bar{\epsilon}}/\epsilon_o \\ \bar{\bar{\xi}}/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\zeta}}/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\mu}}/\mu_o \end{pmatrix}, \quad (1.20)$$

with the 4×4 matrix $\mathcal{Q}(\varphi)$ defined by

$$\mathcal{Q}(\varphi) = \begin{pmatrix} \cos^2 \varphi & \sin \varphi \cos \varphi & \sin \varphi \cos \varphi & \sin^2 \varphi \\ -\sin \varphi \cos \varphi & \cos^2 \varphi & -\sin^2 \varphi & \sin \varphi \cos \varphi \\ -\sin \varphi \cos \varphi & -\sin^2 \varphi & \cos^2 \varphi & \sin \varphi \cos \varphi \\ \sin^2 \varphi & -\sin \varphi \cos \varphi & -\sin \varphi \cos \varphi & \cos^2 \varphi \end{pmatrix}. \quad (1.21)$$

For the simple-isotropic medium with $\bar{\epsilon} = \epsilon_o \bar{\mathbf{I}}$, $\bar{\xi} = \bar{\zeta} = 0$ and $\bar{\mu} = \mu_o \bar{\mathbf{I}}$, from (1.20) we obtain $\bar{\epsilon}_d = \epsilon_o \bar{\mathbf{I}}$, $\bar{\xi}_d = \bar{\zeta}_d = 0$ and $\bar{\mu}_d = \mu_o \bar{\mathbf{I}}$, as required. One can further show that the matrix $\mathcal{Q}(\varphi)$ satisfies

$$\mathcal{Q}(\varphi_1)\mathcal{Q}(\varphi_2) = \mathcal{Q}(\varphi_1 + \varphi_2), \quad (1.22)$$

$$\det \mathcal{Q}(\varphi) = 1, \quad (1.23)$$

$$\mathcal{Q}^{-1}(\varphi) = \mathcal{Q}^T(\varphi) = \mathcal{Q}(-\varphi), \quad (1.24)$$

i.e., it is an orthogonal matrix. The proof is left as an exercise.

Expressing the PEMC admittance parameter M in terms of another parameter φ as

$$M\eta_o = \tan \vartheta, \quad (1.25)$$

the transformation rule (1.16) takes the simple form

$$M_d\eta_o = \frac{M\eta_o + \tan \varphi}{1 - M\eta_o \tan \varphi} = \tan(\vartheta + \varphi). \quad (1.26)$$

Thus, for the choice $\varphi = -\vartheta$ of the duality parameter, the PEMC boundary is transformed to the PMC boundary ($M_d = 0$), while for $\varphi = \pi/2 - \vartheta$, it is transformed to the PEC boundary ($1/M_d = 0$).

1.3 Plane Waves

Basic Conditions

Let us consider time-harmonic plane waves with $\exp(j\omega t)$ dependence in a simple isotropic medium, in front of a boundary surface defined by $\mathbf{n} \cdot \mathbf{r} = 0$. For simplicity we assume that the surface is a plane, i.e., that $\mathbf{n}$ is a constant unit vector. The electric and magnetic field components of waves incident to the boundary are expressed by

$$\mathbf{E}^i(\mathbf{r}) = \mathbf{E}^i \exp(-j\mathbf{k}^i \cdot \mathbf{r}), \quad (1.27)$$

$$\mathbf{H}^i(\mathbf{r}) = \mathbf{H}^i \exp(-j\mathbf{k}^i \cdot \mathbf{r}), \quad (1.28)$$

and the reflected fields are

$$\mathbf{E}^r(\mathbf{r}) = \mathbf{E}^r \exp(-j\mathbf{k}^r \cdot \mathbf{r}), \quad (1.29)$$

$$\mathbf{H}^r(\mathbf{r}) = \mathbf{H}^r \exp(-j\mathbf{k}^r \cdot \mathbf{r}). \quad (1.30)$$

To satisfy the Maxwell equations, the two wave vectors must satisfy the dispersion equation as

$$\mathbf{k}^i \cdot \mathbf{k}^i = \mathbf{k}^r \cdot \mathbf{k}^r = k_o^2 = \omega^2 \mu_o \epsilon_o. \quad (1.31)$$

If the boundary condition is linear, the $\mathbf{k}^i$ and $\mathbf{k}^r$ vectors must have the same components tangential to the boundary surface, i.e., they can be represented as

$$\mathbf{k}^i = \mathbf{k}_t - k_n \mathbf{n}, \quad \mathbf{k}^r = \mathbf{k}_t + k_n \mathbf{n}. \quad (1.32)$$

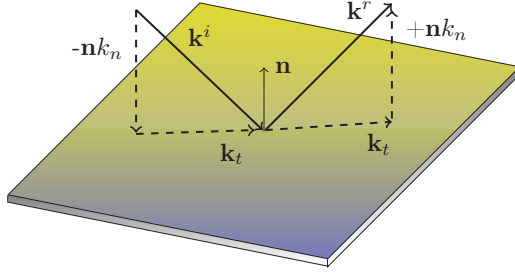


Figure 1.1: The incident and reflected plane waves have the same wave-vector components $\mathbf{k}_t$ tangential to the boundary surface.

A class of boundaries with conditions deviating from linear ones is considered in Chapter 6.

Substituting the plane-wave fields in the Maxwell equations (1.1) and (1.2), equations relating the electric and magnetic fields are obtained as

$$\mathbf{k}^i \times \mathbf{E}^i = k_o \eta_o \mathbf{H}^i, \quad (1.33)$$

$$\mathbf{k}^i \times \eta_o \mathbf{H}^i = -k_o \mathbf{E}^i, \quad (1.34)$$

and

$$\mathbf{k}^r \times \mathbf{E}^r = k_o \eta_o \mathbf{H}^r, \quad (1.35)$$

$$\mathbf{k}^r \times \eta_o \mathbf{H}^r = -k_o \mathbf{E}^r. \quad (1.36)$$

The field components normal to the boundary can be expressed in terms of their tangential components from the orthogonality conditions as

$$\mathbf{k}^i \cdot \mathbf{E}^i = 0, \quad \Rightarrow \quad \mathbf{n} \cdot \mathbf{E}^i = \frac{1}{k_n} \mathbf{k}_t \cdot \mathbf{E}_t^i, \quad (1.37)$$

$$\mathbf{k}^r \cdot \mathbf{E}^r = 0, \quad \Rightarrow \quad \mathbf{n} \cdot \mathbf{E}^r = -\frac{1}{k_n} \mathbf{k}_t \cdot \mathbf{E}_t^r, \quad (1.38)$$

valid for $k_n \neq 0$. Similar conditions are valid for the magnetic fields.

Field Relations

From (1.33) and (1.34) we obtain the following relations between the tangential components of the incident fields:

$$\mathbf{n} \cdot \mathbf{k}_t \times \mathbf{E}_t^i = \frac{k_o}{k_n} \mathbf{k}_t \cdot \eta_o \mathbf{H}_t^i, \quad (1.39)$$

$$\mathbf{n} \cdot \mathbf{k}_t \times \eta_o \mathbf{H}_t^i = -\frac{k_o}{k_n} \mathbf{k}_t \cdot \mathbf{E}_t^i. \quad (1.40)$$

Assuming also $\mathbf{k}_t \cdot \mathbf{k}_t = k_t^2 \neq 0$, the vectors $\mathbf{k}_t$ and $\mathbf{n} \times \mathbf{k}_t$ serve as a 2D basis, in terms of which we can expand the 2D unit dyadic as

$$\bar{\bar{\mathbf{l}}}_t = \bar{\bar{\mathbf{l}}} - \mathbf{nn} = \frac{1}{k_t^2}(\mathbf{k}_t \mathbf{k}_t + (\mathbf{n} \times \mathbf{k}_t)(\mathbf{n} \times \mathbf{k}_t)). \quad (1.41)$$

The relation between the tangential electric and magnetic fields can be expanded as

$$\begin{aligned} \mathbf{E}_t^i &= \frac{1}{k_t^2}(\mathbf{k}_t(\mathbf{k}_t \cdot \mathbf{E}_t^i) + (\mathbf{n} \times \mathbf{k}_t)(\mathbf{n} \times \mathbf{k}_t \cdot \mathbf{E}_t^i)) \\ &= \frac{1}{k_t^2}\left(-\frac{k_n}{k_o}\mathbf{k}_t(\mathbf{n} \times \mathbf{k}_t) + \frac{k_o}{k_n}(\mathbf{n} \times \mathbf{k}_t)\mathbf{k}_t\right) \cdot \eta_o \mathbf{H}_t^i \\ &= \frac{1}{k_o k_n} \mathbf{n} \times (\mathbf{k}_t \mathbf{k}_t + k_n^2 \bar{\bar{\mathbf{l}}}_t) \cdot \eta_o \mathbf{H}_t^i. \end{aligned} \quad (1.42)$$

The dyadic in this relation is of importance in the analysis of plane waves. Denoted by

$$\bar{\bar{\mathbf{J}}}_t = \frac{1}{k_o k_n} \mathbf{n} \times (\mathbf{k}_t \mathbf{k}_t + k_n^2 \bar{\bar{\mathbf{l}}}_t), \quad (1.43)$$

it can be shown to satisfy the properties

$$\text{tr} \bar{\bar{\mathbf{J}}}_t = 0, \quad (1.44)$$

$$\bar{\bar{\mathbf{J}}}_t^2 = -\bar{\bar{\mathbf{l}}}_t, \quad (1.45)$$

$$\bar{\bar{\mathbf{J}}}_t^{-1} = -\bar{\bar{\mathbf{J}}}_t, \quad (1.46)$$

derivations of which are left as exercises.

The dyadic $\bar{\bar{\mathbf{J}}}_t$ allows one to write relations between the tangential electric and magnetic fields of a plane wave in compact form as

$$\mathbf{E}_t^i = \bar{\bar{\mathbf{J}}}_t \cdot \eta_o \mathbf{H}_t^i \quad \eta_o \mathbf{H}_t^i = -\bar{\bar{\mathbf{J}}}_t \cdot \mathbf{E}_t^i. \quad (1.47)$$

For the reflected fields, similar expressions are valid when replacing k_n by $-k_n$:

$$\mathbf{E}_t^r = -\bar{\bar{\mathbf{J}}}_t \cdot \eta_o \mathbf{H}_t^r, \quad \eta_o \mathbf{H}_t^r = \bar{\bar{\mathbf{J}}}_t \cdot \mathbf{E}_t^r. \quad (1.48)$$

The expressions (1.47) and (1.48) will be needed in the analysis of reflections from various boundaries. In this, we can also make use of the following set of relations:

$$\mathbf{k}_t \cdot \bar{\bar{\mathbf{J}}}_t = -\frac{k_n}{k_o} \mathbf{n} \times \mathbf{k}_t, \quad \bar{\bar{\mathbf{J}}}_t \cdot \mathbf{k}_t = \frac{k_o}{k_n} \mathbf{n} \times \mathbf{k}_t, \quad (1.49)$$

$$(\mathbf{n} \times \mathbf{k}_t) \cdot \bar{\bar{\mathbf{J}}}_t = \frac{k_o}{k_n} \mathbf{k}_t, \quad \bar{\bar{\mathbf{J}}}_t \cdot (\mathbf{n} \times \mathbf{k}_t) = -\frac{k_n}{k_o} \mathbf{k}_t, \quad (1.50)$$

and,

$$\bar{\bar{\mathbf{J}}}_t^{(2)} = \frac{1}{2} \bar{\bar{\mathbf{J}}}_t \times \bar{\bar{\mathbf{J}}}_t = \mathbf{nn}, \quad \det_t \bar{\bar{\mathbf{J}}}_t = \text{tr} \bar{\bar{\mathbf{J}}}_t^{(2)} = 1. \quad (1.51)$$

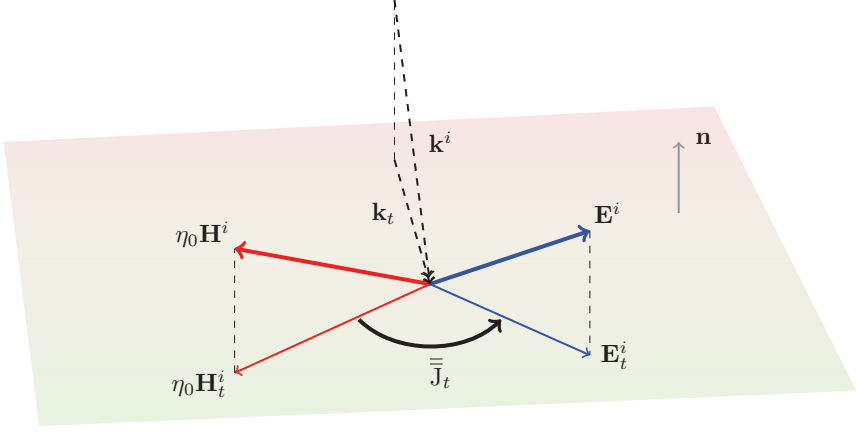


Figure 1.2: The dyadic $\bar{\bar{J}}_t$ maps the tangential component of the magnetic field of an incident plane wave to the tangential component of the electric field. The vectors $\mathbf{E}^i$, $\mathbf{H}^i$ and $\mathbf{k}^i$ are orthogonal to one another.

The dyadic $\bar{\bar{J}}_t$ and the antisymmetric dyadic $\mathbf{n} \times \bar{\bar{I}}$ share the property $(\mathbf{n} \times \bar{\bar{I}})^2 = \bar{\bar{J}}_t^2 = -\bar{\bar{I}}_t$. While $\mathbf{n} \times \bar{\bar{I}}$ rotates vectors on the tangent plane by $\pi/2$, the dyadic $\bar{\bar{J}}_t$ "rotates" the tangential part of the electric field to the tangential part of the magnetic field of the same plane wave. The $\bar{\bar{J}}_t$ dyadic is determined by $\mathbf{k}_t$, the tangential part of the wave vector in the simple-isotropic medium.

1.4 TE/TM Decomposition

It turns out that the field vectors of a plane wave, propagating with the wave vector $\mathbf{k} = \mathbf{n}k_n + \mathbf{k}_t$, can be expressed in terms of their normal components $\mathbf{n} \cdot \mathbf{E}$ and $\mathbf{n} \cdot \mathbf{H}$ as

$$\mathbf{E} = \frac{1}{k_t^2} ((k_t^2 \mathbf{n} - k_n \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{E}) + k_o (\mathbf{n} \times \mathbf{k})(\mathbf{n} \cdot \eta_o \mathbf{H})) \quad (1.52)$$

$$\eta_o \mathbf{H} = \frac{1}{k_t^2} ((k_t^2 \mathbf{n} - k_n \mathbf{k}_t)(\mathbf{n} \cdot \eta_o \mathbf{H}) - k_o (\mathbf{n} \times \mathbf{k})(\mathbf{n} \cdot \mathbf{E})). \quad (1.53)$$

Details of the derivation are left as an exercise. Applying (1.52) and (1.53), any plane wave can be decomposed in two parts as

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{TE} \\ \mathbf{H}_{TE} \end{pmatrix} + \begin{pmatrix} \mathbf{E}_{TM} \\ \mathbf{H}_{TM} \end{pmatrix}, \quad (1.54)$$

with

$$\mathbf{E}_{TE} = \frac{k_o}{k_t^2}(\mathbf{n} \times \mathbf{k})(\mathbf{n} \cdot \eta_o \mathbf{H}), \quad (1.55)$$

$$\mathbf{H}_{TE} = \frac{1}{k_t^2}(k_t^2 \mathbf{n} - k_n \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{H}), \quad (1.56)$$

$$\mathbf{E}_{TM} = \frac{1}{k_t^2}(k_t^2 \mathbf{n} - k_n \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{E}), \quad (1.57)$$

$$\eta_o \mathbf{H}_{TM} = -\frac{k_o}{k_t^2}(\mathbf{n} \times \mathbf{k})(\mathbf{n} \cdot \mathbf{E}). \quad (1.58)$$

The partial fields are called tranverse electric (TE) and transverse magnetic (TM) because the two parts are restricted by the respective conditions

$$\mathbf{n} \cdot \mathbf{E}_{TE} = 0, \quad \mathbf{n} \cdot \mathbf{H}_{TM} = 0. \quad (1.59)$$

The decomposed fields satisfy the Maxwell equations separately. To check this, we can expand

$$\nabla \times \mathbf{E}_{TE} = -j\mathbf{k} \times \mathbf{E}^{TE} \quad (1.60)$$

$$= -j \frac{k_o \eta_o}{k_t^2} \mathbf{k} \times (\mathbf{n} \times \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{H}) \quad (1.61)$$

$$= -j \frac{k_o \eta_o}{k_t^2} (\mathbf{n} k_t^2 - k_n \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{H}) \quad (1.62)$$

$$= -j k_o \eta_o \mathbf{H}^{TE}, \quad (1.63)$$

$$\nabla \times \mathbf{H}_{TM} = -j\mathbf{k} \times \mathbf{H}^{TM} \quad (1.64)$$

$$= j \frac{k_o}{\eta_o k_t^2} \mathbf{k} \times (\mathbf{n} \times \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{E}) \quad (1.65)$$

$$= j \frac{k_o}{\eta_o k_t^2} (\mathbf{n} k_t^2 - k_n \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{E}) \quad (1.66)$$

$$= j \frac{k_o}{\eta_o} \mathbf{E}_{TM}. \quad (1.67)$$

The decomposition is unique for $\mathbf{k}_t \neq 0$. For $\mathbf{k}_t = 0$ the expansions (1.52) and (1.53) are not applicable. In this case the TE/TM decomposition is still possible but not unique. Actually, for $\mathbf{k}_t = 0$ the plane-wave has TEM polarization, $\mathbf{n} \cdot \mathbf{E} = \mathbf{n} \cdot \mathbf{H} = 0$.

Because the TE/TM decomposition of a plane wave does not depend on the choice of the $\mathbf{k}$ vector, it is valid for a sum of plane waves and, ultimately, for an integral of plane waves in the space of $\mathbf{k}$ vectors which may have complex values. Thus, any fields which can be expressed as an integral of plane waves can be decomposed in two independent TE and TM parts with respect to a given $\mathbf{n}$ vector.

1.5 Problems

- 1.1** For a given duality transformation, find a PEMC admittance which is invariant in the transformation. Apply the result to the special transformation (1.19).
- 1.2** Derive the transformation rule (1.17) for the medium dyadics and show that the dyadic $\bar{\bar{\xi}} - \bar{\bar{\zeta}}$ is invariant in the transformation.
- 1.3** Derive the transformation rule (1.20) from (1.17).
- 1.4** Show that the matrix $\mathcal{Q}(\varphi)$ in (1.21) is orthogonal, i.e., that it satisfies
- $$\mathcal{Q}^{-1}(\varphi) = \mathcal{Q}^T(\varphi) = \mathcal{Q}(-\varphi).$$
- 1.5** Show that it is possible to find a duality transformation which maps two given PEMC admittances M_1 and M_2 so that $M_{1d} = -M_{2d} = M_d$ and leaves the simple-isotropic medium invariant. Find the relation between M_1 , M_2 and M_d .
- 1.6** Derive the properties (1.45), (1.46) and (1.51) of the dyadic $\bar{\bar{\mathbf{J}}}_t$.
- 1.7** Derive the decomposition rules (1.52) and (1.53).
- 1.8** Applying (1.20), find another expression for the duality transformation of the medium dyadics in the form

$$\begin{pmatrix} \bar{\bar{\epsilon}}_d/\epsilon_o - \bar{\bar{\mu}}_d/\mu_o \\ (\bar{\bar{\xi}}_d - \bar{\bar{\zeta}}_d)/\sqrt{\mu_o\epsilon_o} \\ (\bar{\bar{\xi}}_d + \bar{\bar{\zeta}}_d)/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\epsilon}}_d/\epsilon_o + \bar{\bar{\mu}}_d/\mu_o \end{pmatrix} = \mathcal{Q}'(\varphi) \begin{pmatrix} \bar{\bar{\epsilon}}/\epsilon_o - \bar{\bar{\mu}}/\mu_o \\ (\bar{\bar{\xi}} - \bar{\bar{\zeta}})/\sqrt{\mu_o\epsilon_o} \\ (\bar{\bar{\xi}} + \bar{\bar{\zeta}})/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\epsilon}}/\epsilon_o + \bar{\bar{\mu}}/\mu_o \end{pmatrix}$$

in terms of a matrix $\mathcal{Q}'(\varphi)$. Which dyadics appear invariant in the transformation?

Chapter 2

Perfect Electromagnetic Conductor Boundary

2.1 PEMC Conditions

The condition (1.8), defining the perfect electromagnetic conductor (PEMC) boundary, actually represents one of the most basic conditions in electromagnetic theory. In fact, applying the four-dimensional formalism of Appendix A, the PEMC medium (also called the axion medium [20, 72, 97, 99]) is defined by the simplest possible linear relation between the electromagnetic two-forms $\Phi = \mathbf{B} + \mathbf{E} \wedge \varepsilon_4$ and $\Psi = \mathbf{D} - \mathbf{H} \wedge \varepsilon_4$ as

$$\Psi = M\Phi. \quad (2.1)$$

In terms of the spatial two-form components $\mathbf{D}, \mathbf{B}$ and spatial one-form components $\mathbf{H}, \mathbf{E}$, (2.1) equals the conditions

$$\mathbf{D} = M\mathbf{B}, \quad \mathbf{H} = -M\mathbf{E}. \quad (2.2)$$

In Gibbsian 3D formalism, the corresponding PEMC medium conditions between the field vectors¹ are

$$\begin{pmatrix} \mathbf{H} \\ \mathbf{D} \end{pmatrix} = M \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix}. \quad (2.3)$$

There is no natural way to express these conditions in the form (1.3) in terms of finite medium dyadics. However, they can be represented as the limit [46]

$$\begin{pmatrix} \bar{\bar{\xi}} & \bar{\bar{\xi}} \\ \bar{\bar{\xi}} & \bar{\bar{\mu}} \end{pmatrix} = \lim_{q \rightarrow \infty} q \begin{pmatrix} M & 1 \\ 1 & 1/M \end{pmatrix} \bar{\bar{\mathbf{I}}}. \quad (2.4)$$

¹Although the Gibbsian field vectors and the field one- and two-forms are here denoted by similar symbols, they obey different algebraic rules.

A slightly more natural form with the same limit is [83, 72]

$$\begin{pmatrix} \bar{\bar{\epsilon}} & \bar{\bar{\xi}} \\ \bar{\bar{\xi}} & \bar{\bar{\mu}} \end{pmatrix} = \lim_{q \rightarrow \infty} q \begin{pmatrix} M(1 + (\mu_o \epsilon_o / q^2)) & 1 \\ 1 & 1/M \end{pmatrix} \bar{\bar{\mathbf{I}}}. \quad (2.5)$$

In both of these representations all four medium dyadics $\bar{\bar{\epsilon}} \cdots \bar{\bar{\mu}}$ become ultimately infinite in magnitude. However, while in the case (2.4) the matrix has no inverse for any q , in the case of (2.5), the matrix does have an inverse for finite values of q , whence it corresponds to a more ordinary bi-anisotropic medium.

Continuity of the tangential components of $\mathbf{E}$ and $\mathbf{H}$, and the normal components of $\mathbf{D}$ and $\mathbf{B}$, at an interface of a PEMC medium, yields the PEMC boundary conditions,

$$\mathbf{n} \times (\mathbf{H} + M\mathbf{E}) = 0, \quad (2.6)$$

$$\mathbf{n} \cdot (\mathbf{D} - M\mathbf{B}) = 0. \quad (2.7)$$

Actually, the condition (2.7) is not necessary, because it can be shown to follow from (2.6) and the Maxwell equations. The proof is left as an exercise. Although (2.7) carries no additional information beyond (2.6), it can be useful in reducing computational effort in some practical cases.

The PEMC boundary can be realized by an interface of the PEMC medium. However, it is not obvious how to realize the PEMC medium. A more practical realization can be based on a slab of medium with suitable properties. Since the PEMC boundary is isotropic and nonreciprocal, it would be natural to apply a slab of uniaxial gyrotropic medium with axis parallel to the vector $\mathbf{n}$ [47]. In Chapter 3 such a realization (3.259), (3.260) will be considered as associated with the realization of more general impedance-boundary surfaces [51].

2.2 Eigenproblem of Dyadic $\bar{\bar{\mathbf{J}}}_t$

Assuming a field consisting of incident and reflected plane waves, the PEMC condition (2.6) can be written for the tangential field components as

$$\eta_o(\mathbf{H}_t^r + \mathbf{H}_t^i) + M\eta_o(\mathbf{E}_t^r + \mathbf{E}_t^i) = 0. \quad (2.8)$$

Substituting the plane-wave relations (1.47) and (1.48), we can write

$$\bar{\bar{\mathbf{J}}}_t \cdot (\mathbf{E}_t^r - \mathbf{E}_t^i) + M\eta_o(\mathbf{E}_t^i + \mathbf{E}_t^r) = 0. \quad (2.9)$$

It appears convenient to expand the tangential field components in terms of the eigenvectors of the $\bar{\bar{\mathbf{J}}}_t$ dyadic (1.43),

$$\bar{\bar{\mathbf{J}}}_t = \frac{1}{k_o k_n} \mathbf{n} \times (\mathbf{k}_t \mathbf{k}_t + k_n^2 \bar{\bar{\mathbf{I}}}_t). \quad (2.10)$$

After some algebraic steps we can derive the relations

$$\bar{\bar{\mathbf{J}}}_t \cdot (k_n \mathbf{k}_t \mp j k_o \mathbf{n} \times \mathbf{k}_t) = \pm j (k_n \mathbf{k}_t \mp j k_o \mathbf{n} \times \mathbf{k}_t), \quad (2.11)$$

details of which are left as an exercise. The solutions for the eigenvalue equation

$$\bar{\mathbf{J}}_t \cdot \mathbf{x}_{t\pm} = J_{\pm} \mathbf{x}_{t\pm} \quad (2.12)$$

can now be identified from (2.11) as

$$J_{\pm} = \pm j, \quad \mathbf{x}_{t\pm} = k_n \mathbf{k}_t \mp j k_o \mathbf{n} \times \mathbf{k}_t. \quad (2.13)$$

Applying the rule

$$\mathbf{x}_{t+} \mathbf{x}_{t-} - \mathbf{x}_{t-} \mathbf{x}_{t+} = -(\mathbf{x}_{t+} \times \mathbf{x}_{t-}) \times \bar{\mathbf{l}} = -X \mathbf{n} \times \bar{\mathbf{l}}, \quad (2.14)$$

$$X = \mathbf{n} \cdot (\mathbf{x}_{t+} \times \mathbf{x}_{t-}) = 2j k_n k_o k_t^2, \quad (2.15)$$

and assuming $k_n k_t^2 \neq 0$, whence $X \neq 0$, we obtain

$$\bar{\mathbf{l}}_t = \frac{1}{X} (\mathbf{x}_{t+} \mathbf{x}_{t-} - \mathbf{x}_{t-} \mathbf{x}_{t+}) \times \mathbf{n}. \quad (2.16)$$

The eigenvectors make a 2D basis, in terms of which we can expand

$$\bar{\mathbf{J}}_t = \bar{\mathbf{J}}_t \cdot \bar{\mathbf{l}}_t = \frac{j}{X} (\mathbf{x}_{t+} \mathbf{x}_{t-} + \mathbf{x}_{t-} \mathbf{x}_{t+}) \times \mathbf{n}. \quad (2.17)$$

The tangential component of the incident electric field can now be decomposed in eigenvectors as

$$\mathbf{E}_t^i = \mathbf{E}_{t+}^i + \mathbf{E}_{t-}^i = A_+ \mathbf{x}_{t+} + A_- \mathbf{x}_{t-}, \quad (2.18)$$

with

$$A_+ = \frac{1}{X} (\mathbf{x}_{t-} \times \mathbf{n}) \cdot \mathbf{E}_t^i, \quad A_- = -\frac{1}{X} (\mathbf{x}_{t+} \times \mathbf{n}) \cdot \mathbf{E}_t^i. \quad (2.19)$$

Applying $\mathbf{k}^i \cdot \mathbf{E}_{\pm}^i = 0$, the corresponding total incident fields can be expressed as

$$\begin{aligned} \mathbf{E}_{\pm}^i &= \frac{\mathbf{n}}{k_n} (\mathbf{k}_t \cdot \mathbf{E}_{t\pm}^i) + \mathbf{E}_{t\pm}^i \\ &= A_{\pm} (\mathbf{n} k_t^2 + k_n \mathbf{k}_t \mp j k_o \mathbf{n} \times \mathbf{k}_t). \end{aligned}$$

One can easily verify that they satisfy

$$\mathbf{E}_+^i \cdot \mathbf{E}_+^i = \mathbf{E}_-^i \cdot \mathbf{E}_-^i = 0, \quad (2.20)$$

whence the incident fields $\mathbf{E}_+^i$ and $\mathbf{E}_-^i$, whose tangential components are eigenvectors of the dyadic $\bar{\mathbf{J}}_t$, must be circularly polarized. From (1.47) the corresponding tangential and total magnetic eigenfields satisfy

$$\eta_o \mathbf{H}_{t\pm}^i = -\bar{\mathbf{J}}_t \cdot \mathbf{E}_{t\pm}^i = \mp j \mathbf{E}_{t\pm}^i, \quad (2.21)$$

$$\eta_o \mathbf{H}_{\pm}^i = \mp j \mathbf{E}_{\pm}^i, \quad (2.22)$$

whence also the magnetic fields $\mathbf{H}_+^i, \mathbf{H}_-^i$ are circularly polarized. Plane-wave fields reflected from the PEMC boundary can be decomposed similarly in two circularly polarized parts.

Assuming real vector of propagation $\mathbf{k}^i$, from (2.13) we have $\mathbf{x}_{t\pm}^* = \mathbf{x}_{t\mp}$, whence $\mathbf{E}_+^{i*}$ and $\mathbf{E}_-^i$ are multiples of the same vector. Expanding

$$\begin{aligned}\mathbf{E}_\pm^i \times \mathbf{E}_\pm^{i*} &= \pm j 2 |A_\pm|^2 (\mathbf{n} k_t^2 + k_n \mathbf{k}_t) \times (k_o \mathbf{n} \times \mathbf{k}_t) \\ &= \mp j 2 k_o k_t^2 |A_\pm|^2 \mathbf{k}^i,\end{aligned}\quad (2.23)$$

$$\mathbf{E}_\pm^i \cdot \mathbf{E}_\pm^{i*} = |A_\pm|^2 (k_t^4 + k_n^2 k_t^2 - k_o^2 k_t^2) = 2 k_o^2 k_t^2 |A_\pm|^2, \quad (2.24)$$

the polarization vectors (see Appendix B) of the two field vectors $\mathbf{E}_\pm^i$ become

$$\mathbf{p}(\mathbf{E}_\pm^i) = \frac{\mathbf{E}_\pm^i \times \mathbf{E}_\pm^{i*}}{j \mathbf{E}_\pm^i \cdot \mathbf{E}_\pm^{i*}} = \mp \mathbf{u}^i, \quad (2.25)$$

where $\mathbf{u}^i = \mathbf{k}^i / k_o$ is a real unit vector. Because the time-harmonic vectors $\mathbf{E}_\pm^i(t)$ have right-handed rotation when looking into the directions of $\mathbf{p}(\mathbf{E}_\pm^i)$, the fields $\mathbf{E}_+^i$ and $\mathbf{E}_-^i$ have respectively left and right-handed circular polarizations when looking in the direction of propagation $\mathbf{u}^i$.

As a consequence of the property

$$\mathbf{E}_\pm^i \cdot \mathbf{E}_\pm^i = 0 \quad \Rightarrow \quad \mathbf{E}_\pm^i \cdot \mathbf{E}_\mp^{i*} = 0, \quad (2.26)$$

valid for real $\mathbf{k}^i$, the Poynting vector of the incident wave becomes

$$\begin{aligned}\mathbf{S}^i &= \frac{1}{2} \mathbf{E}^i \times \mathbf{H}^{i*} = \frac{1}{2 k_o \eta_o} \mathbf{E}^i \times (\mathbf{k}^i \times \mathbf{E}^{i*}) \\ &= \frac{\mathbf{k}^i}{2 k_o \eta_o} |\mathbf{E}^i|^2 = \frac{\mathbf{u}^i}{2 \eta_o} |\mathbf{E}_+^i + \mathbf{E}_-^i|^2 \\ &= \frac{\mathbf{u}^i}{2 \eta_o} (|\mathbf{E}_+^i|^2 + |\mathbf{E}_-^i|^2),\end{aligned}\quad (2.27)$$

whence the incident eigenfields are power orthogonal, i.e., they carry power independently. The same applies for the reflected fields. Depending on the nature of the boundary, there may be power exchange between the incident and reflected eigenfields at the boundary. However, it turns out that, at the PEMC boundary, there is no power coupling between the eigenfields. Because of $\mathbf{n} \cdot \mathbf{k}^r = -\mathbf{n} \cdot \mathbf{k}^i$, handedness of the wave is changed in reflection, whence $\mathbf{E}_+^r$ is right handed and $\mathbf{E}_-^r$ is left handed.

2.3 Reflection from PEMC Boundary

Let us consider the problem of plane-wave reflection from the PEMC boundary defined by the boundary conditions (2.6). From (2.9) we obtain

$$(\bar{\mathbf{J}}_t + M \eta_o \bar{\mathbf{l}}_t) \cdot \mathbf{E}_t^r = (\bar{\mathbf{J}}_t - M \eta_o \bar{\mathbf{l}}_t) \cdot \mathbf{E}_t^i. \quad (2.28)$$