Springer Proceedings in Mathematics & Statistics

Mohd Hafiz Mohd
Norazrizal Aswad Abdul Rahman
Nur Nadiah Abd Hamid
Yazariah Mohd Yatim *Editors*

Dynamical Systems, Bifurcation Analysis and Applications

Penang, Malaysia, August 6–13, 2018

Springer

Springer Proceedings in Mathematics & Statistics

Volume 295

Springer Proceedings in Mathematics & Statistics

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including operation research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

More information about this series at http://www.springer.com/series/10533

Mohd Hafiz Mohd ·
Norazrizal Aswad Abdul Rahman ·
Nur Nadiah Abd Hamid ·
Yazariah Mohd Yatim
Editors

Dynamical Systems, Bifurcation Analysis and Applications

Penang, Malaysia, August 6–13, 2018

Springer

Editors
Mohd Hafiz Mohd
School of Mathematical Sciences
Universiti Sains Malaysia
USM, Penang, Malaysia

Norazrizal Aswad Abdul Rahman
School of Mathematical Sciences
Universiti Sains Malaysia
USM, Penang, Malaysia

Nur Nadiah Abd Hamid
School of Mathematical Sciences
Universiti Sains Malaysia
USM, Penang, Malaysia

Yazariah Mohd Yatim
School of Mathematical Sciences
Universiti Sains Malaysia
USM, Penang, Malaysia

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-981-32-9831-6 ISBN 978-981-32-9832-3 (eBook)
https://doi.org/10.1007/978-981-32-9832-3

Mathematics Subject Classification (2010): 70G60, 37H20, 97M10, 92B05, 97N80

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Organization

Southeast Asian Mathematical Society (SEAMS) School 2018 on Dynamical Systems and Bifurcation Analysis is organized by the School of Mathematical Sciences, Universiti Sains Malaysia.

Organizing Committee

Advisor:	Rosihan M. Ali
	Hailiza Kamarulhaili
	Ahmad Izani Md. Ismail
Chair:	Mohd Hafiz Mohd
Secretary:	Yazariah Mohd Yatim
	Norshafira Ramli
Treasurer:	Nur Nadiah Abd Hamid
	Noor Saifurina Nana Khurizan
Programme:	Maisarah Haji Mohd
	Siti Amirah Abd Rahman
Accommodation:	Farah Aini Abdullah
	Siti Zulaikha Mohd Jamaludin
Technical and Logistic:	Majid Khan Majahar Ali
	Hartini Ahmad
	Syed Mohamed Hussain Syed Osman
Publicity:	Ong Wen Eng
	Md Yushalify Misro
Secretariat:	Amirah Azmi
	Shamani Supramaniam
Liaison Officer:	Syakila Ahmad
	Noor Atinah Ahmad
Publication:	Norazrizal Aswad Abdul Rahman

Preface

This volume collects research papers and survey articles by participants and speakers in the SEAMS School on Dynamical Systems and Bifurcation Analysis (DySBA), which was held in Penang, Malaysia, from 6 to 13 August 2018. The event was organized by the School of Mathematical Sciences, Universiti Sains Malaysia (USM), under the auspices of the Southeast Asian Mathematical Society (SEAMS), Centre International de Mathématiques Pures et Appliquées (CIMPA) and Commission for Developing Countries (CDC) of the International Mathematical Union (IMU). The SEAMS School 2018 on DySBA was designed as part of a series of mathematics study programmes that aims to provide the opportunity for an advanced learning experience in the field of dynamical systems via planned lectures, contributed talks and hands-on workshops. It also aimed to introduce research-based learning for advanced undergraduate students, postgraduate students and young academics.

The main topics highlighted in SEAMS School 2018 on DySBA were differential equations and discrete dynamical systems with applications to mathematical biology. The event brought together both experts and novices in the theory and applications of dynamical systems and bifurcation analysis. The scientific objective of this school was to form a research network among ASEAN mathematicians to collaborate within the broader research community, as well as to open up new opportunities for researchers to link up and collaborate in these fields. The school also intended to foster a joint scientific collaboration between the School of Mathematical Sciences, USM and other international institutions. In general, this school was the first ever SEAMS school concerning dynamical systems and bifurcation theory in Malaysia. The successful organization of this event demonstrates the ability of the School of Mathematical Sciences, USM to participate actively in research engagement activities and scientific collaboration at the regional and international levels.

This edited volume is prepared as a first step in the joint collaboration between ASEAN mathematicians and international researchers to work together in the fields of dynamical systems and bifurcation analysis. It will serve as a unified survey on the development and recent progress in research on dynamical systems and bifurcation from the Southeast Asia region. The authors contributing to the chapters in this book are also active researchers in these fields. This volume intends to give an exposure to prospective readers regarding the theoretical and practical aspects of dynamical systems and bifurcation analysis, as well as recent techniques on numerical continuation and computational methods. While other existing books in the field often give more attention on a particular topic in dynamical systems, our book focuses on the state-of-the-art research on dynamical systems and bifurcation analysis in a broader sense. This is crucial to ensure that readers can grasp different concepts in dynamical systems and understand how the techniques from differential equations, bifurcation analysis and numerical continuation have been employed in analysing the dynamical behaviours of the models under consideration.

Among the modelling frameworks that have been employed in this volume are *Ordinary Differential Equations* (Part I), *Fractional Differential Equations* (Part II), *Delay and Partial Differential Equations* (Part III) and *Discrete Dynamical Systems* (Part IV). This volume also discusses the importance of bifurcation analysis and some numerical continuation packages that can be used to track both stable and unstable steady states together with bifurcation points to gain a better understanding of the dynamics of the systems; these concepts are highlighted in Part V under *Computational Dynamical Systems*. Special attention is also given to applications of dynamical systems and bifurcation analysis in mathematical biology and ecological problems. Several biological examples from recent research are employed as illustrations. Through these examples, it is shown how the techniques from dynamical systems and bifurcation theory have been applied to these different mathematical models in order to answer a range of biologically inspired questions encompassing topics such as species biodiversity and environmental issues. This book will be of great interest to researchers, scientists and educators who work in the fields of differential equations, dynamical systems, bifurcation theory and their applications.

We would like to thank all participants, course lecturers and invited speakers for making the school a great success. Thanks to the organizing committee as well for their great efforts in conducting a well-run school. We are grateful to all of the authors for their contributions to this volume and to all of the reviewers for their timely and detailed feedback on the manuscripts. Additionally, we would like to extend our deepest gratitude to the School of Mathematical Sciences, Universiti Sains Malaysia, SEAMS, CIMPA, CDC-IMU, Division of Research and Innovation USM, Department of Higher Education, Ministry of Education Malaysia and Malaysian Mathematical Sciences Society for their valuable support and commitment to make the event successful.

USM, Penang, Malaysia
April 2019

Mohd Hafiz Mohd
Norazrizal Aswad Abdul Rahman
Nur Nadiah Abd Hamid
Yazariah Mohd Yatim

Contents

Ordinary Differential Equations

Mathematical Modeling and Stability Analysis of Population Dynamics

Auni Aslah Mat Daud

Abstract This study provides a brief introduction of important terminologies and methodologies in the mathematical modelling and stability analysis of the population dynamics. As an example, a mathematical model of population dynamics for thyroid disorder during pregnancy is developed and analysed. The disorders are the second most common endocrine disorders among women in childbearing age, where inadequate or excessive amount of thyroid hormones are produced due to various causes. Thyroid disorders during pregnancy and postpartum can be divided into three types: hyperthyroidism, hypothyroidism and postpartum thyroiditis. They may lead to numerous complications to both mothers and foetuses, such as heart failure, pre-eclampsia, miscarriage, premature birth, and perinatal mortality. The model is described using a system of first order linear ordinary differential equations. Its stability is studied using Routh-Hurwitz criteria. It is found that the model has only one non-negative equilibrium, which is locally asymptotically stable.

Keywords Mathematical model · Thyroid disorders · Equilibrium · Stability analysis · Routh Hurwitz criteria

1 Introduction

A mathematical model is a representation of a real-world system. Mathematical modelling is an important mean for analysing the spread and control of diseases, determining dominant parameters and sensitivities to changes in parameter values. Mathematical modelling has at least two main objectives. Firstly, the models are

A. A. Mat Daud (✉)
Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, 21030 Kuala, Nerus, Malaysia
e-mail: auni_aslah@umt.edu.my

M. H. Mohd et al. (eds.), *Dynamical Systems, Bifurcation Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 295,
https://doi.org/10.1007/978-981-32-9832-3_1

beneficial in understanding the behaviour of the system under study and the identification of important parameters in the system, which can then be used to control the outcome. Secondly, a mathematical model can be employed to predict the future behaviour (such as weather) or measure the outcomes of experiments that cannot be performed in reality (too expensive, not reproducible, dangerous, impractical, or unethical). For example, it would be ethical and non-detrimental for researchers to kill half the female members of a population to measure the population recovery time if they do so using mathematical models instead of real populations.

The solution of a differential equation may not be so important if the solution never appears in the practical, or is only physically realizable under exceptional circumstances. The equilibrium solutions of the model correspond to configurations in which the physical system does not change over time and only occur in the real world if they are stable. Meanwhile, unstable equilibria are not significant and unlikely to occur in reality because these equilibria can only be observed if the initial condition is exactly right while slight perturbations in the system or its physical surroundings will immediately dislodge the system far away from equilibrium.

Biological models often study how the quantities—such as population size changes over time. Mathematically, changes in continuous variables can be described using differential equations. For simple and low-dimensional models, the ideal way to know the values of each variable at all times is by finding the general solution of the differential equations. However, mathematical models in biology are usually governed by differential equations which are very difficult or impossible to solve. Nevertheless, the stability analysis of a model enables us to obtain sufficient information about the behavior of the model without finding the explicit solution of the differential equations. This is important because generally, analyzing a differential equation qualitatively is much easier than finding its analytical solution.

The objective of this study is to propose a mathematical model which can be employed to understand and predict how the subpopulations of patients with thyroid disorders (namely the number of normal non-pregnant women, the number of non-pregnant women with thyroid disorders, the number of pregnant or postpartum women with thyroid disorders without complications and the number of pregnant or postpartum women with thyroid disorders with complications) change over time. To achieve this objective, we will formulate a mathematical model for the population dynamics of the disease using compartmental modelling, and carry out the stability analysis of its equilibrium point.

Thyroid Disorders

The thyroid gland is an endocrine organ which produce thyroid hormones known as thyroxine (T4) and triiodothyronine (T3) which are important in regulating metabolism and development [1]. In pregnancy and postpartum, thyroid disorders are the second commonest endocrine disorder that may affect women in the form of hypothyroidism, hyperthyroidism and postpartum thyroiditis. Hypothyroidims is a condition where there is insufficient thyroid hormone which may be due to iodine deficiency, anti-thyroid antibodies, iatrogenic effects or congenital anomalies. On the contrary, an oversupply of thyroid hormones known as hyperthyroidism may result from autoimmune disorder, pituitary tumour, thyroid nodule, thyroid cancer, thyroiditis, multiple pregnancy and molar pregnancy. These conditions may result in adverse effects to both mothers and foetoses such as maternal heart failure, maternal hypertension, maternal haemorrhage, preeclampsia, anaemia, placental abruption, miscarriage, stillbirth preterm delivery, intrauterine growth restriction, low birth weight, congenital malformation, foetal neurocognitive deficit and foetal thyroid dysfunction. Postpartum thyroiditis is an autoimmune inflammation of the thyroid gland after giving birth and is manifested as hyperthyroidism, hypothyroidism or hyperthyroidism followed by hypothyroidism [2]. If the thyroid dysfunction persists, the patient (depending on the thyroid condition) has higher risks of cardiovascular diseases, mental disorders, paralysis, infertility, osteoporosis, thyroid storm and myxoedema coma (see [1, 3, 4]).

Thyroid disorders may happen to anyone at any age. A woman can only be pregnant during the childbearing age. A woman may acquire the disorder prior to or during her childbearing age and the condition may be treated, depending on the causes. Thyroid disorders during pregnancy and postpartum may be a chronic condition (where the disorder is present before the pregnancy or postpartum) or acquired during the pregnancy and postpartum period. As stated above, the condition may give rise to complications in some women. Similarly, the condition may be treated, depending on the causes. Once the postpartum period ends, the women may or may not continue to have the thyroid disorder, depending on the causes and treatments received (see [1–4]).

2 Methodology

2.1 Mathematical Modelling

In this study, the modeling process begins with the construction of the flow diagram. It is a means of illustrating the model population, which is compartmentalized into state variables. In a flow diagram, the boxes indicate the state variables while the movement of people in the population is indicated by arrows. An ordinary differential equation refers to an instantaneous rate of change in a state variable with respect to the independent variable (time). The flows shown as arrows are calculated using the terms on the right-hand side of the equations: a flow out of a box is taken away from a state variable and is a negative term while a flow into a box is added and is a positive term, as discussed in Appendix A. Some parameters will be introduced to represent the proportionality rates of the flows.

2.2 *Stability Analysis*

An equilibrium point is stable if the system near to the equilibrium point will approach to that equilibrium point while a system is unstable if the system near to the equilibrium point moves away from that equilibrium point [5]. In this study, the stability analysis is conducted using Routh-Hurwitz criteria, which is discussed in detail in Appendix C.

3 Results and Discussion

3.1 *Model*

The flow diagram which describes the population dynamics for thyroid disorders during pregnancy is shown in Fig. 1. There are four dependent variables which represent different stages of the disease, namely the number of normal non-pregnant women $A(t)$, the number of non-pregnant women with thyroid disorders $B(t)$, the number of pregnant or postpartum women with thyroid disorders without complications $C(t)$ and the number of pregnant or postpartum women with thyroid disorders with complications $D(t)$. There are nineteen constant parameters in this study, with values between 0 and 1, representing the following per capita rates or probability per unit time (except $P$and Q which represent the numbers of individuals per unit time):
μ : death due to causes other than thyroid disorders,
τ : pregnant/postpartum women with thyroid disorders who die due to thyroid disorders complications,
κ : pregnant/postpartum women with thyroid disorders who die due to thyroid disorders,
ρ : non-pregnant women with thyroid disorders who die due to thyroid disorders,
P : women entering the childbearing age population without thyroid disorders,
Q : women entering the childbearing age population with thyroid disorders,
ν_A : women of subpopulation A leaving the childbearing age population,
ν_B : women of subpopulation B leaving the childbearing age population,
ν_C : women of subpopulation C leaving the childbearing age population,
ν_D : women of subpopulation D leaving the childbearing age population,
α : normal women in childbearing age having thyroid disorders,
δ : non-pregnant women with thyroid disorders in childbearing age becoming pregnant with thyroid disorders,
ξ : normal women in childbearing age having thyroid disorders during pregnancy/postpartum period,
Υ : non-pregnant women with thyroid disorders recovering from thyroid disorders,
η : pregnant/postpartum women with thyroid disorders recovering from them in non-pregnant state,

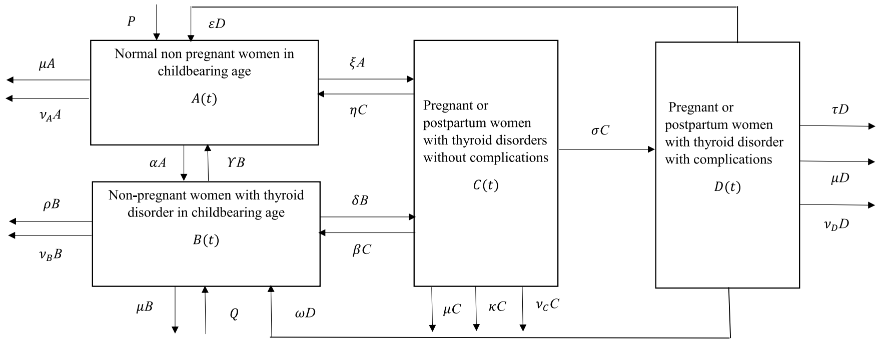

Fig. 1 Flow diagram of the population dynamics for thyroid disorders during pregnancy

β : pregnant/postpartum women with thyroid disorders who continue to have them in non-pregnant state,
σ: pregnant/postpartum women with thyroid disorders having complications during pregnancy/postpartum period,
ε : pregnant/postpartum women with thyroid disorders with complications recovering to become normal non-pregnant women after postpartum period,
ω : pregnant/postpartum women with thyroid disorders with complications recovering from complications but continue to have thyroid disorders in non-pregnant state.

The flow diagram in Fig. 1 is transformed to a system of first order linear ordinary differential equations. The mathematical model is governed by the following system of differential equations:

$$\frac{dA}{dt} = P - \mu A - \nu_A A - \alpha A - \xi A + \Upsilon B + \eta C + \varepsilon D \tag{1}$$

$$\frac{dB}{dt} = Q - \nu_B B - \rho B + \beta C - \delta B - \mu B + \alpha A - \Upsilon B + \omega D \tag{2}$$

$$\frac{dC}{dt} = \xi A - \eta C + \delta B - \beta C - \sigma C - \kappa C - \nu_C C - \mu C \tag{3}$$

$$\frac{dD}{dt} = \sigma C - \omega D - \tau D - \nu_D D - \mu D - \varepsilon D \tag{4}$$

with initial conditions $A(0) = A_0,\ \ B(0) = B_0,\ \ C(0) = C_0$ and $D(0) = D_0$. Note that the terms in Eqs. (1)–(4) which involve the per capita rates are the products of probability per unit time and one of the state variables, which represents the number of individuals.

Since Eqs. (1)–(4) represent interaction between individuals in the population, it makes sense to state that all the parameters involved are non-negative. It is also pertinent to show that all the state variables of the model are non-negative for all time. Hence, we have the following theorem (see the proof in Appendix D):

Theorem 1 *The solution set* $\{A, B, C, D\}$ *of Eqs. (1)–(4) with non-negative initial conditions* $A(0) = A_0$, $B(0) = B_0$, $C(0) = C_0$ *and* $D(0) = D_0$ *remains non-negative for all time* $t > 0$.

4 Stability Analysis

By taking Eqs. (1)–(2) equal to zero and solve the resulting equations simultaneously we found that the equilibrium point (A^*, B^*, C^*, D^*) as follow:

$$A^* = \frac{MN^2Q\,(M\eta\delta + \varepsilon\sigma\delta + NM\Upsilon) - MNP\left(NM\beta\delta + N\omega\sigma\delta - MN^2S\right)}{[(M\eta\xi+\varepsilon\sigma\xi - NMR)\left(NM\beta\delta + N\omega\sigma\delta - MN^2S\right) - \left(MN^2\alpha + NM\beta\xi+N\omega\sigma\xi\right)(M\eta\delta + \varepsilon\sigma\delta + NM\Upsilon)]}$$

$$B^* = \frac{-MN^2Q - \left(MN^2\alpha + NM\beta\xi+N\omega\sigma\xi\right)A^*}{\left(NM\beta\delta + N\omega\sigma\delta - MN^2S\right)}$$

$$C^* = \frac{\xi A^* + \delta B^*}{N}$$

$$D^* = \frac{\sigma C^*}{M}$$

where

$$R = \mu + \nu_A + \alpha + \xi,$$

$$S = \nu_B + \rho + \delta + \Upsilon + \mu,$$

$$N = \eta + \beta + \sigma + \kappa + \nu_C + \mu,$$

$$M = \omega + \tau + \nu_D + \mu + \varepsilon.$$

Conducting the stability analysis of the equilibrium point (A^*, B^*, C^*, D^*) using *Routh-Hurwitz* criteria for $n = 4$, the equilibrium point is asymptotically stable if

$$a_i > 0,\ i = 1, 2, 3, 4$$

$$a_1a_2a_3 > a_3^2 + a_1^2a_4$$

where $a_i,\ i = 1, 2, 3, 4$ are the coefficients in the characteristic polynomial,

$$\left|J^* - \lambda I\right| = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$$

where λ are the eigenvalues, I is the matrix identity, and J^* is the Jacobian matrix of the model, evaluated at the equilibrium point.

The model in this study can be helpful to the policy makers in setting the aim for each parameter in order to achieve certain targets for the number of patients. The parameter values of the developed model can be estimated using the available data from the literature, in the context of Malaysia in order to forecast the number of patients in Malaysia. These parameter values can also be estimated for any other domain in the world using the corresponding prior data for that particular domain. Using appropriate estimated parameter values, one may predict the number of patients for all stages of the disease for the coming few years for any geographical domain.

5 Conclusion

In this study, a mathematical model that describes the population dynamics of thyroid disorders during pregnancy has been formulated. The model was governed by a system of linear first order ordinary differential equations. The equilibrium point of the proposed model is determined and it is asymptotically stable. It is important to note that the research of equilibrium point and stability analysis in this study were found directly by hand. Therefore, in future, an interactive tool can be developed by modelers (for example, using computational tools such as Wolfram MATHEMATICA) for non-modellers to experiment scenarios and visualize the possible consequences under various actions.

Acknowledgements This research is funded by the Ministry of Higher Education, Government of Malaysia under the Research Acculturation Grant Scheme (RAGS 57108). The results of the study were partially presented during The 4th International Conference on Mathematical Sciences (ICMS4) on November 15–17, 2016 at Palm Garden Hotel, Putrajaya, Malaysia and the SEAMS School on Dynamical Systems and Bifurcation Analysis (DySBA)on August 6–13, 2018 at Universiti Sains Malaysia (USM), Penang, Malaysia.

Finally, I would like to express my gratitude to Salilah Saidun and Kritika Manimaran, who have contributed in this study.

6 Appendixes

Appendix A: Formulating ODEs Based on a Flow Diagram

The state variables are depicted by the boxes in the flow diagram while the arrows illustrate the movement of people between different states in the population. The flows shown as arrows are calculated using the terms on the right-hand side of the equations: a flow pointing out of a box is taken away from a state variable and is a negative term while a flow pointing into a box is added and is a positive term [6].

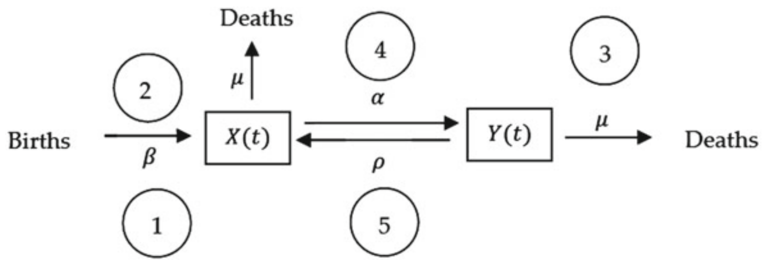

Fig. 2 An example of a flow diagram

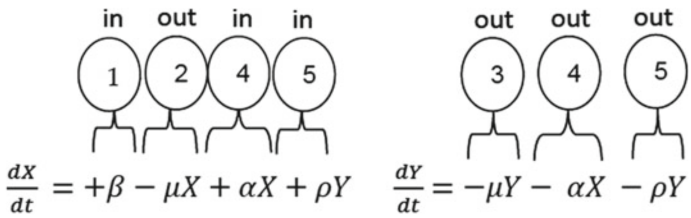

Fig. 3 The resulting governing ODEs (corresponding to the flow diagram in Fig. 2)

Some constant parameters are introduced adjacent to the arrows to represent the proportionality rates of the flows.

An illustration of an example of a flow diagram is provided in Fig. 2, while the formulation of differential equations (the governing equations of the model) based on the flow diagram is shown in Fig. 3.

where β is the birth rate, μ is the death rate, α is the transformation rate from $X(t)$ to $Y(t)$ and ρ is the transformation rate from $X(t)$ to Y(t).

Appendix B: The Computation of Equilibrium Point

Consider a mathematical model governed by a system of differential equations:

$$
\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, x_{2,\ldots,} x_n) \\
\frac{dx_2}{dt} &= f_2(x_1, x_{2,\ldots,} x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(x_1, x_{2,\ldots,} x_n)
\end{aligned}
$$

By definition, the equilibrium values of the model are all values that cause the variables to remain unchanged, or mathematically:

$$f_1\left(\widehat{x_1},\ \widehat{x_2,}\ \ldots,\ \widehat{x_n}\right) = 0$$

$$f_2\left(\widehat{x_1},\ \widehat{x_2,}\ \ldots,\ \widehat{x_n}\right) = 0$$

$$\vdots$$

$$f_n\left(\widehat{x_1},\ \widehat{x_2,}\ \ldots,\ \widehat{x_n}\right) = 0$$

The equilibrium point $\left(\widehat{x_1},\ \widehat{x_2,}\ \ldots,\ \widehat{x_n}\right)$ is obtained by solving the equations above simultaneously or using any methods of solving system of equations in linear algebra.

Appendix C: The Stability Analysis Using Routh Hurwitz Criteria

In this study, we will study a multiple variables model with continuous time. Therefore, the stability analysis is performed using the following steps:

Step 1: Calculating a Jacobian matrix.

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_1,\ x_2,\ \ldots, x_n) & \frac{\partial f_1}{\partial x_2}(x_1,\ x_2,\ \ldots, x_n) & \cdots & \frac{\partial f_1}{\partial x_n}(x_1,\ x_2,\ \ldots, x_n) \\ \frac{\partial f_2}{\partial x_1}(x_1,\ x_2,\ \ldots, x_n) & \frac{\partial f_2}{\partial x_2}(x_1,\ x_2,\ \ldots, x_n) & \cdots & \frac{\partial f_2}{\partial x_n}(x_1,\ x_2,\ \ldots, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x_1,\ x_2,\ \ldots, x_n) & \frac{\partial f_n}{\partial x_2}(x_1,\ x_2,\ \ldots, x_n) & \cdots & \frac{\partial f_n}{\partial x_n}(x_1,\ x_2,\ \ldots, x_n) \end{pmatrix}$$

where $\frac{\partial f_i}{\partial x_j}(x_1,\ x_2 \ldots, x_n)$ is the partial derivative of f_i with respect to its variable, x_j $(i, j = 1,\ 2,\ \ldots,\ n)$.

Step 2: Find the Jacobian matrix.
The Jacobian matrix is evaluated at the equilibrium values, $\widehat{x_1},\ \widehat{x_2,}\ \ldots,\ \widehat{x_n}$. A local stability matrix, $\hat{J} = J|_{x_1=\widehat{x_1},\ x_2=\widehat{x_2},\ \ldots,\ x_n=\widehat{x_n}}$ is obtained. Then find the characteristic polynomial using $\det\left(\hat{J} - \lambda I\right) = 0$, where I is the identity matrix, and rewrite in the following form:

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

with real coefficients a_i for $i = 1,\ 2,\ \ldots,\ n$.

Step 3: Use *Routh-Hurwiz* criteria.
The n Hurwitz matrices are defined as follow:

$$H_1 = (a_1),\ H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix},\ \text{and } H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{pmatrix}$$

Note that if $j > n$, then $a_j = 0$. If and only if all $\det H_j > 0$ with $j = 1, 2, \ldots, n$, then $P(\lambda)$ has roots that are negative or have negative real part and hence the equilibrium point is said to be asymptotically stable. The *Routh-Hurwitz* criteria for polynomials of degree, $n = 4$ are [7]: $a_1 > 0$, $a_3 > 0$, $a_4 > 0$, and $a_1 a_2 a_3 > a_3^2 + a_1^2 a_4$.

Appendix D: Proof of Theorem 1

Given that the initial conditions $A(0) = A_0$, $B(0) = B_0$, $C(0) = C_0$ and $D(0) = D_0$ are non-negative. It is clear from Eq. (1) that

$$\frac{dA}{dt} + [\xi + \mu + \nu + \alpha] A(t) \geq 0,$$

so that,

$$\frac{d}{dt}\left[A(t) \exp\left(\xi + \mu + \nu + \alpha\right) t\right] \geq 0. \tag{A1}$$

Integrating (A1) with respect to t, gives

$$A(t) \geq A(0) \exp\left[-\left(\xi + \mu + \varphi\nu + \alpha\right) t\right] > 0, \quad \forall t \geq 0.$$

Similarly, it can be shown that $B(t) > 0$, $C(t) > 0$, $D(t) > 0$ for all time $t > 0$. This completes the proof.

It is crucial to note that Eqs. (1)–(4) will be analyzed in a feasible region D given by

$$D = \left\{(A, B, C, D) \in R_+^4 : A + B + C + D = N\right\},$$

which can be easily verified to be positively invariant with respect to Eqs. (1)–(4), In what follows, the model is epidemiologically and mathematically well posed in D. (see [8]).

References

1. Boelaert, K., Franklyn, J.A.: Thyroid hormone in health and disease. J. Endocrinol. **187**(1), 115 (2005). https://doi.org/10.1677/joe.1.06131
2. Carney, L.A., Quinlan, J.D., West, J.M.: Thyroid disease in pregnancy. Am. Family Physician **89**(4), 273–278 (2014)
3. Cooper, D., McDermott, M., Wartofsky, L.: Hypothyroidism. J. Clin. Endocrinol. Metab. **89**(11), E2 (2004). https://doi.org/10.1210/jcem.89.11.9990
4. Leo, S.D., Lee, S.Y., Braverman, L.E.: Hyperthyroidism. Lancet. **388**(10047), 906918 (2016). https://doi.org/10.1016/S0140-6736(16)00278-6
5. Kopp, M.: Equilibria and stability analysis. Universitet Wien (2011) http://www.mabs.at/koop/teaching/modelling2011/files/stab.pdf
6. Garnett, G.P.: An introduction to mathematical models in sexually transmitted disease epidemiology. Sex Transm Inf. **78**, 712 (2002)
7. Hethcote, H.W.: The mathematics of infectious diseases. SIAM Rev. **42**, 599653 (2000). https://doi.org/10.1137/S0036144500371907
8. Allen, L.J.S.: Introduction to Mathematics Biology. Pears Education, Slough (2006)

Optimal Harvesting Regions of a Polluted Predator-Prey Fishery System

Tau Keong Ang, Hamizah M. Safuan, Ummu Atiqah M. Roslan and Mohd Hafiz Mohd

Abstract The present paper examines a predator-prey fishery system by taking into account the toxin released by the fish which can lead to polluted system. Both predator and prey fish species obey the logistic population growth with their respective environmental carrying capacities. In the proposed model, both fish species produce toxins that contribute to mutual infection which is detrimental to each other. Different harvesting efforts are applied on the predator and prey fish populations, respectively. The equilibria existed in the model are studied together with the local stability properties. We consider the threshold conditions which trigger the bifurcation that occurred in the steady states. The global stability properties of coexistence equilibrium are studied by constructing an appropriate Lyapunov function. Bendixson-Dulac criterion is applied to rule out the existence of limit cycle in the system. From the bifurcation analysis, the dynamical behaviors of the system are observed as well as the persistence and extinction properties. It is shown that harvesting parameters are most likely to drive a fish population towards extinction compared to toxicant parameters which are less influential. Regions of optimal harvesting strategies were found to guarantee the persistence of both fisheries. Finally, the existence of a bionomic equilibrium solution has been examined with three possible cases.

Keywords Fishery · Prey-predator · Harvesting · Toxicant · Bionomic equilibrium

---

T. K. Ang (✉) · H. M. Safuan
Universiti Tun Hussein Onn Malaysia, 84600 Pagoh, Johor, Malaysia
e-mail: taukeong1018@gmail.com

U. A. M. Roslan
Universiti Malaysia Terengganu, 21030 Kuala Nerus, Terengganu, Malaysia

M. H. Mohd
Universiti Sains Malaysia, 11800 Penang, Malaysia

M. H. Mohd et al. (eds.), *Dynamical Systems, Bifurcation Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 295,
https://doi.org/10.1007/978-981-32-9832-3_2

1 Introduction

Due to the developing global concerns on the management strategies to eradicate the problems of over-exploitation and over-utilization of natural resources, ecological modelling is proved to be one of the efficient and economic approaches to investigate the possible cause and effect [1, 2]. Natural resource such as fishery is an essential source of protein to human being. However, the depletion of fisheries has distressed the ecological balance and the effects can be detrimental if the resource becoming scarce and eventually extinct.

In population ecology, the interaction between a species and the associated environments is one of the crucial elements to consider since the dynamical behaviors of the species is strictly dependent on the environment. The presence of toxin is claimed to have a hazardous effect on the population's mortality rate and some of the pioneering work on the toxic in population models were studied by Chattopadhyay [3] and Samanta [4]. Both researchers proposed a two competing species model with toxin produced by two competitors. The toxin produced can be harmful to each other and the ratio of toxin is a decisive factor for the stability properties exhibited in the system. The abnormal level of toxic substances can induce critical fluctuation on the dynamics from both the biological and ecological perspectives, causing the system to become unstable.

As the impacts of pollutant are getting immediate attention in population dynamics, numerous research were done to investigate the behaviors or dynamics caused by toxin in a more specific terrestrial community and also the aquatic environment. For instance, a series of research works were done by Huang et al. [5, 6] that emphasized on the dynamics of fish population in the presence of pollutants. Both research showed the changes in dynamics of fishery models in the presence of contaminated environment, as compared to the conventional fishery models. Both qualitative and quantitative approaches showed the consistent result that the concentration of toxin is the key factor for the population persistence.

On the other hand, the commercial values of fishery resources had drawn the interest of researchers on the farming effects of fish population. Kar and Chaudhuri [7] explored the exploitation of fisheries by developing a competing fishery model which is influenced by toxicity while Das et al. [8] contributed his idea on harvesting activities of predator-prey fishery model. Similar research was studied by Ang et al. [9]. The research [7–9] examined the properties of equilibria in the system and found that harvesting activities can give rise to great impacts both biologically and economically to the fishery systems.

The principal aim of the present research is to work on the dynamical nature of a toxin-polluted fishery model where mutual infection occurs among the predator and prey fish populations. Both the predator and prey fish species are subjected to nonlinear logistic growth rate with different harvesting efforts. Bifurcation analysis is performed with respect to the harvesting parameters to determine the influence of harvesting activities on the persistence and extinction properties. In the present work, we found that the effect of harvesting is more significant than that of toxin in inducing the state that lead to the extinction of the fish population.