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The Dual of $L_\infty(X, \mathcal{L}, \lambda)$, Finitely Additive Measures and Weak Convergence

A Primer

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Preface

Assuming some familiarity with Lebesgue measure, integration and related functional analysis summarised in Chap. 2, this is an exposition of topics that arise when identifying elements of the dual space of $L_\infty(X, \mathcal{L}, \lambda)$ with finitely additive measures on a σ-algebra $\mathcal{L}$ when the measure λ is complete and σ-finite. Such a representation has its origins in the independent work of Fichtenholz and Kantorovitch [14] and Hildebrandt [20] and culminated in a general abstract theory due to Yosida and Hewitt [35] in 1952. However, even now it is not unusual ([16] is an exception) for books on measure theory to give a detailed account of the dual space of $L_p(X, \mathcal{L}, \lambda)$, $1 \leqslant p < \infty$, while relegating the case $p = \infty$ to references, e.g. [12, p. 296] which invokes the theory of finitely additive measures on algebras such as is developed in [35].

An explanation may be that a study of finitely additive measures on algebras necessitates a possibly unwelcome diversion from the mainstream theory of countably additive measures that suffices for $p \in [1, \infty)$. Whatever the reason, a consequence is that $L_\infty(X, \mathcal{L}, \lambda)^*$ has acquired an aura of mystery, to the extent that it is often not very clear beyond the mere definition what is meant by saying that a bounded sequence in $L_\infty(X, \mathcal{L}, \lambda)$ is weakly convergent.

The aim here is to take Yosida and Hewitt theory on σ-algebras beyond the representation theorem for $L_\infty(X, \mathcal{L}, \lambda)^*$, pointing out some of its consequences for measurable functions generally and in particular for weak convergence of sequences in $L_\infty(X, \mathcal{L}, \lambda)$. The target audience is anyone who feels nervous about representing the dual of $L_\infty(X, \mathcal{L}, \lambda)$ by finitely additive measures in the knowledge that *there exist uncountably many, linearly independent, finitely additive measures* $\nu \geqslant 0$ *defined on the Lebesgue* σ*-algebra of* $(0, 1)$ *with the property that*

$$\int_0^{1-\frac{1}{k}} u\, d\nu = 0 \text{ for all } u \in L_\infty(0,1) \text{ and } k \in \mathbb{N}, \text{ but } \int_0^1 1\, d\nu = 1. \qquad (\dagger)$$

An essential goal will be to come to terms with observations such as this one.

In their seminal work, Yosida and Hewitt [35] studied general Banach spaces $L_\infty(X, \mathcal{M}, \mathcal{N})$ of essentially bounded measurable functions, where measurability is determined by an algebra $\mathcal{M}$ (closed under complementation and finite unions) and essential boundedness is defined in terms of a family $\mathcal{N} \subset \mathcal{M}$ (closed under countable unions with the added property that $A \subset B \in \mathcal{N}$ implies $A \in \mathcal{N}$) that mimics null sets. Obviously, $L_\infty(X, \mathcal{L}, \lambda)$ is a special case of $L_\infty(X, \mathcal{M}, \mathcal{N})$ but in general no measure of any kind is involved in the definition of $L_\infty(X, \mathcal{M}, \mathcal{N})$. However, although [35] shows that the dual of $L_\infty(X, \mathcal{M}, \mathcal{N})$ can be expressed in terms of finitely additive measures, the exposition here is restricted to $L_\infty(X, \mathcal{L}, \lambda)$ because

> properties of finitely additive measures on σ-algebras are less circumscribed by hypotheses than on algebras, and replacing the algebra $\mathcal{M}$ by a σ-algebra $\mathcal{L}$ and $\mathcal{N}$ by the family of null sets $\{E \in \mathcal{L} : \lambda(E) = 0\}$, where λ is complete and σ-finite, yields a theory which is relevant in applications, including when X is a Lebesgue measurable subset of $\mathbb{R}^n$ or a differentiable manifold, or when $X = \mathbb{N}$ with counting measure.

For a σ-finite measure space the ultimate aim is to develop theory sufficient to characterise weakly convergent sequences in $L_\infty(X, \mathcal{L}, \lambda)$ in terms of their λ-almost-everywhere pointwise behaviour. However, in the process, when (X, ρ) is a locally compact Hausdorff topological space and $(X, \mathcal{B}, \lambda)$ is a corresponding Borel measure space, there emerges a natural way to localise weak convergence. A sequence is weakly convergent in $L_\infty(X, \mathcal{B}, \lambda)$ if and only if it is weakly convergent at every point x_0 in the one-point compactification of (X, ρ). Here, weak convergence at x_0 is defined in terms of functionals which are zero outside every neighbourhood of x_0; for an example of such, see (†).

The essential range $\mathcal{R}(u)(x_0)$[1] of a Borel measurable function u at x_0 is similarly defined in terms of those elements of $L_\infty(X, \mathcal{B}, \lambda)^*$ which are localised at x_0. Since it need not be a singleton, $\mathcal{R}(u)(x_0)$ can be interpreted as a multivalued representation of the fine structure at x_0 of $u \in L_\infty(X, \mathcal{B}, \lambda)$ which is intimately related to weak convergence.

The Literature

In her Foreword to the monograph by Bhaskara Rao and Bhaskara Rao [6], Dorothy Maharam Stone cites Salomon Bochner as having said that "contrary to popular mathematical opinion finitely additive measures were more interesting, more difficult to handle, and perhaps more important than countably additive measures". Oxtoby [25] described [6] as a comprehensive account of finitely additive measures which effectively organises a large body of material that is widely scattered in the literature and deserves to be better known, and in their preface the authors themselves described it as a reference book as well as a textbook.

[1] $\mathcal{R}(u)(x_0)$ is sometimes referred to as the cluster set of u at x_0.

The origins of this theory are to be found in the early days of modern integration theory when there were many contributors: see [12, §III.15, p. 233 and §IV.16, p. 388[2]] and the comprehensive bibliography with notes in [6]. However, presumably because they could not match the versatility of Lebesgue's theory of integration and the power of its convergence theorems, finitely additive measures seem to have fallen out of fashion. Nevertheless, they continue to have significant roles in, for example, mathematical economics, probability, statistics, optimization, control theory and analysis [7, 9, 25, 35].

In a series of three papers on additive set functions on abstract topological spaces, A. D. Alexandroff [2] studied bounded regular finitely additive measures that represent linear functionals on spaces of continuous functions. On a similar theme, but in a more general setting, a much-cited reference for the dual of $L_\infty(X, \mathcal{L}, \lambda)$ is Dunford and Schwartz [12, p. 296] which covers the theory of finitely additive set functions on algebras and includes extensive historical notes. For a recent account, see Fonseca and Leoni [16, Theorem 2.24], and Aliprantis and Border [3] for the abstract theory in which it is embedded.

It will soon be apparent that key results, including (†), rely on the axiom of choice. For a discussion of the role of the axiom of choice, geometrical and paradoxical aspects of finitely additive measures, and their invariance under group actions on X, see Tao [32]. Oxtoby's commentary [25] is of independent interest.

A key role is played throughout by the set $\mathfrak{G}$ of finitely additive measures that take only the values 0 and 1 and the observation that $L_\infty(X, \mathcal{L}, \lambda)$ is isometrically isomorphic to a space of real-valued continuous functions on $(\mathfrak{G}, \tau)$ with the maximum norm, where τ is a compact Hausdorff topology. Further analysis of $\mathfrak{G}$ in a Borel setting then leads to localization, and to other developments mentioned above and outlined in the Introduction.

What follows is in large part an extension of a simplified version of Yosida and Hewitt [35], set out in the notation and terminology of Chap. 2.

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[2]The reference to Theorem 8.15 on p. 388 is a misprint; 8.15 is a Definition and obviously Theorem 8.16 was intended.

Contents

Chapter 1
Introduction

Overview

In a normed linear space V, a sequence $\{v_k\}$ converges weakly to v ($v_k \rightharpoonup v$) if $v^*(v_k) \to v^*(v)$ for all $v^* \in V^*$, the dual space of V and, from the uniform boundedness principle, weakly convergent sequences are bounded in norm. However, it has been known since the work of Banach that when V is a complete normed linear space it may not be necessary to use all elements of V^* when testing for weak convergence. Indeed, when $C(Z)$ denotes the space of real-valued continuous functions on a compact metric space Z with the maximum norm, he showed that $v_k \rightharpoonup v$ in $C(Z)$ if and only if $v_k(z) \to v(z)$ for all $z \in Z$ and $\{\|v_k\|\}$ is bounded. To do so he observed [5, Annexe, Thm. 7] that Dirac δ-functions satisfy conditions for a set W^* in the dual space of a Banach space to have the property that

$$\{\|v_k\|\} \text{ bounded and } w^*(v_k) \to 0 \text{ for all } w^* \in W^* \text{ imply } v_k \rightharpoonup 0. \tag{W}$$

When (X, ϱ) is a locally compact Hausdorff space and $(C_0(X, \varrho), \|\cdot\|_\infty)$ is the Banach space of real-valued continuous functions on X that vanish at infinity (see (2.9)), weakly convergent sequences are pointwise convergent because δ-functions belongs to the dual space of $C_0(X, \varrho)$, and bounded by the uniform boundedness principle. Conversely, from Theorem 2.37 (Riesz) and Lebesgue's Dominated Convergence Theorem [15, Thm. 2.24], sequences that are norm-bounded and pointwise convergent on X are weakly convergent.

In particular, when $\mathcal{Z}$ is a compact Hausdorff space, for $\{v_k\} \subset C(\mathcal{Z})$ (the space of real-valued continuous functions on $\mathcal{Z}$ with the maximum norm)

$$v_k \rightharpoonup v_0 \text{ in } C(\mathcal{Z}) \Leftrightarrow \sup_k \|v_k\| < \infty \text{ and } v_k(z) \to v_0(z) \text{ for all } z \in \mathcal{Z}. \tag{V}$$

The possibility of usefully extending these observations to $L_\infty(X, \mathcal{L}, \lambda)$ (the real Banach space of essentially bounded real-valued functions defined by (2.8)) at first appears limited because, for example, in an open set $\Omega \subset \mathbb{R}^n$ with Lebesgue measure,

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