

Walter Lacarbonara
Balakumar Balachandran
Jun Ma · J. A. Tenreiro Machado
Gabor Stepan *Editors*

Nonlinear Dynamics of Structures, Systems and Devices

Proceedings of the First International
Nonlinear Dynamics Conference
(NODYCON 2019), Volume I



Springer

Nonlinear Dynamics of Structures, Systems and Devices


Walter Lacarbonara • Balakumar Balachandran
Jun Ma • J. A. Tenreiro Machado • Gabor Stepan
Editors

Nonlinear Dynamics of Structures, Systems and Devices

Proceedings of the First International
Nonlinear Dynamics Conference
(NODYCON 2019), Volume I

 Springer

Editors

Walter Lacarbonara 
Department of Structural and Geotechnical
Engineering
Sapienza University of Rome
Rome, Italy

Balakumar Balachandran
Department of Mechanical Engineering
University of Maryland
College Park, MD, USA

Jun Ma
Department of Physics
Lanzhou University of Technology
Lanzhou, Gansu, China

J. A. Tenreiro Machado
Department of Electrical Engineering
Polytechnic of Porto - School
of Engineering (ISEP)
Porto, Portugal

Gabor Stepan
Department of Applied Mechanics
Budapest University of Technology
and Economics
Budapest, Hungary

ISBN 978-3-030-34712-3 ISBN 978-3-030-34713-0 (eBook)
<https://doi.org/10.1007/978-3-030-34713-0>

© Springer Nature Switzerland AG 2020

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This volume is part of three volumes collecting the *Proceedings of the First International Nonlinear Dynamics Conference (NODYCON 2019)* held in Rome, February 17–20, 2019. NODYCON was launched to foster the tradition of the conference series originally established by Prof. Ali H. Nayfeh in 1986 at Virginia Polytechnic Institute and State University (Virginia Tech), Blacksburg, VA, USA, as the Nonlinear Vibrations, Stability, and Dynamics of Structures Conference. With the passing in 2017 of Prof. Nayfeh, who was also the founder of the Springer journal *Nonlinear Dynamics* in 1990, NODYCON 2019 was organized as a collective tribute of the community to Prof. Nayfeh for being one of the most influential leaders of nonlinear dynamics. NODYCON 2019 was also established to look to and dream about the future. The call for papers attracted contributions dealing with established nonlinear dynamics research topics as well as with the latest trends and developments. At the same time, to reflect the rich spectrum of topics covered by the journal *Nonlinear Dynamics*, the call included diverse and multidisciplinary topics, to mention a few, multi-scale dynamics, experimental dynamics, dynamics of structures/industrial machines/equipment/facilities, dynamics of adaptive, multifunctional, metamaterial structures, dynamics of composite/nanocomposite structures, reduced-order modeling, nonsmooth dynamics, fractional-order system dynamics, nonlinear interactions and parametric vibrations, computational techniques, nonlinear system identification, dynamics of NEMS/MEMS/nanomaterials, multibody dynamics, fluid/structure interaction, influence of nonlinearities on vibration control systems, human–machine interaction, nonlinear wave propagation in discrete and continuous media, chaotic map-based cryptography, ecosystem dynamics, social media dynamics, complexity in engineering, and network dynamics.

For NODYCON 2019, the organizers received 450 two-page abstracts and based on 467 reviews from the Program Committee, the Steering and Advisory Committees, and external reviewers, 391 papers and 17 posters were accepted, published in the Book of Abstracts (NODYS Publications, Rome, ISBN 978-88-944229-0-0), and presented by nearly 400 participants from 68 countries. The diverse topics covered by the papers were organized along four major themes to organize the technical sessions:

- (a) Concepts and methods in nonlinear dynamics
- (b) Nonlinear dynamics of mechanical and structural systems
- (c) Nonlinear dynamics and control
- (d) Recent trends in nonlinear dynamics

The authors of a selection of approximately 60 papers were invited to publish in the Special Issue of *Nonlinear Dynamics* entitled “NODYCON 2019 First International Nonlinear Dynamics Conference.” Over 200 full papers were submitted to the *Proceedings of the First International Nonlinear Dynamics Conference* (NODYCON 2019) and only 121 of them were accepted. These papers have been collected into three volumes, which are listed below together with a sub-topical organization.

Volume 1: Nonlinear Dynamics of Structures, Systems, and Devices

- (a) Methods for nonlinear dynamics
- (b) Bifurcations and nonsmooth systems
- (c) Nonlinear phenomena in mechanical systems and structures
- (d) Experimental dynamics, system identification and monitoring
- (e) Fluid–structure interaction, multibody system dynamics
- (f) Turning processes, rotating systems, and systems with time delays

Volume 2: Nonlinear Dynamics and Control

- (g) Vibration absorbers and isolators
- (h) Control of nonlinear systems
- (i) Sensors and actuators
- (j) Network synchronization

Volume 3: New Trends in Nonlinear Dynamics

- (k) Smart materials, metamaterials, composite and nanocomposite materials, and structures
- (l) MEMS/NEMS and energy harvesters
- (m) Nonlinear phenomena in bio- and ecosystem dynamics
- (n) Chaos in electronic systems
- (o) Fractional-order systems

I wish to acknowledge the work of the Co-Editors of the NODYCON 2019 Proceedings: Prof. Balakumar Balachandran (University of Maryland, College Park, MD, USA), Prof. Jun Ma (Lanzhou University of Technology, China), Prof. J. A. Tenreiro Machado (Instituto Superior de Engenharia do Porto, Portugal), Prof. Gabor Stepan (Budapest University of Technology and Economics, Hungary).

The success of NODYCON 2019 relied primarily on the efforts, talent, energy, and enthusiasm of researchers in the field of nonlinear dynamics who wrote and submitted these papers. Special praise is also deserved for the reviewers who invested significant time in reading, examining, and assessing multiple papers, thus ensuring a high standard of quality for this conference proceedings.

Rome, Italy
August 2019

Walter Lacarbonara

Preface for Volume 1: Nonlinear Dynamics of Structures, Systems, and Devices

Volume 1 of the NODYCON Proceedings is composed of 55 papers, which are spread across the following groupings: (a) methods for nonlinear dynamics (8 papers), (b) bifurcations and nonsmooth systems (8 papers), (c) nonlinear phenomena in mechanical systems and structures (12 papers), (d) experimental dynamics, system identification, and monitoring (8 papers), (e) fluid–structure interaction and multibody system dynamics (9 papers), and (f) turning processes, rotating systems, and systems with time delays (10 papers). It is acknowledged that a paper placed in one grouping could have easily been placed in another grouping as well. As one reads through these 55 contributions, one will note the use of a wide range of experimental, analytical, and numerical techniques for study of the nonlinear dynamics of a wealth of systems across different length and time scales.

In the work of N. Potosakis, E. Paraskevopoulos, and S. Natsiavas, an augmented Lagrangian formulation is used to construct models of nonlinear mechanical systems such as vehicle systems subjected to bilateral scleronomic motion constraints. A. O. Belyakov and A. P. Seyranian study domains of stability for parametrically excited systems by using high-order approximations of the monodromy matrix. A. Liu and D. Wagg examine normal form analysis for a two-degree-of-freedom system. E. Kremer investigates system vibratory responses to amplitude-modulated and phase-modulated excitations. A. M. Bersani, A. Borri, A. Milanese, G. Tomasetti, and P. Vellucci summarize some recent results obtained from studies of the asymptotic properties of important enzyme reactions. Z. Wang, Z. Tang, J. H. Park, and Y. Wang study system modeling for the Hammerstein nonlinear model with unknown but bounded noise. For strongly nonlinear systems, H.-E. Du, G.-K. Er, and V. P. Iu consider construction of frequency-response curves and determination of unstable response regions. Soliton solutions of the Korteweg-de Vries equation are investigated by S. Carillo, M. L. Schiavo, and C. Schiebold.

Two types of bifurcations related to limit directions of nonsmooth vector fields are examined by M. Anatali and G. Stepan. S. Natsiavas and E. Paraskevopoulos study dynamics of multibody systems experiencing impacts with friction. H. Xu and J. Ji use state feedback control to create Neimark–Sacker bifurcation in a vibro-impact system. H. Z. Horvath and D. Takacs consider the stability of suitcases

and trailers that are subjected to non-holonomic constraints. I. D. Atanasovska, K. R. Hedrih, and D. B. Momcilovic study the vibro-impact dynamics of spur gears with wear. M. Ramírez, J. Collado, and F. Dohnal provide an efficient means for computing the stability transition curves of coupled Mathieu equations. M. Yadav, S. S. Chaurasia, and S. Sinha study the symmetry in a group of Stuart–Landau oscillators; in particular, with regard to oscillation death states. Dynamics and bifurcations of solutions of the so-called Sprott A system are studied by M. Messias and A. C. Reinol.

J. Awrejcewicz, R. Starosta, and G. Sypniewska-Kamińska examine the dynamics of a three-degree-of-freedom pendulum-spring-damper system in the presence of an external resonance and a two-to-one internal resonance. In the presence of internal resonances, the influence of initial geometric imperfections on the dynamics of a slender cylindrical panel nonlinear response is investigated by F. M. A. Silva, W. A. Vaz, and P. B. Gonçalves. M. Farid and O. V. Gendelman treat sloshing dynamics in partially filled storage tanks. Free oscillations of arbitrarily sagged and inclined cables oscillating around a catenary static profile are studied by A. Mansour, G. Rega, and O. B. Mekki. Nonlinear interactions between flexural and sway oscillations of a laterally braced column are examined by D. Orlando, J. M. P. Raimundo, and P. B. Gonçalves. The influence of temperature on cable dynamics under different excitations is considered by Y. Zhao, H. Lin, L. Chen, and Z. Guo. Behavior of a harmonically forced nonlinear system with a shape memory alloy-based spring is investigated by S. Ramnarace and J. Bridge. Static behavior and free oscillations of shallow circular arches are treated by U. Eroglu and G. Ruta. G. Liu and W. Zhang investigate the response of a composite cantilever plate placed in a subsonic air flow and study the frequency response of this system. A hysteresis damper is examined by D. Li and H. Fang. Nonlinear oscillations of an axially excited beam embedded in a viscoelastic medium are treated by E. Babilio. Energy localization in a chain of coupled, nonlinear oscillators is investigated by A. Kovaleva.

N. Barbieri, M. J. Mannala, R. Barbieri, and G. Barbieri use experimental data based on torsional response to tune nonlinear models of transmission line cables. Nonlinear, stochastic dynamics of a Duffing oscillator are studied by L. G. G. Villani, S. Silva, and A. Cunha, Jr. Hysteretic behavior of wire rope isolators is considered by N. Vaiana, F. Marmo, S. Sessa, and L. Rosati. Nonlinear behavior of rubber shear springs is experimentally examined by S. Gong, S. Oberst, and X. Wang. Experimental studies on the responses of a six-degree-of-freedom parallel manipulator are reported by K. Ringgaard and O. Balling. N. Cavalagli, M. Ciano, G. Fagotti, M. Gioffrè, V. Gusella, and C. Pepi discuss experimental investigations into the dynamic response of a two-storey masonry structure mounted on a shaker table and subjected to a transient excitation with a strong vertical component. Rate dependence of a hysteretic device is experimentally investigated and characterized by M. Antonelli, B. Carboni, W. Lacarbonara, D. Bernardini, and T. Kalmár-Nagy. K. R. Hedrih, S. V. Brčić, and S. Paunović discuss the use of photoelasticity for nonlinear dynamic studies for different applications, including dams.

A combined analytical, experimental, and numerical investigation is used to explore the dynamics of an acoustically levitated sphere by A. Dolev and I. Bucher. Vortex-induced vibrations are studied by V. Kurushina, E. Pavlovskaya, A. Postnikov, G. R. Franzini, and M. Wiercigroch. C. Mannini studies vortex-induced vibrations and galloping dynamics of a cylinder with a rectangular cross section. Dynamics of an aeroelastic system is investigated by C. Demartino, G. Matteoni, and C. T. Georgakis. Computational dynamics of a multibody system in a vertical fluid flow is the subject of the work of Z. Terze, V. Pandža, and D. Zlatar. M. Eugeni, F. Mastroddi, and F. Saltari examine damping models used in flutter analysis of highly flexible aircraft. P. Rosatelli, W. Lacarbonara, A. Arena, and D. J. Inman examine morphing wing dynamics by using computational means. Multibody dynamic studies are carried out by P. Masarati, A. Zanoni, V. Muscarello, R. Paolini, and G. Quaranta to understand how the interactions between the vehicle dynamics and human biomechanics affect helicopter's handling qualities. The use of absolute nodal coordinate formulation for studies of thin plate dynamics is examined by K.-W. Kim, J.-W. Lee, J.-S. Jang, J.-H. Kang, and W.-S. Yoo.

Chatter in turning process is the subject matter of the work presented by B. Beri and G. Stepan. Aperiodic dynamics of spinning shafts with varying rotation speeds are studied by F. Georgiades. M. A. Al-Shudeifat and C. Nataraj study backward whirling of a rotor with fatigue cracks. Inner race defects in rolling element bearings are treated by T. H. Mohamad, S. Ilbeigi, and C. Nataraj. Turning process dynamics are studied by A. Gousskov, G. Panovko, and D. D. Tung. A. Wang, W. Jin, and Q. Lin investigate regenerative effects and friction force effects on chatter during turning operations. Y. Jin and P. Xu study noise-induced transitions in a triple well potential system with a time delay. Dimension reduction for rotor dynamic studies is the topic of the work of K. Lu, H. Zhang, H. Zhou, Y. Jin, Y. Yang, and C. Fu. Drum-type washing machines are studied by C. Baykal, E. Cigeroglu, and Y. Yazicioglu. Wear effects in hydrodynamic bearings used in rotating systems are examined by T. J. Machado and G. C. Stroti.

In conclusion, this volume represents a multifaceted cross section of recent advances in computational methods for nonlinear dynamics, bifurcations, nonlinear phenomena in mechanical systems and structures, experimental dynamics, nonlinear system identification, fluid–structure interaction and multibody system dynamics, turning processes, rotating systems, and nonsmooth system dynamics including systems with time delays. We hope that readers will benefit from the rich work portrayed here on nonlinear dynamics of structures, systems, and devices and that new ideas and future contributions will be inspired.

Rome, Italy
 College Park, MD, USA
 Lanzhou, China
 Porto, Portugal
 Budapest, Hungary
 August 2019

Walter Lacarbonara
 Balakumar Balachandran
 Jun Ma
 J. A. Tenreiro Machado
 Gabor Stepan

Contents

Part I Methods for Nonlinear Dynamics

| | |
|---|----|
| Nonlinear Dynamics of Multibody Systems Using an Augmented Lagrangian Formulation | 3 |
| Nikolaos Potosakis, Elias Paraskevopoulos, and Sotirios Natsiavas | |
| Stability Boundary Approximation of Periodic Dynamics | 13 |
| Anton O. Belyakov and Alexander P. Seyranian | |
| ϵ^2-Order Normal Form Analysis for a Two-Degree-of-Freedom Nonlinear Coupled Oscillator | 25 |
| Xuanang Liu and David J. Wagg | |
| Vibrational Mechanics of Systems with Amplitude and Phase Modulation of Excitation | 35 |
| Eugen Kremer | |
| Singular Perturbation Techniques and Asymptotic Expansions for Some Complex Enzyme Reactions | 43 |
| Alberto Maria Bersani, Alessandro Borri, Alessandro Milanesi, Giovanna Tomassetti, and Pierluigi Vellucci | |
| A Novel Two-Stage Ellipsoid Filtering Based System Modeling Algorithm for a Hammerstein Nonlinear Model with an Unknown Noise Term | 55 |
| Ziyun Wang, Ze Tang, Ju H. Park, and Yan Wang | |
| A New Method for the Frequency Response Curve and Its Unstable Region of a Strongly Nonlinear Oscillator | 65 |
| Hai-En Du, Guo-Kang Er, and Vai Pan Iu | |
| Matrix Soliton Solutions of the Modified Korteweg–de Vries Equation ... | 75 |
| Sandra Carillo, Mauro Lo Schiavo, and Cornelia Schiebold | |

Part II Bifurcations and Nonsmooth Systems

| | |
|---|-----|
| Bifurcations of Limit Directions at Codimension-2 Discontinuities of Vector Fields | 87 |
| Mate Antali and Gabor Stepan | |
| Boundary Layer Dynamics of Multibody Systems Involving Impact and Friction | 97 |
| Sotirios Natsiavas and Elias Paraskevopoulos | |
| Creation of Neimark-Sacker Bifurcation for a Three-Degree-of-Freedom Vibro-Impact System with Clearances | 107 |
| Huidong Xu and Jinchun Ji | |
| Analogue Models of Rocking Suitcases and Snaking Trailers | 117 |
| Zoltan Horvath and Denes Takacs | |
| A Theoretical Model for Vibro-Impact Dynamics of Spur Gears with Tooth Flanks Wear | 127 |
| Ivana D. Atanasovska, Katica R. (Stevanović) Hedrih, and Dejan B. Momcilovic | |
| Stability of Coupled and Damped Mathieu Equations Utilizing Symplectic Properties | 137 |
| Miguel Ramírez Barrios, Joaquín Collado, and Fadi Dohnal | |
| Asymmetry in the Basin Stability of Oscillation Death States Under Variation of Environment-Oscillator Links | 147 |
| Manish Yadav, Sudhanshu Shekhar Chaurasia, and Sudeshna Sinha | |
| The Occurrence of Zero-Hopf Bifurcation in a Generalized Sprott A System | 157 |
| Marcelo Messias and Alisson C. Reinol | |

Part III Nonlinear Phenomena in Mechanical Systems and Structures

| | |
|--|-----|
| External and Internal Resonances in a Mass-Spring-Damper System with 3-dof | 169 |
| Jan Awrejcewicz, Roman Starosta, and Grażyna Sypniewska-Kamińska | |
| Internal Resonances in an Imperfect Circular Cylindrical Panel | 179 |
| Frederico M. A. Silva, Wanclaine A. Vaz, and Paulo B. Gonçalves | |
| Weakly Nonlinear Liquid Sloshing: Modelling and Exploration of Response Regimes | 187 |
| Maor Farid and Oleg V. Gendelman | |
| Catenary-Based Nonlinear Multimodal Theory of Cable Free Vibrations | 197 |
| Achref Mansour, Giuseppe Rega, and Othman Ben Mekki | |

Flexural and Sway Interaction in the Nonlinear Vibrations of a Phenomenological Model of a Laterally Braced Column..... 207
 Diego Orlando, Juliana M. P. Raimundo, and Paulo B. Gonçalves

Influences of Temperature on Dynamic Behaviors of Suspended Cables Under Primary and Sub-Harmonic Excitations Simultaneously .. 217
 Yaobing Zhao, Henghui Lin, Lincong Chen, and Zixiong Guo

Analysis of a Shape Memory Alloy Spring System Under Harmonic Excitation 227
 Shivan Ramnarace and Jacqueline Bridge

Vibration of Pre-Loaded Shallow Circular Arches..... 237
 Uğurcan Eroğlu and Giuseppe Ruta

Nonlinear Vibration Responses of Laminated Composite Cantilever Plate in Subsonic Air Flow 247
 Gen Liu and Wei Zhang

Adaptation of Energy Dissipation in a Laminated Module with Tunable Twin Wells 257
 Dejian Li and Hui Fang

The Duffing–Mathieu Equation Arising from Dynamics of Post-Buckled Beams 267
 Enrico Babilio

Resonance-Induced Energy Localization in Weakly Dissipative Anharmonic Chains..... 277
 Agnessa Kovaleva

Part IV Experimental Dynamics, System Identification and Monitoring

Torsional Analysis of Transmission Line Cables 289
 Nilson Barbieri, Marcos José Mannala, Renato Barbieri, and Gabriel de Sant’ Anna Vitor Barbieri

Application of a Stochastic Version of the Restoring Force Surface Method to Identify a Duffing Oscillator 299
 Luis G. G. Villani, Samuel da Silva, and Americo Cunha Jr.

Modeling of the Hysteretic Behavior of Wire Rope Isolators Using a Novel Rate-Independent Model 309
 Nicolò Vaiana, Francesco Marmo, Salvatore Sessa, and Luciano Rosati

A Non-linear Model of Rubber Shear Springs Validated by Experiments 319
 Sanpeng Gong, Sebastian Oberst, and Xinwen Wang

| | |
|--|-----|
| Operational Modal Analysis on a Six-Degree-of-Freedom Parallel Manipulator: Reproducibility, Excitation and Pose Dependency | 329 |
| Kasper Ringgaard and Ole Balling | |
| Shaking Table Investigation on the Masonry Structures Behaviour to Earthquakes with Strong Vertical Component | 337 |
| Nicola Cavalagli, Matteo Ciano, Gianluca Fagotti, Massimiliano Giofrè, Vittorio Gusella, and Chiara Pepi | |
| Quantifying Rate-Dependence of a Nonlinear Hysteretic Device | 347 |
| Marco Antonelli, Biagio Carboni, Walter Lacarbonara, Davide Bernardini, and Tamás Kalmár-Nagy | |
| Application of Photoelasticity to Some Nonlinear Dynamic Problems and Stress State Analysis in Dams: A Brief Overview Inspired by the Results of Prof. Vlatko Brčić | 357 |
| Katica R. (Stevanović) Hedrih, Stanko V. Brčić, and Stepa Paunović | |
| Part V Fluid–Structure Interaction, Multibody System Dynamics | |
| Levitated and Parametrically Excited Sphere Dynamics in a Single-Axis Ultrasonic Levitator | 369 |
| Amit Dolev and Izhak Bucher | |
| Modelling VIV of Transversally Oscillating Rigid Structures Using Nonlinear Fluid Oscillators | 379 |
| Victoria Kurushina, Ekaterina Pavlovskaja, Andrey Postnikov, Guilherme Rosa Franzini, and Marian Wiercigroch | |
| Asymptotic Analysis of a Dynamical System for Vortex-Induced Vibration and Galloping | 389 |
| Claudio Mannini | |
| A Quasi-Steady 3-DoFs Sectional Aerodynamic Model: Preliminary Results | 399 |
| Cristoforo Demartino, Giulia Matteoni, and Christos T. Georgakis | |
| Lie Group Dynamics of Multibody System in Vortical Fluid Flow | 409 |
| Zdravko Terze, Viktor Pandža, and Dario Zlatar | |
| Damping Models in Aircraft Flutter Analyses | 419 |
| Marco Eugeni, Franco Mastroddi, and Francesco Saltari | |
| Dynamic Response of a Morphing Wing | 429 |
| Patrizio Rosatelli, Walter Lacarbonara, Andrea Arena, and Daniel J. Inman | |
| Helicopter Pilot Biomechanics by Multibody Analysis | 439 |
| Pierangelo Masarati, Andrea Zanoni, Vincenzo Muscarello, Rita Paolini, and Giuseppe Quaranta | |

Derivation of Non-dimensional Equation of Motion for Thin Plate in Absolute Nodal Coordinate Formulation 449
 Kun-Woo Kim, Jae-Wook Lee, Jin-Seok Jang, Ji-Heon Kang, and Wan-Suk Yoo

Part VI Turning Processes, Rotating Systems, and Systems with Time Delays

Approximated Dynamics of Chatter in Turning Processes 463
 Bence Beri and Gabor Stepan

Chaotic Dynamics in Spinning Shafts with Non-constant Rotating Speed Described by Variant Lyapunov Exponents 471
 Fotios Georgiades

Post-Resonance Backward Whirl in a Jeffcott Rotor with a Breathing Crack Model 485
 Mohammad A. Al Shudeifat and C. Nataraj

Proper and Smooth Orthogonal Decompositions for Detection of Inner Race Defects in Rolling Element Bearings with Variable Rotational Speeds 493
 Turki H. Mohamad, Shahab Ilbeigi, and C. Nataraj

Effect of the Compliance of the Part on the Double-Turning Process 503
 Alexander Gouskov, Grigory Panovko, and Dinh Duc Tung

Effect of the Regenerative and Frictional Force on Chatter in Turning Process 513
 An Wang, Wuyin Jin, and Qian Lin

Noise-Induced Transitions and Resonances in a Delayed Triple-Well Potential System 523
 Yanfei Jin and Pengfei Xu

Application of the Second Dimension Reduction Method in Nonlinear Rotor Dynamic System 533
 Kuan Lu, Haopeng Zhang, Hao Zhou, Yulin Jin, Yongfeng Yang, and Chao Fu

Vibration Analysis of Washing Machines in the Drum Plane 549
 Cem Baykal, Ender Cigeroglu, and Yiğit Yazicioglu

Nonlinear Model for Wear Effects in Hydrodynamic Bearings Applied to Rotating Systems 561
 Tiago Henrique Machado and Gustavo Chaves Storti

Author Index 569

Subject Index 583

Contributors

Mohammad A. Al Shudeifat Department of Aerospace Engineering, Khalifa University, Abu Dhabi, UAE

Mate Antali Budapest University of Technology and Economics, Budapest, Hungary

Marco Antonelli Department of Structural and Geotechnical Engineering, Sapienza University of Rome, Rome, Italy

Andrea Arena Department of Structural and Geotechnical Engineering, Sapienza University of Rome, Rome, Italy

Ivana D. Atanasovska Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia

Jan Awrejcewicz Department of Automation Biomechanics and Mechatronics, Lodz University of Technology, Lodz, Poland

Enrico Babilio Department of Structures for Engineering and Architecture (DiSt), University of Naples “Federico II”, Naples, Italy

Balakumar Balachandran Department of Mechanical Engineering, University of Maryland, College Park, MD, USA

Ole Balling Mechanical and Materials Engineering, Department of Engineering, Aarhus University, Aarhus, Denmark

Nilson Barbieri Pontifícia Universidade Católica do Paraná-PUCPR, Curitiba, Paraná, Brazil
Universidade Tecnológica Federal do Paraná-UTFPR, Curitiba, Paraná, Brazil

Renato Barbieri Universidade do Estado de Santa Catarina-UDESC, Joinville, Santa Catarina, Brazil

Cem Baykal Middle East Technical University, Ankara, Turkey

Anton O. Belyakov Moscow School of Economics, Lomonosov Moscow State University, Moscow, Russia
National Research Nuclear University “MEPhI”, Moscow, Russia
Central Economics and Mathematics Institute, Russian Academy of Sciences, Moscow, Russia

Bence Beri Department of Applied Mechanics, Budapest University of Technology and Economics, Budapest, Hungary

Davide Bernardini Department of Structural and Geotechnical Engineering, Sapienza University of Rome, Rome, Italy

Alberto Maria Bersani Department of Mechanical and Aerospace Engineering, Sapienza University, Roma, Italy

Alessandro Borri Institute for System Analysis and Computer Science “Antonio Ruberti”, Roma, Italy

Stanko V. Brčić Faculty of Civil Engineering, University of Belgrade, Belgrade, Serbia

Jacqueline Bridge Department of Mechanical and Manufacturing Engineering, The University of the West Indies, St. Augustine, Trinidad and Tobago

Izhak Bucher Faculty of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa, Israel

Biagio Carboni Department of Structural and Geotechnical Engineering, Sapienza University of Rome, Rome, Italy

Sandra Carillo Università di Roma “La Sapienza”, Dip. S.B.A.I., Rome, Italy
I.N.F.N.—Sezione Roma1, Gr. IV—M.M.N.L.P., Rome, Italy

Nicola Cavalagli Department of Civil and Environmental Engineering, University of Perugia, Perugia, Italy

Sudhanshu Shekhar Chaurasia Indian Institute of Science Education and Research (IISER), Mohali, Punjab, India

Lincong Chen College of Civil Engineering, Huaqiao University, Xiamen, Fujian, China

Matteo Ciano Department of Civil and Environmental Engineering, University of Perugia, Perugia, Italy

Ender Cigeroglu Middle East Technical University, Ankara, Turkey

Joaquín Collado Automatic Control Department, CINVESTAV, México City, México

Americo Cunha Jr. UERJ—Universidade do Estado do Rio de Janeiro, NUMERICO—Nucleus of Modeling and Experimentation with Computers, Rio de Janeiro, Brazil

Samuel da Silva UNESP – Universidade Estadual Paulista, Faculdade de Engenharia, Departamento de Engenharia Mecânica, São Paulo, Brazil

Cristoforo Demartino Department of Energy, Environment, and Infrastructure Sciences, Zhejiang University-University of Illinois (ZJUI), Zhejiang University, Haining, China

Gabriel de Sant’Anna Vitor Barbieri Pontifícia Universidade Católica do Paraná-PUCPR, Curitiba, Paraná, Brazil
Universidade Tecnológica Federal do Paraná-UTFPR, Curitiba, Paraná, Brazil

Fadi Dohnal Division for Mechatronics Lienz, UMIT – Private University for Health Sciences, Medical Informatics and Technology, Lienz, Austria

Amit Dolev Faculty of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa, Israel

Hai-En Du Department of Civil and Environmental Engineering, University of Macau, Macau SAR, China

Guo-Kang Er Department of Civil and Environmental Engineering, University of Macau, Macau SAR, China

Uğurcan Eroğlu Istanbul Technical University, Istanbul, Turkey

Marco Eugeni Department of Mechanical and Aerospace Engineering, Sapienza University of Rome, Rome, Italy

Gianluca Fagotti Ufficio Speciale Ricostruzione Umbria, Foligno, Italy

Hui Fang Institute of Coastal and Ocean Engineering, School of Engineering, Ocean University of China, Qingdao, China

Maor Farid Faculty of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa, Israel

Guilherme Rosa Franzini Offshore Mechanics Laboratory, Escola Politécnica, University of São Paulo, São Paulo, Brazil

Chao Fu Institute of Vibration Engineering, Northwestern Polytechnical University, Xi’an, P. R. China
MIIT Key Laboratory of Dynamics and Control of Complex Systems, Northwestern Polytechnical University, Xi’an, P. R. China

Oleg V. Gendelman Faculty of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa, Israel

Christos T. Georgakis Department of Engineering-Structural Monitoring and Dynamics, Aarhus University, Aarhus, Denmark

Fotios Georgiades School of Engineering, College of Science, University of Lincoln, Lincoln, UK

Massimiliano Gioffrè Department of Civil and Environmental Engineering, University of Perugia, Perugia, Italy

Paulo B. Gonçalves Department of Civil Engineering, Pontifical Catholic University, Rio de Janeiro, Brazil

Sanpeng Gong School of Chemical and Environmental Engineering, China University of Mining and Technology, Beijing, China
Center for Audio, Acoustics and Vibration, University of Technology, Sydney, NSW, Australia

Alexander Gousov Mechanical Engineering Research Institute of Russian Academy of Sciences, Moscow, Russia

Zixiong Guo College of Civil Engineering, Huaqiao University, Xiamen, Fujian, China

Vittorio Gusella Department of Civil and Environmental Engineering, University of Perugia, Perugia, Italy

Zoltan Horvath Department of Applied Mechanics, Budapest University of Technology and Economics, Budapest, Hungary

Shahab Ilbeigi Villanova Center for Analytics of Dynamic Systems, Villanova University, Villanova, PA, USA

Daniel J. Inman Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI, USA

Vai Pan Iu Department of Civil and Environmental Engineering, University of Macau, Macau SAR, China

Jin-Seok Jang Korea Institute of Industrial Technology, Daegu, South Korea

Jinchen Ji University of Technology Sydney, Broadway, NSW, Australia

Wuyin Jin School of Mechanical and Electrical Engineering, Lanzhou University of Technology, Lanzhou, China

Yanfei Jin Department of Mechanics, Beijing Institute of Technology, Beijing, China

Yulin Jin School of Aeronautics and Astronautics, Sichuan University, Chengdu, P. R. China

Tamás Kalmár-Nagy Department of Fluid Mechanics, Faculty of Mechanical Engineering, University of Technologies and Economics, Budapest, Hungary

Ji-Heon Kang Pusan National University, Busan, South Korea

Kun-Woo Kim Korea Institute of Industrial Technology, Daegu, South Korea

Agnessa Kovaleva Space Research Institute, Moscow, Russia

Eugen Kremer Schaeffler Automotive Buehl GmbH & Co. KG, Buehl, Baden, Germany

Institute for Problems in Mechanical Engineering of Russian Academy of Sciences, St. Petersburg, Russia

Victoria Kurushina Department of Transport of Hydrocarbon Resources, Institute of Transport, Industrial University of Tyumen, Tyumen, Russia

Centre for Applied Dynamics Research, School of Engineering, University of Aberdeen, Kings' College, Aberdeen, Scotland, United Kingdom

Walter Lacarbonara Department of Structural and Geotechnical Engineering, Sapienza University of Rome, Rome, Italy

Jae-Wook Lee Korea Institute of Industrial Technology, Daegu, South Korea

Dejian Li Institute of Coastal and Ocean Engineering, School of Engineering, Ocean University of China, Qingdao, China

Henghui Lin College of Civil Engineering, Huaqiao University, Xiamen, Fujian, China

Qian Lin Editorial Department of Journal of Lanzhou University of Technology, Lanzhou University of Technology, Lanzhou, China

Gen Liu Beijing Key Laboratory of Nonlinear Vibrations and Strength of Mechanical, College of Mechanical Engineering, Beijing University of Technology, Beijing, P. R. China

Xuanang Liu Department of Mechanical Engineering, University of Sheffield, Sheffield, UK

Kuan Lu Institute of Vibration Engineering, Northwestern Polytechnical University, Xi'an, P. R. China

MIIT Key Laboratory of Dynamics and Control of Complex Systems, Northwestern Polytechnical University, Xi'an, P. R. China

Jun Ma Department of Physics, Lanzhou University of Technology, Lanzhou, Gansu, China

Tiago Henrique Machado Department of Integrated Systems, School of Mechanical Engineering, University of Campinas, Campinas, Brazil

J. A. Tenreiro Machado Department of Electrical Engineering, Polytechnic of Porto - School of Engineering (ISEP), Porto, Portugal

Marcos José Mannala Instituto de Tecnologia para o Desenvolvimento-LACTEC, Curitiba, Paraná, Brazil

Claudio Mannini CRIACIV-Department of Civil and Environmental Engineering, University of Florence, Florence, Italy

Achref Mansour Université de Tunis El Manar, Ecole Nationale d'Ingénieurs de Tunis LGC, Tunis, Tunisia
University Grenoble Alpes, Inria, CNRS, Grenoble INP, LJK, Grenoble, France

Francesco Marmo Department of Structures for Engineering and Architecture, University of Naples Federico II, Naples, Italy

Pierangelo Masarati Department of Aerospace Science and Technology, Politecnico di Milano, Milano, Italy

Franco Mastroddi Department of Mechanical and Aerospace Engineering, Sapienza University of Rome, Rome, Italy

Giulia Matteoni Arup, London, UK

Othman Ben Mekki Université de Tunis El Manar, Ecole Nationale d'Ingénieurs de Tunis, LGC, Tunis, Tunisia

Marcelo Messias Universidade Estadual Paulista (UNESP), Faculdade de Ciências e Tecnologia, Departamento de Matemática e Computação, Presidente Prudente, SP, Brazil

Alessandro Milanesi Department of Basic and Applied Sciences for Engineering, Sapienza University, Roma, Italy

Turki H. Mohamad Villanova Center for Analytics of Dynamic Systems, Villanova University, Villanova, PA, USA

Dejan B. Momcilovic Institute for Testing of Materials-IMS Institute, Belgrade, Serbia

Vincenzo Muscarello Department of Aerospace Science and Technology, Politecnico di Milano, Milano, Italy

C. Nataraj Department of Mechanical Engineering, Villanova Center for Analytics of Dynamic Systems, Villanova University, Villanova, PA, USA

Sotirios Natsiavas Department of Mechanical Engineering, Aristotle University, Thessaloniki, Greece

Sebastian Oberst Center for Audio, Acoustics and Vibration, University of Technology, Sydney, NSW, Australia

Diego Orlando State University of Rio de Janeiro, Rio de Janeiro, Brazil

Viktor Pandža Chair of Flight Vehicle Dynamics, Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Zagreb, Croatia

Grigory Panovko Mechanical Engineering Research Institute of Russian Academy of Sciences, Moscow, Russia

Rita Paolini Department of Aerospace Science and Technology, Politecnico di Milano, Milano, Italy

Elias Paraskevopoulos Department of Mechanical Engineering, Aristotle University, Thessaloniki, Greece

Ju H. Park Department of Electrical Engineering, Yeungnam University, Kyongsan, Republic of Korea

Stjepa Paunović Department of Mechanics, Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia

Ekaterina Pavlovskaja Centre for Applied Dynamics Research, School of Engineering, University of Aberdeen, Kings' College, Aberdeen, Scotland, United Kingdom

Chiara Pepi Department of Civil and Environmental Engineering, University of Perugia, Perugia, Italy

Andrey Postnikov School of Engineering, University of Lincoln, Lincoln, United Kingdom

Nikolaos Potosakis Department of Mechanical Engineering, Aristotle University, Thessaloniki, Greece

Giuseppe Quaranta Department of Aerospace Science and Technology, Politecnico di Milano, Milano, Italy

Juliana M. P. Raimundo Pontifical Catholic University of Rio de Janeiro, Rio de Janeiro, Brazil

Miguel Ramírez Barrios Professional Interdisciplinary Unit of Biotechnology, National Polytechnic Institute, Mexico City, México

Shivan Ramnarace Department of Mechanical and Manufacturing Engineering, The University of the West Indies, St. Augustine, Trinidad and Tobago

Giuseppe Rega Dipartimento di Ingegneria Strutturale e Geotecnica, Università di Roma La Sapienza, Roma, Italy

Alisson C. Reinol Universidade Tecnológica Federal do Paraná (UTFPR), Departamento Acadêmico de Matemática, Apucarana, PR, Brazil

Kasper Ringgaard Mechanical and Materials Engineering, Department of Engineering, Aarhus University, Aarhus, Denmark

Patrizio Rosatelli Sapienza University of Rome, Rome, Italy

Luciano Rosati Department of Structures for Engineering and Architecture, University of Naples Federico II, Naples, Italy

Giuseppe Ruta University "La Sapienza" and National Group for Mathematical Physics, Roma, Italy

Francesco Saltari Department of Mechanical and Aerospace Engineering, Sapienza University of Rome, Rome, Italy

Mauro Lo Schiavo Università di Roma “La Sapienza”, Dip. D.I.S.G., Rome, Italy

Cornelia Schiebold Department of Mathematics and Science Education, Mid Sweden University, Sundsvall, Sweden
Instytut Matematyki, Uniwersytet Jana Kochanowskiego w Kielcach, Kielce, Poland

Salvatore Sessa Department of Structures for Engineering and Architecture, University of Naples Federico II, Naples, Italy

Alexander P. Seyranian Institute of Mechanics, Lomonosov Moscow State University, Moscow, Russia

Frederico M. A. Silva School of Civil and Environmental Engineering, Federal University of Goiás, Goiânia, Brazil

Sudeshna Sinha Indian Institute of Science Education and Research (IISER), Mohali, Punjab, India

Roman Starosta Institute of Applied Mechanics, Poznań University of Technology, Poznań, Poland

Gabor Stepan Department of Applied Mechanics, Budapest University of Technology and Economics, Budapest, Hungary

Katica R. (Stevanović) Hedrih Department of Mechanics, Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, Serbia

Gustavo Chaves Storti Department of Integrated Systems, School of Mechanical Engineering, University of Campinas, Campinas, Brazil

Grażyna Sypniewska-Kamińska Institute of Applied Mechanics, Poznań University of Technology, Poznań, Poland

Dénes Takacs MTA-BME Research Group on Dynamics of Machines and Vehicles, Budapest, Hungary

Ze Tang Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), Jiangnan University, Wuxi, Jiangsu, PR China

Zdravko Terze Chair of Flight Vehicle Dynamics, Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Zagreb, Croatia

Giovanna Tomassetti Department of Basic and Applied Sciences for Engineering, Sapienza University, Roma, Italy

Dinh Duc Tung Bauman Moscow State Technical University, Moscow, Russia

Nicolò Vaiana Department of Structures for Engineering and Architecture, University of Naples Federico II, Naples, Italy

Wanclaine A. Vaz School of Civil and Environmental Engineering, Federal University of Goiás, Goiânia, Brazil

Pierluigi Vellucci Department of Economics, Roma Tre University, Rome, Italy

Luis G. G. Villani UERJ – Universidade do Estado do Rio de Janeiro, NUMERICO – Nucleus of Modeling and Experimentation with Computers, Rio de Janeiro, Brasil

David J. Wagg Department of Mechanical Engineering, University of Sheffield, Sheffield, UK

An Wang School of Mechanical and Electrical Engineering, Lanzhou University of Technology, Lanzhou, China

Xinwen Wang School of Chemical and Environmental Engineering, China University of Mining and Technology, Beijing, China

Yan Wang Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), Jiangnan University, Wuxi, Jiangsu, PR China

Ziyun Wang Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), Jiangnan University, Wuxi, Jiangsu, PR China

Marian Wiercigroch Centre for Applied Dynamics Research, School of Engineering, University of Aberdeen, Kings' College, Aberdeen, Scotland, United Kingdom

Huidong Xu Taiyuan University of Technology, Taiyuan, P. R. China

Pengfei Xu Department of Mechanics, Beijing Institute of Technology, Beijing, China

Department of Mathematics, Changzhi University, Changzhi, China

Manish Yadav Indian Institute of Science Education and Research (IISER), Mohali, Punjab, India

Yongfeng Yang Institute of Vibration Engineering, Northwestern Polytechnical University, Xi'an, P. R. China

MIIT Key Laboratory of Dynamics and Control of Complex Systems, Northwestern Polytechnical University, Xi'an, P. R. China

Yiğit Yazicioglu Middle East Technical University, Ankara, Turkey

Wan-Suk Yoo Pusan National University, Busan, South Korea

Andrea Zanoni Department of Aerospace Science and Technology, Politecnico di Milano, Milano, Italy

Haopeng Zhang Institute of Vibration Engineering, Northwestern Polytechnical University, Xi'an, P. R. China

MIIT Key Laboratory of Dynamics and Control of Complex Systems, Northwestern Polytechnical University, Xi'an, P. R. China

Wei Zhang Beijing Key Laboratory of Nonlinear Vibrations and Strength of Mechanical, College of Mechanical Engineering, Beijing University of Technology, Beijing, P. R. China

Yaobing Zhao College of Civil Engineering, Huaqiao University, Xiamen, Fujian, China

Hunan Provincial Key Laboratory of Structures for Wind Resistance and Vibration Control, Hunan University of Science and Technology, Xiangtan, Hunan, China

Hao Zhou Institute of Vibration Engineering, Northwestern Polytechnical University, Xi'an, P. R. China

MIIT Key Laboratory of Dynamics and Control of Complex Systems, Northwestern Polytechnical University, Xi'an, P. R. China

Dario Zlatar Chair of Flight Vehicle Dynamics, Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Zagreb, Croatia

Part I
Methods for Nonlinear Dynamics

Nonlinear Dynamics of Multibody Systems Using an Augmented Lagrangian Formulation



Nikolaos Potosakis, Elias Paraskevopoulos, and Sotirios Natsiavas

Abstract A class of multibody systems subject to bilateral scleronomic motion constraints is investigated. The formulation is based on a new set of equations of motion, expressed as a coupled system of strongly nonlinear second-order ordinary differential equations. After putting these equations in a weak form, the position, velocity, and momentum type quantities are assumed to be independent, leading to a three-field set of equations of motion. Next, an equivalent augmented Lagrangian formulation is set up by introducing a set of penalty terms. This final set of equations is then used as a basis for developing a new time integration scheme, which is applied to several example systems. In those examples, special emphasis is put on illustrating the advantages of the new method when applied to mechanical systems, involving redundant constraints or singular configurations.

Keywords Analytical mechanics · Multibody dynamics · Bilateral motion constraints · Weak form of equations of motion · Augmented Lagrangian

1 Introduction

Research on multibody dynamics helps in developing more efficient and robust numerical techniques for solving challenging engineering problems. This in turn yields useful design gains in many areas, including mechanisms, robotics, biomechanics, automotive, railway, and aerospace structures [1–4]. Typically, the equations of motion for this class of systems are derived and cast in the form of a set of differential-algebraic equations (DAEs) of high index. Earlier attempts to solve these equations are based on index reduction or coordinate partitioning techniques [3, 4]. In contrast, the main objective of this chapter is to first create and employ

N. Potosakis · E. Paraskevopoulos · S. Natsiavas (✉)
Department of Mechanical Engineering, Aristotle University, Thessaloniki, Greece
e-mail: natsiava@auth.gr

a better theoretical foundation and then proceed to development of more advanced numerical schemes.

In the new approach, the equations of motion employed are second-order ordinary differential equations (ODEs). This is achieved by combining concepts of Analytical Dynamics and differential geometry and leads to a natural elimination of singularities associated with DAE formulations from the onset [5]. Since the ulterior motive is the development of an efficient numerical integration scheme, these equations are first put in a convenient weak form. Moreover, the position, velocity, and momentum type quantities are assumed to be independent, forming a three-field set of equations [6]. Finally, the set of equations obtained is solved by application of an augmented Lagrangian formulation, which is set up after introducing appropriate penalty terms [7]. Next, the validity and efficiency of this scheme is tested and illustrated by applying it to a number of characteristic example mechanical systems.

The set of equations of motion employed is included in Sect. 2. Originally, they appear in a strong form and are subsequently put in a three-field weak form. After introducing penalty terms, they are cast eventually in an augmented Lagrangian form. Then, a temporal discretization scheme is developed and numerical results are presented for two mechanical examples in Sect. 3.

2 Equations of Motion: Augmented Lagrangian Formulation

This chapter employs a new set of equations of motion, obtained for a class of multibody mechanical systems subject to equality constraints. The motion is described by a finite number of generalized coordinates $q = (q^1 \dots q^n)$, at any time t [1, 2]. In this way, it can be represented by the motion of a fictitious point, say p , along a curve on the n -dimensional configuration manifold M of the system. Moreover, the tangent vector \underline{v} to this curve belongs to an n -dimensional vector space T_pM , the tangent space of manifold M at p [2]. The systems examined are subject to a set of k motion constraints. For simplicity, these constraints are assumed to be scleronomic, with form

$$\dot{\psi}^R \equiv a_i^R(q)\dot{q}^i = 0. \quad (1)$$

When a constraint is holonomic, its equation can be integrated in the algebraic form

$$\phi^R(q) = 0. \quad (2)$$

The equations of motion of the class of systems examined can be cast in the form

$$\tilde{h}^* \equiv \tilde{h}_M^* - \tilde{h}_C^* = 0 \quad (3)$$

on manifold M , where

$$\tilde{h}_M^* = h_i e^i \quad \text{with} \quad h_i = \left(g_{ij} v^j \right)' - \Lambda_{\ell i}^m g_{m j} v^j v^\ell - f_i \quad (4)$$

and

$$\tilde{h}_C^* = \sum_{R=1}^k h_R a_i^R e^i \quad \text{with} \quad h_R = \left(\bar{m}_{RR} \dot{\lambda}^R \right)' + \bar{c}_{RR} \dot{\lambda}^R + \bar{k}_{RR} \lambda^R - \bar{f}_R. \quad (5)$$

In Eq. (5), the summation convention on repeated indices does not apply to index R . Moreover, the coefficients \bar{m}_{RR} , \bar{c}_{RR} , \bar{k}_{RR} , and \bar{f}_R are determined by the constraints [5]. Equation (3) represents a set of $n + k$ unknowns q^i and λ^R . A complete mathematical formulation is obtained by incorporating the k equations of the constraints, which are expressed originally by Eqs. (1) and (2). In particular, these equations are replaced eventually by

$$g_R = \left(\bar{m}_{RR} \dot{\phi}^R \right)' + \bar{c}_{RR} \dot{\phi}^R + \bar{k}_{RR} \phi^R = 0 \quad \text{and} \quad g_R = \left(\bar{m}_{RR} \dot{\psi}^R \right)' + \bar{c}_{RR} \dot{\psi}^R = 0, \quad (6)$$

respectively, for $R = 1, \dots, k$ [5].

Taking into account Eq. (3) leads first to

$$\int_{t_1}^{t_2} \left(\tilde{h}_M^* - \tilde{h}_C^* \right) (\underline{w}) dt = 0, \quad \forall \underline{w} \in T_p M, \quad (7)$$

along a natural trajectory on the manifold and within any time interval $[t_1, t_2]$. Moreover, as variation of a function f is defined the derivative of f along vector \underline{w} , by

$$\delta f \equiv \underline{w}(f) = f_i w^i. \quad (8)$$

Then, $w^i = \delta q^i$ for each holonomic coordinate, while a little more involved relation is obtained in case of nonholonomic coordinates [6]. In addition, the position, velocity, and momentum variables are considered as independent quantities in the sequel. For this, a new velocity field \underline{v} is introduced on manifold M , which should eventually be forced to become identical to the true velocity field \underline{v} . This means that

$$v^i = v^i \Rightarrow \delta v^i = \delta v^i, \quad (9)$$

with variations defined through Eq. (8) by $\delta v^i = \underline{w}(v^i)$ and $\delta v^i = \underline{w}(v^i)$. In analogy to the action leading to Eq. (7), conditions (9) are imposed by

$$\int_{t_1}^{t_2} \left[\delta\pi_i (v^i - v^i) + \pi_i (\delta v^i - \delta v^i) \right] dt = 0, \quad (10)$$

where the quantities $\delta\pi_i$ and π_i are components of co-vectors belonging to the cotangent space T_p^*M . In the same spirit, by considering the motion constraints expressed by Eq. (6), the following relation must also be satisfied

$$\int_{t_1}^{t_2} g_R \delta\lambda^R dt = 0, \quad (11)$$

for an arbitrary multiplier $\delta\lambda^R$ and each $R = 1, \dots, k$. Then, integrating by parts the first term in the integrand for a holonomic constraint yields

$$\left(\bar{m}_{RR} \dot{\phi}^R \delta\lambda^R \right) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\bar{m}_{RR} \dot{\phi}^R (\delta\lambda^R)' - (\bar{c}_{RR} \dot{\phi}^R + \bar{k}_{RR} \phi^R) \delta\lambda^R \right] dt = 0, \quad (12)$$

while a similar result is also obtained for a nonholonomic constraint.

Next, a similar action is also taken for the velocity components $\dot{\lambda}^R$, by introducing a new vector field on $T_{p_R}M_R$ for each constraint manifold M_R , with components μ^R , together with a new set of Lagrange multipliers σ_R , belonging to the cotangent space $T_{p_R}^*M_R$. As a consequence, the weak formulation is augmented by the terms

$$\int_{t_1}^{t_2} \left[\delta\sigma_R (\mu^R - \dot{\lambda}^R) + \sigma_R (\delta\mu^R - \delta\dot{\lambda}^R) \right] dt = 0, \quad (R = 1, \dots, k), \quad (13)$$

where $\delta\sigma_R$ represents the component of a co-vector on $T_{p_R}^*M_R$. Then, one can relate the strong time derivatives v^i (of q^i or ϑ^i , for a true or a pseudo-coordinate, respectively) and $\dot{\lambda}^R$ of the position type variables to weak velocities, denoted by v^i and μ^R , through two new sets of Lagrange multipliers, denoted by $\delta\pi_i$ and $\delta\sigma_R$, respectively.

Finally, appending the terms in Eqs. (10)–(13) to Eq. (7) and performing lengthy manipulations leads eventually to an involved three-field set of equations [6]. Then, since the variations w^i , $\delta\lambda^R$, δv^i , $\delta\mu^R$, $\delta\pi_i$, and $\delta\sigma_R$ are independent, collecting the terms in these equations multiplied by these quantities leads to a coupled set of nonlinear algebraic equations. In fact, by adding suitable penalty terms due to the constraints, it is found that the form of these equations remains unaffected, making the substitution

$$\bar{\mu}^R = \mu^R - \xi_R \dot{\phi}^R \quad \text{and} \quad \bar{\lambda}^R = \lambda^R - \xi_R \phi^R, \quad (14)$$

when μ^R and λ^R is multiplied by \bar{m}_{RR} or \bar{c}_{RR} and \bar{k}_{RR} , respectively. This provides a convenient and strong basis for developing an appropriate temporal discretization of the equations of motion. For the purposes of the present chapter, this task was