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Discoveries at the Frontiers of Science

From Nuclear Astrophysics to Relativistic Heavy Ion Collisions

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Preface

This volume addresses problems and current research at the forefront of theoretical physics. It also mirrors the deep and broad interests of Professor Walter Greiner (1935–2016), the founder and senior editor of Springer Nature's 'FIAS Interdisciplinary Science Series'.

In June 2017, the Frankfurt Institute for Advances Studies FIAS hosted the 'International Symposium on Discoveries at the Frontiers of Science' to honor and commemorate the scientific legacy of Walter Greiner. The Symposium succeeded in bringing together many world-class scientists for a lively and inspiring exchange of ideas.

Subsequently, the participants have worked diligently to prepare this collection of overview articles. Their contributions cover and connect together, topics ranging from atomic and molecular physics to quantum field theory and nuclear physics; and from relativistic heavy ion collisions and the Equation of State EoS of hot dense nuclear QCD-matter to general relativistic binary black hole—and neutron star mergers, which are being tested experimentally by gravitational wave observatories like LIGO/VIRGO. Further, forefront topics include fundamental quantum mechanics and possible modifications of Einstein's GR.

The Equation of State (EoS) and the quest for phase transitions of nuclear and neutron star matter to quark matter, or from a pure Yang-Mills gluon plasma to a quark-gluon plasma (QGP) are among the great challenges that can be addressed both in studies of the core of neutron stars or neutron star mergers and in high energy heavy ion collisions, during the ultradense, hot matter formation, expansion, and hadronization.

Related topics include the interaction of heavy quarks with QGP partons, heavy ion collision experiments at GSI, FAIR, NICA, SPS, LHC, and RHIC beam energies to investigate the transition from baryon-dominated to meson-dominated matter, the analysis of baryon, pion, and kaon flow as well as the formation of D- and B-mesonic states in heavy ion collisions.

The volume includes reviews addressing photon scattering experiments, superheavy elements, extended versions of general relativity, new experimental developments in heavy ion collisions, and renewable energy networks, and—finally—tributes to Walter Greiner's scientific career.

Many of these contributions have roots in Walter's work and bear witness to his remarkably productive life as a brilliant scientist.

Walter will continue to be present in our hearts and in our minds.

Frankfurt am Main, Germany
September 2019

Johannes Kirsch
Stefan Schramm (Deceased)
Jan Steinheimer-Froschauer
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What the Azimuthal Distribution of Heavy Mesons Tells Us About the Quark Gluon Plasma?

Joerg Aichelin, Pol B. Gossiaux, Marlene Nahrgang and Klaus Werner

Abstract Heavy mesons (charm and bottom) are one of the few probes which are sensitive to the time evolution of a Quark Gluon Plasma (QGP), light mesons come to a statistical equilibrium latest at the end of the QGP expansion and do therefore not carry information on the QGP properties during the expansion. We discuss here the interaction of the heavy quarks with the QGP partons and how this interaction influences the azimuthal distribution of the heavy mesons. We will argue that there are indications that small p_T heavy quarks equilibrate in the QGP whereas those with a high p_T create a finite azimuthal flow due to the different path lengths in the QGP. These are results of the pQCD based Monte-Carlo (MC@sHQ) approach which is coupled to EPOS 2 modeling the expansion of the QGP.

1 Introduction

I have neither studied in Frankfurt nor have I been employed by Frankfurt. Nevertheless, I owe a lot to Walter. When I appeared the first time in Frankfurt, innocently invited to present my work, which I did together with Horst Stöcker, in an one hour seminar called "Palaver" Walter bombarded me with questions for more than 2 1/2 h. At the end I was completely exhausted and Walter had knowledge about all the details of our work. Obviously me answers satisfied him because from then on we had many discussions, supported also by the red wine which he asked me regularly to bring to him from my French hometown. Among many things I learned from him is that a physicist should work at the frontiers to unknown land, what he always demonstrate by his own work. So I started a couple of years ago to work on heavy quarks and especially on what one can learn from heavy quarks about the quark gluons plasma which is created in heavy ion collisions.

To understand the formation, the expansion and the hadronization of a quark gluon plasma (QGP) created in ultra-relativistic heavy ion collisions is the ultimate

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objective of the heavy ion experiments at the Large Hadron Collider (LHC) at CERN and the Relativistic Heavy Ion Collider (RHIC) at Brookhaven. To achieve this objective turned out to be more complicated than expected because the multiplicities of the observed hadrons indicate that in the end of the expansion the QGP is in statistical equilibrium and therefore their multiplicity follows statistical laws. In addition, the temperature at which the transition to hadrons takes place, extracted from this statistical analysis, is about the same as that obtained by lattice gauge calculation in which the Quantumchromodynamics, the underlying theory for strong interactions, is solved on a computer [1, 2]. Once statistical equilibrium is obtained the information on how the system approaches this equilibrium is lost. Hadrons which are created from the QGP can therefore tell little, in order not to say nothing, about the time evolution of the QGP before hadronization.

If one wants to study the time evolution of the QGP one has to rely on probes which pass the QGP but which do not come to an equilibrium with the QGP. Such probes exist. One may study electromagnetic probes like photons or dileptons or hard probes which are created in the first interaction between projectile and target nucleons in hard processes, means in those in which the momentum transfer is large. Both of these probes have their advantages and inconveniences. Electromagnetic probes are rare and many processes contribute, before and after the hadronization, to the measured spectra. Many of them are hard to assess because the production cross sections are unknown or only vaguely known and the composition of the hadron gas after the hadronization is debated, depending on how the hadron properties change in a dense environment at high temperature. Despite of these difficulties many interesting features have been discovered by analyzing the electromagnetic probes [3]. As an example, the large anisotropy in the azimuthal plane has for long been a surprise because early produced photons and dileptons should not show such a feature. In the meantime it has been discovered that bremsstrahlung and hadronic interactions are the origin of this anisotropy.

Hard probes, heavy quarks and energetic light quarks or gluons are created in initial hard collisions between projectile and target and have to traverse the plasma. We concentrate here on heavy quarks. The interpretation of jet observables is more complicated because the leading jet parton may change its identity has just started and interesting results are expected for the near future. Being colored objects the interaction of heavy quarks with plasma particles is strong. Due to the propagation through the colored partonic medium high-p_T heavy quarks suffer from a substantial energy loss, while low-p_T heavy quarks are expected to thermalize at least partially within the medium. The nuclear modification factor, R_{AA}, which is the ratio of the spectra measured in heavy-ion collisions to the scaled proton-proton reference, and the elliptic flow, v_2, are traditional observables of heavy-flavor hadrons and decay leptons. A suppression of high-p_T D mesons, heavy-flavor decay electrons and muons has been measured by the STAR [4, 5] and Phenix [6] collaborations at RHIC and the ALICE [7–9] and CMS [10] collaborations at LHC. The v_2 of D mesons, heavy-flavor decay electrons and muons was found to be nonvanishing both at RHIC [11] and at LHC [12].

2 The MC@sHQ Approach

In the MC@sHQ approach [13, 14] the heavy quark ($Q\bar{Q}$) pairs are initialized according to the p_T distribution from FONLL [15–17]. We assume the LO production processes, i.e., an azimuthally back-to-back initialization of the $Q\bar{Q}$ pairs with $\mathbf{p}_{T,\bar{Q}} = -\mathbf{p}_{T,Q}$. The heavy quarks can interact with the plasma constituents purely elastically or in a combination of elastic and inelastic collisions. The elastic cross sections in Born approximation are obtained within a hard thermal loop (HTL) calculation, including a running coupling constant α_s [13, 18]. The contribution from the t-channel is regularized by a reduced Debye screening mass κm_D^2, which is calculated self-consistently [13, 19], yielding a gluon propagator with

$$1/Q^2 \to 1/(Q^2 - \kappa \tilde{m}_D^2(T)) \tag{1}$$

for a momentum transfer Q^2. In this HTL+semihard approach [13], κ is determined such that the average energy loss is maximally insensitive to the intermediate scale between soft (with a HTL gluon propagator) and hard (with a free gluon propagator) processes. The inelastic cross sections include both, the incoherent gluon radiation [20] and the effect of coherence, i.e. the Landaul-Pomeranchuk-Migdal (LPM) effect [21]. In this approach the incoming light partons are considered as massless [22–25].

The fluid dynamical evolution is used as a background providing us with the temperature and velocity fields necessary to sample thermal scattering partners for the heavy quarks. The MC@sHQ approach couples the Monte-Carlo treatment of the Boltzmann equation of heavy quarks (MC@sHQ) [13] to the $3 + 1$ dimensional fluid dynamical evolution of the locally thermalized QGP following the initial conditions from EPOS2 [26, 27]. EPOS2 is a multiple scattering approach which combines pQCD calculations for the hard scatterings with Gribov-Regge theory for the phenomenological, soft initial interactions. Jet components are identified and subtracted while the soft contributions are mapped to initial fluid dynamical fields. By enhancing the initial flux tube radii viscosity effects are mimicked, while the subsequent $3 + 1$ dimensional fluid dynamical expansion itself is ideal. Including final hadronic interactions the EPOS2 event generator has successfully described a variety of bulk and jet observables, both at RHIC and at LHC [26, 27]. For details we refer to the references.

Including elastic and inelastic collisions this approach reproduces quite well the experimental D-meson and non photonic electron data at RHIC and LHC. As an example we display in Fig. 1 the D meson R_{AA} as dashed line, for elastic (coll) as well as for elastic+inelastic collisions (coll+rad) in comparison with ALICE data. Elastic cross sections alone give not sufficient stopping in this approach.

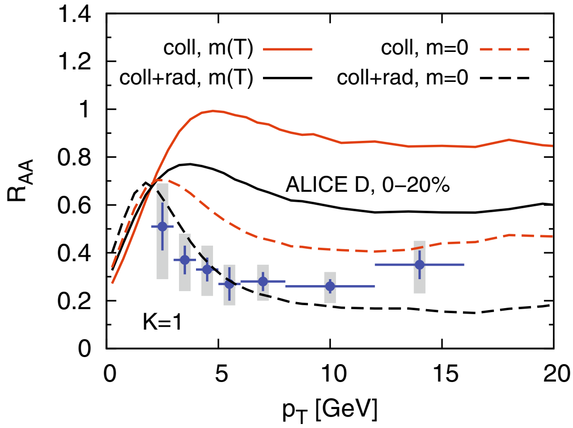

Fig. 1 Comparison of the D meson R_{AA} for a QGP consisting of massive quasiparticles (solid lines) and massless partons (dashed lines). Purely collisional (orange, light) and collisional + radiative (LPM) (blackline) energy loss scenarios are shown

3 Azimuthal Distribution of a Back-to-Back Emitted $Q\bar{Q}$ Pair

The heavy (anti)quarks are propagated through the QGP by means of the coupled MC@sHQ+EPOS approach, which was described above. Here, we track the evolution of the heavy (anti)quark until it leaves the QGP [22]. At this transition point we extract the difference of the azimuthal angles, $\Delta\phi$, of those $Q\bar{Q}$ pairs which were initially produced together. The distributions of $\Delta\phi$ are shown in Fig. 2 for $c\bar{c}$ pairs in the left column and for $b\bar{b}$ pairs in the right column. These pairs are taken into account if both the quark and the antiquark are finally at a rapidity $|y_Q| < 1$ and $|y_{\bar{Q}}| < 1$. The results for the 0–20% most central collisions are plotted in the upper row, while in the middle row we see results for 20–40% centrality and in the lowest row for peripheral collisions (40–60% most central). In each individual plot we show the distribution of azimuthal correlations for three different classes of p_T. The lowest p_T class collects all $Q\bar{Q}$ pairs, where both, the quark and the antiquark, have a final p_T between 1 and 4 GeV. In the intermediate-p_T class quark and antiquark have a final p_T between 4 and 10 GeV and in the high p_T-class the final p_T of the quark and antiquark is in between 10 and 20 GeV. For these calculations we have multiplied our cross section with a K-factor to obtain the best agreement with the experimental R_{AA}. This is due to the fact that the purely collisional interaction mechanism produces a larger average $p_\perp^2$ per unit time [25].

We see, first of all, that the initial correlations are broadened and they are broadened more strongly for the purely collisional interaction mechanism than for the mechanism including radiative corrections.

The systems that are created in the most central collisions are the largest and reach the highest temperatures and densities. Therefore the broadening of the initial delta-function-like correlations is most efficient. We find a substantial broadening of these correlations for all p_T classes and both interaction mechanisms for $c\bar{c}$ pairs in Fig. 2a

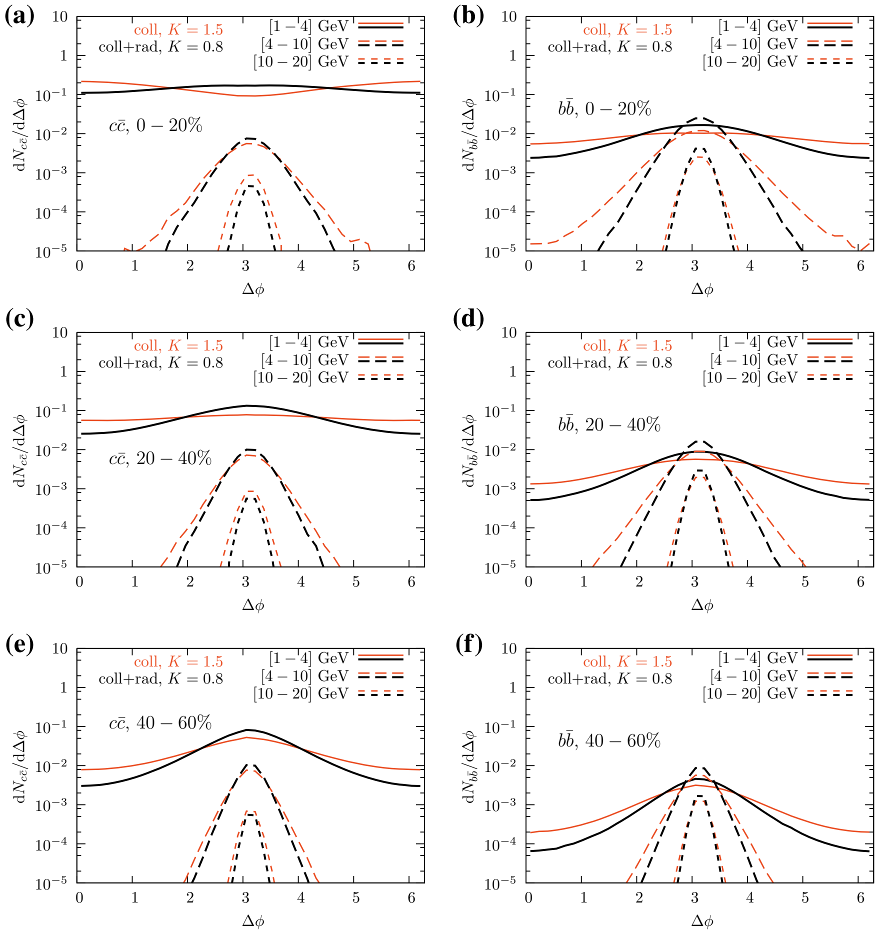

Fig. 2 Azimuthal correlations of initially correlated $Q\bar{Q}$ pairs at the transition temperature. In the left column the azimuthal distributions of $c\bar{c}$ pairs are shown, in the right column those of $b\bar{b}$ pairs at midrapidity. The centralities are 0–20% (upper row), 20–40% (middle row) and 40–60% (lower row). In each plot we compare the purely collisional (orange/light) to the collisional plus radiative (black/dark) interaction mechanism for different classes of final p_T. See text for more details

and for $b\bar{b}$ pairs in Fig. 2b. For the quark with lowest p_T the initial correlations are almost completely washed out. This almost flat $\mathrm{d}N_{Q\bar{Q}}/\mathrm{d}\Delta\phi$ distribution is a strong hint that the heavy quarks are in equilibrium with their environment.

4 v_2 as a Measure of Approaching Equilibrium

This question, whether the heavy quarks approach equilibrium with the QPG partons can be studied by measuring the azimuthal anisotropy of the heavy quarks. Initially, produced in a hard collisions the heavy quarks have no preferred direction in the transverse plane whereas the QGP partons have a fluid dynamical flow. This fluid dynamical flow is the response to the eccentricity in the initial geometry

$$\varepsilon_n = \frac{\sqrt{\langle r^n \cos(n\phi)\rangle^2 + \langle r^n \sin(n\phi)\rangle^2}}{\langle r^n \rangle} \tag{2}$$

where ϕ is the spatial azimuthal angle and $r = \sqrt{x^2 + y^2}$ the distance from the center. The average $\langle \cdot \rangle$ is weighted by the local energy density. Similarly the n-th order angles of the participant plane[1] can be obtained from the initial state via

$$\psi_n^{\mathrm{PP}} = \frac{1}{n} \arctan \frac{\langle r^n \cos(n\phi)\rangle}{\langle r^n \sin(n\phi)\rangle} \,. \tag{3}$$

It has been shown that the flow coefficients

$$v_n^{\mathrm{EP}} = \frac{\int \mathrm{d}\phi \cos\left[n(\phi - \psi_n^{\mathrm{EP}})\right] \frac{\mathrm{d}N}{\mathrm{d}y\mathrm{d}\phi}}{\int \mathrm{d}\phi \frac{\mathrm{d}N}{\mathrm{d}y\mathrm{d}\phi}} \,, \tag{4}$$

taken as the Fourier coefficients of the single-particle azimuthal distribution with respect to the event-plane angle $\psi_n^{\mathrm{EP}} = (1/n)\arctan(\langle p_T \sin(n\phi)\rangle / \langle p_T \cos(n\phi)\rangle)$, where ϕ is the azimuthal angle of the transverse momentum of the measured particles, corresponds very well to the flow coefficients v_n^{PP} obtained from correlating the single particles with the initial participant plane [28].

We now investigate the centrality dependence of the heavy-flavor flow further by plotting the integrated v_n/ε_n, which is dominated by low p_t quarks as a function of the centrality in Fig. 3. We concentrate on the collisional + radiative(LPM) energy loss model at $\sqrt{s} = 2.76$ TeV. Under the assumption that $v_n \propto \varepsilon_n$ (which holds for small and intermediate centralities) the plotted quantity can be identified with the efficiency of the medium to transform an initial geometry into an anisotropy in momentum space. By comparing the D mesons flow to the flow of the light charged hadrons from the bulk and the heavy B mesons we can make the following observations. For all particles we see that the efficiency of the system to respond to the initial geometry decreases toward more peripheral collisions and a mass hierarchy can be observed in the slopes of this decrease. For v_2/ε_2, D mesons and light charged hadrons show a very similar behavior in both the magnitude and the slope, which as such would imply that the overall efficiency of transferring an initial ellipticity to bulk flow

[1]The term "participant plane" is commonly used for the following definition. We would like to point out though, that the initial conditions used here, do not rely on a participant picture.

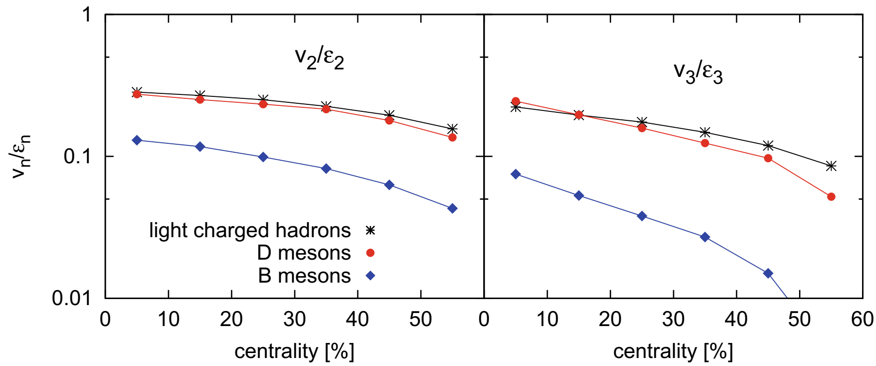

Fig. 3 The centrality dependence of the ratios v_n/ε_n for $n = 2, 3$ in the collisional + radiative (LPM) energy loss model at $\sqrt{s} = 2.76$ TeV collision energies for the light charged hadron, the D and B meson flow. The bulk flow is obtained as $v_n(2, |\Delta\eta| > 1)$ from the full EPOS2 model [27]

and to flow of the charm quarks is of the same order suggesting a perfect coupling of the charm quarks to the bulk. We can see, however, that this does not hold for the third-order Fourier coefficient of the flow where, although being of the same magnitude (within expected errors) in the central collisions, the ratio v_3/ε_3 falls off more quickly for D mesons than for the bulk flow toward more peripheral collisions. For B mesons, the flow is smaller in magnitude and by the steeper decrease one can see a more rapid decoupling from the bulk medium that can be understood. This is a consequence of the large mass of the b quark which leads to larger times needed for equilibration. This time is evidently not available in the rapidly expanding system. We can conclude, that the p_T integrated v_2, which is dominated by low p_T heavy mesons, is another hint that c quarks observed as D-mesons came to an equilibrium with the expanding QGP partons.

5 Eccentricity Is only One Reason for a Finite v_2

We continue our study on the information which is contained in the v_2 observable. In theoretical studies we can artificially switch off the bulk flow by assuming that the local rest frame of the fluid is the same as the laboratory frame. This procedure is of course only a first approximation to a scenario without bulk flow as the temperature field is still taken from an evolution that includes bulk flow, yet it gives an idea of how much of the heavy-flavor flow stems from the path length difference due to the initial eccentricity. For this study we calculate peripheral collisions at $\sqrt{s} = 2.76$ TeV [22]. The results are displayed in Fig. 4. We find that around $p_T \sim 2$ GeV both the v_2 and the v_3 of charm quarks are almost entirely due to the bulk flow of the medium. At $p_T \approx 4$ GeV the charm quark v_2 originating from path length differences is $\sim 50\%$ of the charm quark v_2 produced in a medium with bulk flow. This picture is

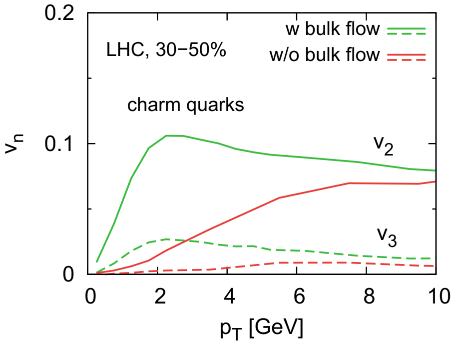

Fig. 4 The contribution of the bulk flow to the charm quark elliptic (solid) and triangular (dashed) flow (right plot) for 30–50% most central Pb+Pb collisions at $\sqrt{s} = 2.76$ TeV

slightly different for the triangular flow v_3. Path length differences seem to be smaller in triangularly shaped event geometries and the corresponding angular sectors are smaller, which diminishes the importance of this contribution to the flow. Up to $p_T \sim 4$ GeV we find that the charm quark v_3 is built up almost exclusively from the bulk flow of the medium, which makes it an excellent probe of the dynamics and interactions of charm quarks in the quark-gluon plasma.

6 Summary

Heavy quarks have been identified as a tool to study the time evolution of a QGP created in ultra-relativistic heavy ion collisions. We demonstrated that calculations using the pQCD based Monte-Carlo (MC@sHQ) approach which is coupled to EPOS 2 for the modeling the expansion of the QGP show that heavy quarks with a small p_T come to an equilibrium with the QPG partons. Therefore their azimuthal distribution is close to that of the light QGP partons. Heavy quarks with a large p_T do not equilibrate. Their azimuthal distribution measures the path length difference of the trajectories of the heavy quarks in the QGP. It depends on the elementary interactions of the heavy quarks with the QGP constituents. The transition between both distributions is smooth, so only theoretical investigations can separate them.

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References

1. S. Borsanyi, Z. Fodor, C. Hoelbling, S.D. Katz, S. Krieg, K.K. Szabo, Phys. Lett. B **730**, 99 (2014)
2. A. Bazavov et al. [HotQCD Collaboration], Phys. Rev. D **90**(9), 094503 (2014)
3. O. Linnyk, E.L. Bratkovskaya, W. Cassing, Prog. Part. Nucl. Phys. **87**, 50 (2016). https://doi.org/10.1016/j.ppnp.2015.12.003, arXiv:1512.08126 [nucl-th]
4. STAR Collaboration, Phys. Rev. Lett. **98**, 192301 (2007); Erratum-ibid. **106**, 159902 (2011)
5. X. Dong [STAR Collaboration], Nucl. Phys. A **904–905**, 19c (2013)
6. PHENIX Collaboration, Phys. Rev. C **84**, 044905 (2011)
7. ALICE Collaboration, JHEP **09**, 112 (2012)
8. A. Dainese [ALICE Collaboration], arXiv:1212.0995 [nucl-ex]
9. B. Abelev et al. [ALICE Collaboration], arXiv:1305.2707 [nucl-ex]
10. CMS Collaboration [CMS Collaboration], CMS-PAS-HIN-15-005
11. L. Adamczyk et al. [STAR Collaboration], arXiv:1405.6348 [hep-ex]
12. J. Adam et al. [ALICE Collaboration], Phys. Lett. B **753**, 41 (2016)
13. P.B. Gossiaux, J. Aichelin, Phys. Rev. C **78**, 014904 (2008), arXiv:0802.2525 [hep-ph]
14. J. Aichelin, P.B. Gossiaux, T. Gousset, Phys. Rev. D **89**(7), 074018 (2014), arXiv:1307.5270 [hep-ph]
15. M. Cacciari, M. Greco, P. Nason, JHEP **9805**, 007 (1998)
16. M. Cacciari, S. Frixione, P. Nason, JHEP **0103**, 006 (2001)
17. M. Cacciari, S. Frixione, N. Houdeau, M.L. Mangano, P. Nason, G. Ridolfi, JHEP **1210**, 137 (2012)
18. A. Peshier, arXiv:hep-ph/0601119
19. S. Peigne, A. Peshier, Phys. Rev. D **77**, 114017 (2008)
20. J.F. Gunion, G. Bertsch, Phys. Rev. D **25**, 746 (1982)
21. P.B. Gossiaux, Nucl. Phys. A **910–911**, 301 (2013), arXiv:1209.0844 [hep-ph]
22. M. Nahrgang, J. Aichelin, S. Bass, P.B. Gossiaux, K. Werner, Phys. Rev. C **91**(1), 014904 (2015). https://doi.org/10.1103/PhysRevC.91.014904, arXiv:1410.5396 [hep-ph]
23. M. Nahrgang, J. Aichelin, S. Bass, P.B. Gossiaux, K. Werner, Nucl. Phys. A **931**, 575 (2014). https://doi.org/10.1016/j.nuclphysa.2014.08.094, arXiv:1409.1464 [hep-ph]
24. M. Nahrgang, J. Aichelin, P.B. Gossiaux , K. Werner, Phys. Rev. C **89**(1), 014905 (2014). https://doi.org/10.1103/PhysRevC.89.014905, arXiv:1305.6544 [hep-ph]
25. M. Nahrgang, J. Aichelin, P.B. Gossiaux, K. Werner, Phys. Rev. C **90**(2), 024907 (2014). https://doi.org/10.1103/PhysRevC.90.024907, arXiv:1305.3823 [hep-ph]
26. K. Werner, I. Karpenko, T. Pierog, M. Bleicher, K. Mikhailov, Phys. Rev. C **82**, 044904 (2010)
27. K. Werner, I. Karpenko, M. Bleicher, T. Pierog, S. Porteboeuf-Houssais, Phys. Rev. C **85**, 064907 (2012)
28. H. Holopainen, H. Niemi, K.J. Eskola, Phys. Rev. C **83**, 034901 (2011). https://doi.org/10.1103/PhysRevC.83.034901, arXiv:1007.0368 [hep-ph]

Limiting Temperature, Phase Transition(s), Crossover, …

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Abstract I present a short review of two physical models devoted to the equation of state at high energy densities: The Hagedorn concept of limiting temperature $T = T_H$ and the statistical bag model of phase transitions at $T = T_c$. The statistical bag model admits the different orders of phase transitions between hadrons (small bags) and quark gluon plasma (infinitely large bags). The crossover transition between hadron resonance gas and cluster quark gluon plasma at $T = T_{\rm cr}$ is also possible within the statistical bag model. For all these different phenomena rather similar values of the temperatures, $T_H \sim T_c \sim T_{\rm cr} = 150$–$160$ MeV, have been assumed.

Keywords Limiting temperature · Quark-gluon bags · Phase transitions · Crossover

1 Introduction

Equation of state and other equilibrium properties of matter are the subjects of statistical mechanics. This physical approach is used to describe a multiparticle system and calculate its partition function, i.e., the sum over all permitted microstates. To make this, one should define particle species, interparticle interactions, conserved charges in the considered system, and the external conditions. All this information requires careful experimental and theoretical investigations, and at present it is rather well known for typical atomic and molecular systems, at least in their gas and liquid phases. These statistical systems are usually defined in terms of several number of atomic and molecular species and two-particle potentials between their different pairs. If chemical reactions are possible, the numbers of conserved atomic species, or corresponding chemical potentials, should be additionally defined. One also chooses an appropriate statistical ensemble which reflects the physical boundary conditions.

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Thus, a statistical description of real gases and liquids becomes a purely mathematical problem. Nevertheless, this mathematical problem remains to be rather difficult. For example, one should still follow some phenomenological models in a description of the liquid-gas phase transition and its critical point as rigorous analytical results in this field are rather poor.

Strongly interacting matter at high energy density is formed at the early stages of nucleus-nucleus collisions and/or in the central regions of neutron stars. What types of particles should be considered as the fundamental ones and what are the composite objects? What are the fundamental forces between the matter constituents? What are the conserved charges? What statistical ensemble should be used for their description? At high energy densities an equilibrium system consists from elementary particles, and the answers to the above questions have been changed in time with increasing of our knowledge about basic physical features of elementary particles.

The fundamental microscopic constituents and basic interactions at high energy density were not well known up to the recent time. First experiments on high energy collisions carried out in the middle of the last century demonstrated large amount of new particle species—hadrons and resonances. These particles are influenced by the strong interactions. And the main feature of the strong interactions is a creation of new and new types of hadrons and resonances with increasing of collision energies. These features of the strong interactions lead Hagedorn to formulation of his statistical model with exponentially increasing spectrum of resonances at large masses. This model considered in Sect. 2 introduced the new hypothetical physical constant—the limiting temperature $T_H \cong 160$ MeV.

Today we know that fundamental constituents of the strongly interacting matter are quarks and gluons, and their interactions are described by the quantum chromodynamics (QCD). This fact is important for our understanding of nature. It becomes clear that hadrons and resonances can not be the point-like objects, as it was assumed in the Hagedorn model. Hadrons and resonances should have an internal quark structure and, thus, non-zero proper volume. This was taken into account by the excluded volume procedure in statistical models with arbitrary number of hadron species discussed in Sect. 3.

An understanding of a fundamental role of quarks and gluons does not make however easier a theoretical description of relativistic nucleus-nucleus collisions. The main feature of these reactions at high collision energy is a production of huge amount of hadrons and resonances. Measured physical quantities are mostly presented in terms of hadrons. One still has to work with composite objects both at the beginning and the end of nucleus-nucleus collisions. Quarks and gluons can appear at the short early stage of nucleus-nucleus reaction and the only straightforward signals of this stage can be its specific electromagnetic radiation. The statistical model discussed in Sect. 4 treats matter at high energy densities in terms of hadron-like degrees of freedom—quark-gluon bags. The notions of hadrons have been changed with increasing of our knowledge on the strong interactions: hadrons, resonances, fireballs, and quark-gluon bags. Quark-gluon bags play a role of the bridge between hadron gas from one side and quarks-gluon plasma (QGP) from the other side. Section 5 summarizes my presentation.

2 Hagedorn Limiting Temperature

The first model of matter at high energy density was formulated in 1950 by Fermi [1]. It was assumed that a system created in high energy proton-proton collisions consists of pions and behaves as the black-body radiation, i.e., pions were considered as non-interacting particles, and the pion mass was neglected as compared to high temperature of the system.

In what follows we consider the system with zero values of the net baryon number, electric charge, and strangeness. These conditions correspond approximately to nucleus-nucleus collisions in Large Hadron Collider at CERN. The temperature T remains then the only independent thermodynamical variable in the thermodynamic limit when the system volume goes to infinity. The system pressure is defined in terms of the grand canonical partition function $Z(T, V)$ as (the system of units with $h/(2\pi) = c = k_B = 1$ will be used),

$$p(T) \;=\; T \lim_{V\to\infty} \frac{\ln Z(T, V)}{V}\;. \tag{1}$$

The pressure function plays a role of the thermodynamical potential in the grand canonical ensemble. The entropy density s and energy density ε can be calculated from $p(T)$ using the thermodynamical identities:

$$s(T) \;\equiv\; \frac{dp}{dT}\;, \qquad \varepsilon(T) \;\equiv\; T\frac{dp}{dT} \;-\; p\;. \tag{2}$$

Let us consider the ideal gas of particle with mass m and degeneracy factor g. The partition function can be calculated in the Boltzmann approximation as

$$Z(T, V) \;=\; \sum_{N=0}^{\infty} \frac{V^N}{N!} \left[g \int_0^{\infty} \frac{k^2 dk}{2\pi^2} \exp\left(-\frac{\sqrt{k^2 + m^2}}{T} \right) \right]^N \;\equiv\; \exp\left[V\phi_m(T)\right]\;, \tag{3}$$

where

$$\phi_m(T) \;=\; \frac{g}{2\pi^2} \int_0^{\infty} k^2 dk\, \exp\left(-\frac{\sqrt{k^2 + m^2}}{T} \right) \;=\; g\,\frac{m^2 T}{2\pi^2}\, K_2(m/T)\;, \tag{4}$$

with K_2 being the modified Bessel function. The function $\phi_m(T)$ has physical meaning as particle number density. Using Eqs. (1) and (2) one finds

$$p \;=\; T\phi_m(T)\;, \qquad \varepsilon \;=\; T^2\,\frac{d\phi_m}{dT}\;. \tag{5}$$

At $T/m \gg 1$ one obtains

$$p \cong \frac{g}{\pi^2} T^4 \,, \quad \varepsilon \cong \frac{3g}{\pi^2} T^4 \,, \tag{6}$$

i.e., in the high temperature limit a behavior of the energy density has the familiar Stephan-Boltzmann (S-B) form, $\varepsilon = \sigma T^4$ with S-B constant $\sigma = 3g/\pi^2 \cong 0.30\, g$. The Bose and Fermi statistics lead to the same behavior $\varepsilon \sim T^4$ with the corresponding S-B constants $\sigma_{\text{Bose}} = \pi^2 g/30 \cong 0.33\, g$ and $\sigma_{\text{Fermi}} = 7\pi^2 g/240 \cong 0.29\, g$ which are only slightly different from their Boltzmann approximation.

In the opposite limit, $m/T \gg 1$, one obtains from Eq. (4)

$$\phi_m(T) \cong g \left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T}\right) \,. \tag{7}$$

In this large mass limit the quantum statistics effects play no role at all. In what follows we will neglect quantum statistics effects. Note that this Boltzmann approximation is rather reasonable for the analysis of nucleus-nucleus collisions at very high collision energies where the baryon chemical potential is approximately equal to zero. This classical approximation is however violated for systems with large baryon densities and moderate temperatures, e.g., Fermi statistics effects may play a crucial role in a description of nuclear matter created in nucleus-nucleus reactions at small collisions energies or matter formed inside neutron stars.

The number of particles N is a random variable and has the Poisson probability distribution with the average value

$$\langle N \rangle = V\, \phi_m(T) \,, \tag{8}$$

which leads to a most familiar form of the ideal gas equation of state

$$pV = \langle N \rangle T \,. \tag{9}$$

The only difference from the school textbook formula is that $\langle N \rangle$ in Eq. (9) is not a constant number but depends on the system volume and temperature according to Eq. (8).

One can generalize the above equations to the system of several particle species with masses $m_1, \ldots, m_n$:

$$p(T; m_1, \ldots, m_n) = \sum_{i=1}^{n} p(T; m_i) \,, \quad \varepsilon(T; m_1, \ldots, m_n) = \sum_{i=1}^{n} \varepsilon(T; m_i) \,, \tag{10}$$

where $p(T; m_i)$ and $\varepsilon(T; m_i)$ are given by Eq. (5) with $m = m_i$ and $g = g_i$. These expressions look again similar to the ideal gas formulae for a mixture of different atoms. However, in contrast to non-relativistic physics, the numbers of each particle species marked by the mass m_i are not constant values but are changed with the system temperature according to Eq. (8). This is a feature of relativistic physics. Only conserved charges, not just the number particles, are really conserved. Adding the energy, i.e., by increasing the temperature, one observes more and more new particles in the system. Note again that our discussion does not introduce any types of conserved charges.

Let us fix $m_1 < \cdots < m_N$ in Eq. (10) and extend the sums in Eq. (10) to infinity. For these series to exist it is necessary that nth term goes to zero at $n \to \infty$, this can be only achieved if $m_n \to \infty$ at $n \to \infty$, otherwise the infinite sums in Eq. (10) would be divergent. The lightest particle in the hadron spectrum is the pion with $m_\pi \cong 140$ MeV. Then one needs to add all known particles (and antiparticles). The particle degeneracy factor g_m, i.e., the number of the internal degrees of freedom for the particle with mass m, assumes that g_m equal terms are present in Eq. (10) for each m value.

It is convenient to introduce the mass spectrum density $\rho(m)$, i.e., $\rho(m)dm$ gives the number of different particle mass states including their internal degeneracies g_m in the interval $[m, m + dm]$. The partition function of the system takes then the following form

$$Z(T,V) = \sum_{N=0}^{\infty} \frac{V^N}{N!} \int_0^\infty \rho(m_1)dm_1 \int_0^\infty \frac{k_1^2 dk_1}{2\pi^2} \exp\left(-\frac{\sqrt{k_1^2+m_1^2}}{T}\right) \times \cdots$$
$$\times \int_0^\infty \rho(m_N)dm_N \int_0^\infty \frac{k_N^2 dk_N}{2\pi^2} \exp\left(-\frac{\sqrt{k_N^2+m_N^2}}{T}\right) = \exp\left[V \int_0^\infty \rho(m)\,\phi_m(T)\right] . \tag{11}$$

The system pressure p, particle number density $n \equiv \langle N\rangle/V$, and energy density ε read

$$p = T\,n = T \int_0^\infty dm\,\rho(m)\,\phi_m(T) , \quad \varepsilon = T^2 \int_0^\infty dm\,\rho(m)\,\frac{d\phi_m}{dT} . \tag{12}$$

It was suggested by Hagedorn [2] that hadron mass spectrum increases exponentially at $m \to \infty$

$$\rho(m) \sim C m^{-a} \exp(bm) , \tag{13}$$

with model parameters C, a, and b. The spectrum (13) was originally motivated by the data for the spectrum of known particles and resonances, and the exponential behavior of transverse momentum spectra of secondary particles in high energy collisions. Both arguments lead to the same value of $T_H \equiv 1/b \cong 160$ MeV. Later, Eq. (13) was also supported by the theoretical arguments formulated within the statistical bootstrap model [3]. From Eqs. (7) and (13) it follows that the integrands in Eq. (12) behave as $m^{3/2-a+l}\exp[-m(1/T - 1/T_H)]$ at $m \to \infty$ with $l = 0$ for p and $l = 1$ for ε. The values of $T_H = 1/b$ becomes, therefore, a limiting temperature (Hagedorn temperature), i.e., at $T > T_H$ the integrals in Eq. (12) become divergent. At $T \to T_H - 0$ a behaviour of the thermodynamical functions (12) with mass spectrum (13) depends crucially on parameter a in Eq. (13). For $T = T_H$ the exponential part of the integrands in Eq. (12) vanishes and the convergence or divergence of the integrals at their upper limit are defined by the parameter a:

$$p,\ n,\ \varepsilon\ \to \infty\ , \quad \text{for}\ \ a \le 5/2\ , \tag{14}$$

$$p,\ n,\ \to \text{const}\ ,\ \ \varepsilon\ \to \infty\ , \quad \text{for}\ \ 5/2 < a \le 7/2\ , \tag{15}$$

$$p,\ n,\ \varepsilon\ \to \text{const}\ , \quad \text{for}\ \ a > 7/2\ . \tag{16}$$

The system properties at T near T_H are dependent on contributions of heavy particles with $m \to \infty$. This contribution, in turn, are defined by the value of a. The limiting temperature singularity $T = T_H$ appears because of the exponential increasing factor, $\exp(m/T_H)$, of the mass spectrum (13), but the specific behaviour near this singular point is defined by the power factor, m^{-a}, in Eq. (13). The mass spectrum (13) was suggested by Hagedorn more than 50 years ago. A special name—fireball—was introduced for the heavy hadrons. They were considered as an extensions of hadron resonances to the high mass region. A clear experimental identification of the individual fireball states in the region of very large masses is rather problematic. The mass distinction of these excited states at high m becomes smaller then their expected decay widths. Even today, a presence of the fireball-like states in nature remains as the open question.

One feature of these states does not look however as the physical one. All particles including fireballs with $m \to \infty$ were treated as point-like objects. We overcome this unrealistic feature of the statistical bootstrap model. By taking into account particle proper volumes we hope to transform a limiting temperature T_H into the temperature of a phase transition.

3 Excluded Volume Effects

We introduce now the particle proper volume effects. For fixed particle number, N, Eq. (8), $pV = NT$, will be modified according to the van der Waals excluded volume procedure with v_0 being the proper volume particle parameter

$$p(V - v_0 N)\ =\ NT\ . \tag{17}$$

The grand canonical partition function (3) is then transformed into

$$Z(T, V)\ =\ \sum_{N=0}^{\infty} \frac{[(V - v_0 N)\,\phi_m]^N}{N!}\,\theta(V - v_0 N)\ . \tag{18}$$

To proceed further one can use the Laplace transformation of Eq. (18):

$$\hat{Z}(T,s) \equiv \int_0^{\infty} ds \, \exp(-sV)\, Z(T,V) = \left[s - \exp(-v_0 s)\phi_m(T)\right]^{-1} . \quad (19)$$

An exponentially increasing part of the partition function behaves as $Z(T, V) \sim \exp(pV/T)$ and generates the singularity of the function $\hat{Z}$ in variable s. The farthest-right singularity s^* gives us the system pressure,

$$p(T) = T\, s^* . \quad (20)$$

This is because at $s^* > p(T)/T$ the V-integral in Eq. (19) diverges at its upper limit. The connection of the farthest-right s-singularity of $\hat{Z}(T, s)$ to the asymptotic $V \to \infty$ behaviour of $Z(T, V)$ given by Eq. (19) is a general mathematical property of the Laplace transform. The farthest-right s-singularity of the function (19) is a simple pole. It leads to the following transcendental equation for the system pressure [4]

$$p(T) = \exp\left(-\frac{v_0 p(T)}{T}\right) T\phi_m(T) . \quad (21)$$

At $v_0 = 0$ Eq. (21) is reduced to the ideal gas result of Eq. (5).

Equation (19) can be generalize for an arbitrary number of types of particles $(m_1, v_1), \ldots, (m_n, v_n)$:

$$\hat{Z}(T,s) = \left[s - \sum_{j=1}^{n} \exp(-v_j s)\phi_{m_j}(T)\right]^{-1} . \quad (22)$$

As long as the number n is finite, the farthest-right singularity s^* of the function (22) is always the pole. We denote this pole point as $s_H(T)$. The equation of state of the system reads then as

$$p(T) = \sum_{j=1}^{n} \exp\left(-\frac{v_j\, p(T)}{T}\right) T\phi_{m_j}(T) . \quad (23)$$

Our final step in the model formulation is to extend the summation over different (m, v)-types of particles up to infinity. As before, it is convenient to work with an integral over particle mass-volume spectrum. The spectrum function $\rho(m, v)$ is introduced, so that $\rho(m, v)dmdv$ gives the number of different mass-volume states. The sum $\sum_{j=1}^{n} \ldots$ over different particle species in (22) is then replaced by $\int dmdv\rho(m, v) \ldots$ integral.

4 Statistical Models of Bags

The statistical model with $\rho(m, v)$ mass-volume spectrum is defined by the following formulae [5, 6]:

$$\hat{Z}(T,s) \equiv \int_0^\infty dV \exp(-sV)\, Z(T,V) = \left[s - \int dm dv\, \rho(m,v)\, \exp(-vs)\phi_m(T)\right]^{-1}$$
$$= [s - f(T,s)]^{-1} , \tag{24}$$
$$p(T) \equiv T \lim_{V\to\infty} \frac{\ln Z(T,V)}{V} = T\, s^*(T) , \tag{25}$$

where $s^*(T)$ is the farthest-right s-singularity of the function $\hat{Z}(T,s)$. Note that all models discussed in the previous sections can be obtained using the particular choices of the $\rho(m, v)$ spectrum when this spectrum is reduced to corresponding special forms of either $\rho(m)$ or to $\rho(v)$ functions.

One possible singular point of the function $\hat{Z}$ in variable s is evidently the pole singularity, $s^* = s_H$ defined by the transcendental equation

$$s_H(T) = f(T, s_H) . \tag{26}$$

If the mass-volume spectrum $\rho(m, v)$ is restricted by a finite number of states, (m_j, v_j), the farthest-right singularity s^* is always the pole s_H given by Eq. (26). For an infinite number of states, another singular point $s^* = s_Q$ can emerge. This is a possible singularity of the function $f(T,s)$ itself, which can appear due to a divergence of the $dmdv$-integrals at their upper limits. The equation of state takes then the form:

$$p(T) = \max\{s_H(T), s_Q(T)\} . \tag{27}$$

The mathematical mechanism for possible phase transitions in our model is the ‘coalescence’ (coincidence) of the two singularities $s_H(T)$ and $s_Q(T)$. Note that the possible phase transitions reveal themselves as the singularities of the $p(T)$ function, and they can only appear in the thermodynamical limit $V \to \infty$.

The mass-volume spectrum $\rho(m, v)$ is supposed to reproduce the known low-lying hadron states. The region where both m and v are large will be described within the bag model [7]. The density of states of quarks and gluons with the total energy $m - Bv$ ($B = \text{const} > 0$) can be then presented at large m and v as

$$\rho(m,v) \cong C\, v^\gamma\, (m - Bv)^\delta \exp\left[\frac{4}{3}\sigma_Q^{1/4} m^{1/4} (m - Bv)^{3/4}\right] , \tag{28}$$

where C, γ, and δ are the model parameters, and σ_Q is the Stefan-Boltzmann constant counting gluons (spin, color) and (anti-)quarks (spin, color, flavor) states inside the bag.