CLASSICS IN MATHEMATICS

Max Karoubi

K-Theory

An Introduction

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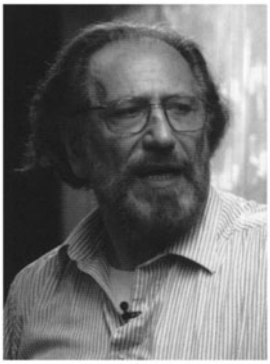

Max Karoubi received his PhD in mathematics (Doctorat d'Etat) from Paris University in 1967, while working in the CNRS (Centre National de la Recherche Scientifique), under the supervision of Henri Cartan and Alexander Grothendieck. After his PhD, he took a position of "Maître de Conférences" at the University of Strasbourg until 1972. He was then nominated full Professor at the University of Paris 7-Denis Diderot until 2007. He is now an Emeritus Professor there.

Max Karoubi

K-Theory

An Introduction

Reprint of the 1978 Edition

With a New Postface by the Author and a List of Errata

With 26 Figures

Springer

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Max Karoubi

K-Theory

An Introduction

With 26 Figures

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A Pierre et Thomas

Foreword

K-theory was introduced by A. Grothendieck in his formulation of the Riemann–Roch theorem (cf. Borel and Serre [2]). For each projective algebraic variety, Grothendieck constructed a group from the category of coherent algebraic sheaves, and showed that it had many nice properties. Atiyah and Hirzebruch [3] considered a topological analog defined for any compact space X, a group $K(X)$ constructed from the category of vector bundles on X. It is this "topological K-theory" that this book will study.

Topological K-theory has become an important tool in topology. Using K-theory, Adams and Atiyah were able to give a simple proof that the only spheres which can be provided with H-space structures are S^1, S^3 and S^7. Moreover, it is possible to derive a substantial part of stable homotopy theory from K-theory (cf. J. F. Adams [2]). Further applications to analysis and algebra are found in the work of Atiyah–Singer [2], Bass [1], Quillen [1], and others. A key factor in these applications is Bott periodicity (Bott [2]).

The purpose of this book is to provide advanced students and mathematicians in other fields with the fundamental material in this subject. In addition, several applications of the type described above are included. In general we have tried to make this book self-contained, beginning with elementary concepts wherever possible; however, we assume that the reader is familiar with the basic definitions of homotopy theory: homotopy classes of maps and homotopy groups (cf. collection of spaces including projective spaces, flag bundles, and Grassmannians. Hilton [1] or Hu [1] for instance). Ordinary cohomology theory is used, but not until the end of Chapter V. Thus this book might be regarded as a fairly self-contained introduction to a "generalized cohomology theory".

The first two chapters ("Vector bundles" and "First notions in K-theory") are chiefly expository; for the reader who is familiar with this material, a brief glance will serve to acquaint him with the notation and approach used. Chapter III is devoted to proving the Bott periodicity theorems. We employ various techniques following the proofs given by Atiyah and Bott [1], Wood [1] and the author [2], using a combination of functional analysis and "algebraic K-theory".

Chapter IV deals with the computation of particular K-groups of a large The version of the "Thom isomorphism" in Section IV.5 is mainly due to Atiyah, Bott and Shapiro [1] (in fact they were responsible for the introduction of Clifford algebras in K-theory, one of the techniques which we employ in Chapter III).

Chapter V describes some applications of K-theory to the question of H-space structures on the sphere and the Hopf invariant (Adams and Atiyah [1]), and to the solution of the vector field problem (Adams [1]). We also present a sketch of the theory of characteristic classes, which we apply in the proof of the Atiyah–Hirzebruch integrality theorems [1]. In the last section we use K-theory to make some computations on the stable homotopy groups of spheres, via the groups $J(X)$ (cf. Adams [2], Atiyah [1], and Kervaire–Milnor [1]).

In spite of its relative length, this book is certainly not exhaustive in its coverage of K-theory. We have omitted some important topics, particularly those which are presented in detail in the literature. For instance, the Atiyah–Singer index theorem is proved in Cartan–Schwartz [1], Palais [1], and Atiyah–Singer [2] (see also appendix 3 in Hirzebruch [2] for the concepts involved). The relationship between other cohomology theories and K-theory is only sketched in Sections V.3 and V.4. A more complete treatment can be found in Conner–Floyd [1] and Hilton [2] (Atiyah–Hirzebruch spectral sequence). Finally algebraic K-theory is a field which is also growing very quickly at present. Some of the standard references at this time are Bass's book [1] and the Springer Lecture Notes in Mathematics, Vol. 341, 342, and 343.

I would like to close this foreword with sincere thanks to Maria Klawe, who greatly helped me in the translation of the original manuscript from French to English.

Paris, Summer 1977

Max Karoubi

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Remarks on Notation and Terminology

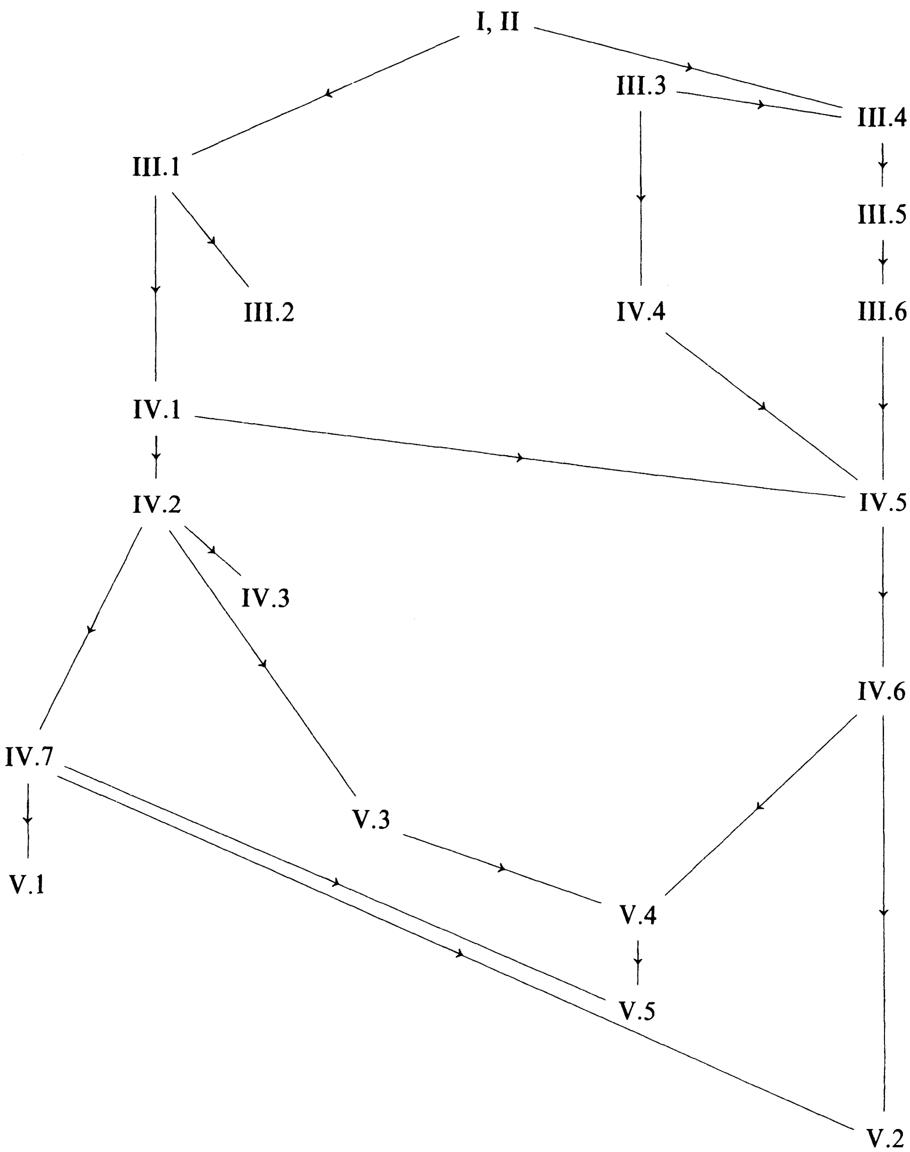
The following notation is used throughout the book: $\mathbb{Z}$ integers, $\mathbb{Q}$ rational numbers, $\mathbb{R}$ real numbers, $\mathbb{C}$ complex numbers, $\mathbb{H}$ quaternions; $\mathrm{GL}_n(A)$ denotes the group of invertible $n \times n$ matrices with coefficients in the ring A. The notation $*\cdots*$ signifies an assertion in the text which is not a direct consequence of the theorems proved in this book, but which may be found in the literature; these assertions are not referred to again, except occasionally in exercises.

If $\mathscr{C}$ is a category, and if E and F are objects of $\mathscr{C}$, then the symbol $\mathscr{C}(E, F)$ or $\mathrm{Hom}_{\mathscr{C}}(E, F)$ means the set of morphisms from E to F.

More specific notation is listed at the end of the book.

A reference to another part of the book is usually given by two numbers (e.g. 5.21) if it is in the same chapter, or by three numbers (e.g. IV.6.7) if it is in a different chapter.

Interdependence of Chapters and Sections



Summary of the Book by Sections

Chapter I. Vector Bundles

1. *Quasi-vector bundles.* This section covers the general concepts and definitions necessary to introduce Section 2. Theorem 1.12 is particularly important in the sequel.

2. *Vector bundles.* The "vector bundles" considered here are locally trivial vector bundles whose fibers are finite dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$. To be mentioned: Proposition 2.7 and Examples 2.3 and 2.4 will be referred to in the sequel.

3. *Clutching theorems.* This technical section is necessary to provide a bridge between the theory of vector bundles and the theory of "coordinate bundles" of N. Steenrod [1]. The clutching theorems are useful in the construction of the tangent bundle of a differentiable manifold (3.18) and in the description of vector bundles over spheres (3.9; see also I.7.6).

4. *Operations on vector bundles.* Certain "continuous" operations on finite dimensional vector spaces: direct sum, tensor product, duality, exterior powers, etc. . . . can be also defined on the category of vector bundles.

5. *Sections of vector bundles.* Only continuous sections are considered here. The major topic concerns the solution of problems involving extensions of sections over paracompact spaces.

6. *Algebraic properties of the category of vector bundles.* In this section we prove that the category $\mathscr{E}(X)$ of vector bundles over a compact space X, is a "pseudo-abelian additive" category. Essentially this means that one has direct sums of vector bundles (the "Whitney sum"), and that every projection operator has an image. From this categorical description (6.13), we deduce the theorem of Serre and Swan (6.18): The category $\mathscr{E}(X)$ is equivalent to the category $\mathscr{P}(A)$, where A is the ring of continuous functions on X, and $\mathscr{P}(A)$ is the category of finitely generated projective modules over A.

7. *Homotopy and representability theorems.* This section is essential for the following chapters. We prove that the problem of classification of vector bundles with compact base X depends only on the homotopy type of X (7.2). We also prove that $\Phi_n^k(X)$ (the set of isomorphism classes of k-vector bundles, over X of rank n for $k=\mathbb{R}$ or $\mathbb{C}$), considered as a functor of X, is a direct limit of representable functors. This takes the concrete form of Theorems 7.10 and 7.14.

8. *Metrics and forms on vector bundles.* It is sometimes important to have some additional structure on vector bundles, such as bilinear forms, Hermitian forms, etc. With the exception of Theorem 8.7, this section is not used in the following chapters (except in the exercises).

Chapter II. First Notions of *K*-Theory

1. *The Grothendieck group of an additive category. The group* K(X). Starting with the simple notion of symmetrization of an abelian monoid, we define the group $K(\mathscr{C})$ of an additive category using the monoid of isomorphism classes of objects of $\mathscr{C}$. Considering the case where $\mathscr{C}$ is $\mathscr{E}(X)$ and X is compact, we obtain the group $K(X)$ (actually $K_{\mathbb{R}}(X)$ or $K_{\mathbb{C}}(X)$ according to which theory of vector bundles is considered). We prove that $K_{\mathbb{R}}(X) \approx [X, \mathbb{Z} \times BO]$ and $K_{\mathbb{C}}(X) \approx [X, \mathbb{Z} \times BU]$ (1.33).

2. *The Grothendieck group of an additive functor. The group $K(X, Y)$.* In order to obtain a "reasonable" definition of the Grothendieck group $K(\varphi)$ for an additive functor $\varphi: \mathscr{C} \to \mathscr{C}'$, which generalizes the definition of $K(\mathscr{C})$ when $\mathscr{C}'=0$, we assume some topological conditions on the categories $\mathscr{C}$ and $\mathscr{C}'$ and on the functor φ (2.6). Since these conditions are satisfied by the "restriction" functor $\mathscr{E}(X) \to \mathscr{E}(Y)$ where Y is closed in X, we then define the "relative group" $K(X, Y)$ to be the K-group of this functor. In fact, $K(X, Y) \approx \tilde{K}(X/Y)$ (2.35). This isomorphism shows that essentially we do not obtain a new group; however, the groups $K(\varphi)$ and $K(X, Y)$ will be important technical tools later on.

3. *The group K^{-1} of a Banach category. The group $K^{-1}(X)$.* This section represents the first step towards the construction of a cohomology theory h^* where the term h^0 is the group $K(X, Y)$ (also denoted by $K^0(X, Y)$) considered in II.2. The group $K^{-1}(\mathscr{C})$, where $\mathscr{C}$ is a Banach category, is obtained from the automorphisms of objects of $\mathscr{C}$. Again, if we consider the case where $\mathscr{C}$ is $\mathscr{E}(X)$, we obtain the group called $K^{-1}(X)$. We prove that if Y is a closed subspace of X then the sequence

$$K^{-1}(X) \to K^{-1}(Y) \to K(X, Y) \to K(X) \to K(Y) \text{ is exact.}$$

We also prove that $K_{\mathbb{R}}^{-1}(X) \approx [X, \mathrm{O}]$ and $K_{\mathbb{C}}^{-1}(X) \approx [X, \mathrm{U}]$ (3.19).

4. *The groups $K^{-n}(X)$ and $K^{-n}(X, Y)$.* The aim of this section is to define the groups $K^{-n}(X, Y)$ for $n \geqslant 2$ and to establish the exact sequence

$$K^{-n-1}(X) \to K^{-n-1}(Y) \to K^{-n}(X, Y) \to K^{-n}(X) \to K^{-n}(Y), \quad \text{for } n \geqslant 1$$

One possible definition is $K^{-n}(X, Y)=\tilde{K}(S^n(X/Y))$ (4.12). We prove some "Mayer–Vietoris exact sequences" (4.18 and 4.19) which will be very useful later on.

5. *Multiplicative structures.* The tensor product of vector bundles provides the group $K(X)$ with a ring structure. It is more difficult to define a "cup-product"

$$K(X, Y)\times K(X', Y')\to K(X\times X', X\times Y'\cup Y\times X')$$

or more generally

$$K^{-n}(X, Y)\times K^{-n'}(X', Y')\to K^{-n-n'}(X\times X', X\times Y'\cup Y\times X')$$

when Y and Y' are non-empty. This is accomplished in a theoretical sense in proposition 5.6; however, in applications it is often useful to have more explicit formulas. For this we introduce another definition of the group $K(X, Y)$ by putting metrics on the vector bundles involved (5.16). This will not be used before Chapter IV. The existence of such cup-products shows that there is a direct splitting $K(X)\approx H^0(X; \mathbb{Z})\oplus K'(X)$ where $K'(X)$ is a nil ideal (cf. 5.9; note that $K'(X)\approx \tilde{K}(X)$ if X is connected).

Chapter III. Bott Periodicity

1. *Periodicity in complex K-theory.* In this section we define an isomorphism $K_{\mathbb{C}}^{-n}(X, Y)\approx K_{\mathbb{C}}^{-n-2}(X, Y)$. The method (due to Atiyah, Bott, and Wood) is to reduce this isomorphism for general n, to a theorem on Banach algebras (1.11): If A is a complex Banach algebra, the group $K(A)$ (defined as $K(\mathscr{P}(A))$ is naturally isomorphic to $\pi_1(\mathrm{GL}(A))$ where $\mathrm{GL}(A)=\mathrm{inj\ lim}\,\mathrm{GL}_n(A)$. This theorem is proved using the Fourier series of a continuous function with values in a complex Banach space, and some classical results in Algebraic K-theory on Laurent polynomials. The original theorem follows when we let A be the ring of complex continuous functions on a compact space.

2. *First applications of Bott periodicity theorem in the complex case.* As a first application we obtain the classical theorem of Bott: for $n>i/2$, we have $\pi_i(\mathrm{U}(n))\approx\mathbb{Z}$ if i is odd and $\pi_i(\mathrm{U}(n))=0$ for i even. We also prove that real K-theory is periodic of period 4 mod. 2-torsion: $K_{\mathbb{R}}^{-n}(X, Y)\otimes_{\mathbb{Z}}\mathbb{Z}'\approx K_{\mathbb{R}}^{-n-4}(X, Y)\otimes_{\mathbb{Z}}\mathbb{Z}'$, where $\mathbb{Z}'=\mathbb{Z}[\frac{1}{2}]$. This theorem will be strengthened in III.5.

3. *Clifford algebras.* These algebras play an important role in real K-theory and will be used in Chapter IV in both real and complex K-theory. This section is purely algebraic. The essential result is Theorem 3.21, which establishes a kind of periodicity for Clifford algebras. This "algebraic" periodicity will be effectively used in III.5 to prove the "topological" periodicity of real K-theory and at the same time give another proof of the periodicity of complex K-theory.

4. *The functors $K^{p,q}(\mathscr{C})$ and $K^{p,q}(X)$.* The idea of this section is to use the Clifford algebras $C^{p,q}$ to algebraicly define new functors $K^n(X)=K^{p,q}(X)$ for $n=p-q\in\mathbb{Z}$. We prove that these functors are *by definition* periodic, of period 8 in the real case, and of period 2 in the complex case, and that $K^0(X)$ and $K^{-1}(X)$ are indeed the functors defined in Chapter II. Bott periodicity will then be proved if we show that the two definitions of $K^n(X)$ agree for negative values of n. This is done in the next two sections.

5. *The functors $K^{p,q}(X, Y)$ and the isomorphism t. Periodicity in real K-theory.* After some preliminaries introducing the relative groups $K^{p,q}(X, Y)$ we present the fundamental theorem of this chapter: The groups $K^{p,q+1}(X, Y)$ and $K^{p,q}(X\times B^1, X\times S^0\cup Y\times B^1)$ are isomorphic. Assuming this theorem (the proof follows in Section III.6), we prove that $K_{\mathbb{R}}^{-n}(X, Y)\approx K_{\mathbb{R}}^{-n-8}(X, Y)$ with the definitions of Chapter II. At the same time we prove the periodicity in complex K-theory (5.17) once more. Moreover, using Propositions 4.29 and 4.30 we prove the existence of weak homotopy equivalences between the iterated loop spaces $\Omega^r(0)$ and certain homogeneous spaces (5.22). We also compute the homotopy groups $\pi_i(0(n))$ for $n>i+1$ (5.19) with the help of Clifford algebras.

6. *Proof of the fundamental theorem.* The pattern of this section is analogous to that of Section III.1, since the main theorem is likewise a consequence of a general theorem on Banach algebras (6.12). Moreover the proof of this general theorem uses the same ideas as the proof of Theorem 1.11.

Chapter IV. Computations of Some K-Groups

1. *The Thom isomorphism in complex K-theory for complex vector bundles.* The purpose of this section is to compute the complex K-theory of the Thom space of a complex vector bundle (1.9). In this computation a key role is played by bundles of exterior algebras. Theorem 1.3. is particularly important in the sequel.

2. *Complex K-theory of complex projective spaces and complex projective bundles.* In this section (classical in style), we construct a method which may also be used for ordinary cohomology (see V.3). Using the technical Proposition 2.4 we are able to compute the K-theory of $P_n=P(\mathbb{C}^{n+1})$ and more generally of $P(V)$ where V is a complex vector bundle (2.13). The "splitting principle" (2.15) is used frequently later on. With this principle we are able to make the multiplicative structure of $K^*(P(V))$ explicit (2.16).

3. *Complex K-theory of flag bundles and Grassmann bundles. K-theory of a product.* This section is also classical in style, but is not essential to the sequel. We explicitly compute $K^*(F(V))$ where $F(V)$ is the flag bundle of a complex vector bundle V. We also compute $K^*(G_p(V))$ where $G_p(V)$ is the fiber bundle of p-subspaces in V (3.12). These results are used to compute $\mathscr{K}(BU(n)) = \text{proj lim } K(G_p(\mathbb{C}^n)$ (3.22), and the K-theory of a product (3.27).

4. *Complements in Clifford algebras.* The concept of "spinors" was not introduced in Section III.3, since it is not essential in proving Bott periodicity. However we now need this concept to prove the analog of Thom's theorem in K-theory (for real or complex vector bundles). After some algebraic preliminaries we study the possibilities of lifting the structural group of a real vector bundle to the spinorial group Spin(n) or Spin^c(n). Theorem 4.22 is particularly important for our purpose.

5. *The Thom isomorphism in real and complex K-theory for real vector bundles.* As in IV.1, the purpose of this section is to compute the K-theory of the Thom space of a vector bundle, but now the vector bundle is real, and the K-theory used is real or complex. With an additional spinorial hypothesis, we prove that $K(V) \approx K^{-n}(X)$ if n is the rank of V. If the base is compact and n is a multiple of 8 (of 2 in complex K-theory), we prove that $K(V)$ is a $K(X)$-module of rank one generated by the "Thom class" T_V. Finally, if $f\colon X \to Y$ is a proper continuous map between differentiable manifolds and if $\text{Dim}(Y) - \text{Dim}(X) = 0 \bmod 8$ (mod 2 in the complex case), we define, with an additional spinorial hypothesis, a "Gysin homomorphism" $f_*\colon K(X) \to K(Y)$ which is analogous to the Gysin homomorphism in ordinary cohomology. This homomorphism is only used in V.4.

6. *Real and complex K-theory of real projective spaces and real projective bundles.* This section is much more technical than the others (the results are only used in V.2). After some easy but tedious lemmas making systematic use of Clifford algebras, we are able to compute (up to extension) the real and complex K-theory of a real projective bundle (6.40 and 6.42). In the case of real projective spaces, the K-theory is completely determined (6.46 and 6.47).

7. *Operations in K-theory.* One of the charms of K-theory is that we are able to define some very nice operations. For example, there are the exterior power operations λ^k (due to Grothendieck). By a method due to Atiyah we determine all the operations in complex K-theory. With this method we show that the "Adams operations" ψ^k are the only ring operations in complex K-theory (7.13). They will be very useful in applications.

The operations λ^k and ψ^k may also be defined in real K-theory. However, their properties are more difficult to prove. We must refer to Adams [3] or Exercise 8.5 for a complete proof. From the operations ψ^k, we obtain the operations ρ^k, which will be very useful in V.2 and V.5.

Chapter V. Some Applications of K-Theory

1. *H-space structures on spheres and the Hopf invariant.* Using the Adams operations in complex K-theory, we prove that the only spheres which admit an H-space structure are S^1, S^3, and S^7. In fact, we prove more: if $f: S^{2n-1} \to S^n$ is a map of odd Hopf invariant, then n must be 2, 4 or 8.

2. *The solution of the vector field problem on the sphere.* Let us write every integer t in the form $(2\alpha-1)\cdot 2^\beta$, for $\beta=\gamma+4\delta$ with $0\leqslant\gamma\leqslant 3$, and define $\rho(t)=2^\gamma+8\delta$. Then the maximum number of independent vector fields on the sphere S^{t-1} is exactly $\rho(t)-1$ (2.10). The proof of this classical theorem is "elementary" (in the context of this book) and uses essentially the operations ρ^k in the real K-theory of real projective spaces.

3. *Characteristic classes and the Chern character.* For each complex vector bundle V, we define "Chern classes" $c_i(V) \in H^{2i}(X; \mathbb{Z})$ in an axiomatic way (3.15). The construction of these classes is analogous to the construction of classes done in Section IV.3. By means of these classes, we construct a fundamental homomorphism, the "Chern character", from $K_{\mathbb{C}}(X)$ to $H^{\text{even}}(X; \mathbb{Q})$. The Chern character induces an isomorphism between $K_{\mathbb{C}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $H^{\text{even}}(X; \mathbb{Q})$ for every compact X.

4. *The Riemann–Roch theorem and integrality theorems.* To each complex stable vector bundle (resp. ^{c}spinorial real stable bundle) we associate an important characteristic class $\tau(V)$, called the Todd class (resp. $A(V)$, called the Atiyah–Hirzebruch class). These classes play an important role in the "differentiable Riemann–Roch theorem": For each suitably continuous map $f: X \to Y$ and for each element x of $K_{\mathbb{C}}(X)$, we have the formula $ch(f_*^K(x)) = f_*^H(A(v_f)\cdot ch(x))$ where $A(v_f)$ denotes the Atiyah–Hirzebruch class of the stable bundle $f^*(TY)-TX$ (assuming that $\mathrm{Dim}(Y)=\mathrm{Dim}(X)$ mod 2 and that there is a stable ^{c}spinorial structure on v_f). From this theorem we obtain integral theorems for characteristic classes (4.21) and the homotopy invariance of certain characteristic classes (4.24).

5. *Applications of K-theory to stable homotopy.* In this section we explain how K-theory may be applied to obtain some interesting information about the stable homotopy groups of spheres. We only include those partial results which can be obtained from the material in this book. More complete results are found in the series of J. F. Adams on the groups $J(X)$ [2], and in Husemoller's book [1].

Chapter I

Vector Bundles

1. Quasi-Vector Bundles

1.1. Let k be the field of real numbers or complex numbers[1], and let X be a topological space.

1.2. Definition. A *quasi-vector bundle with base X* is given by

1) a finite dimensional k-vector space E_x for every point x of X,

2) a topology on the disjoint union $E = \bigsqcup E_x$ which induces the natural topology on each E_x, such that the obvious projection $\pi: E \to X$ is continuous.

1.3. Example. Let X be the sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. For every point x of S^n we choose E_x to be the vector space orthogonal to x. Then $E = \bigsqcup E_x$ is naturally a subspace of $S^n \times \mathbb{R}^{n+1}$ and may be provided with the induced topology.

1.4. Example. Starting from the preceding example, let us arbitrarily choose a vector space $F_x \subset E_x$ for each $x \in S^n$; then if F is given the induced topology again we have a quasi-vector bundle on X.

More examples are given in the following sections.

1.5. A quasi-vector bundle is denoted by $\xi = (E, \pi, X)$ or simply by E if there is no risk of confusion. The space E is the *total space* of ξ and E_x is the *fiber* of ξ at the point x.

1.6. Let $\xi = (E, \pi, X)$ and $\xi' = (E', \pi', X')$ be quasi-vector bundles. A *general morphism* from ξ to ξ' is given by a pair (f, g) of continuous maps $f: X \to X'$ and $g: E \to E'$ such that

1) the diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{f} & X' \end{array}$$

is commutative.

---

[1] In general, these are the most interesting cases; however, sometimes we will use the field of quaternions $\mathbb{H}$.

2) The map $g_x\colon E_x \to E'_{f(x)}$ induced by g is k-linear.

General morphisms can be composed in an obvious way. In this way we construct a category whose objects are quasi-vector bundles and whose arrows are general morphisms.

1.7. If ξ and ξ' have the same base $X=X'$, a *morphism* between ξ and ξ' is a general morphism (f, g) such that $f=\mathrm{Id}_X$. Such a morphism will be simply called g in the sequel. The quasi-vector bundles with the same base X are the objects of a subcategory, whose arrows are the morphisms we have just defined.

1.8. Example. Let us return to Example 1.3, and let $n=1$. Let $\xi'=(E', \pi', X')$ where $X=X'=S^1$, and $E'=S^1\times\mathbb{R}$ with the product topology. If we identify $\mathbb{R}^2$ with the complex numbers as usual, we can define a continuous map $g\colon E\to E'$ by the formula $g(x, z)=(x, iz/x)$ (this is well defined because x is orthogonal to z in $\mathbb{R}^2=\mathbb{C}$). In fact g is an isomorphism between E and E' in the category described in 1.7.

1.9. Example. Let E'' be the quotient of $E'=S^1\times\mathbb{R}$ by the equivalence relation $(x, t)\sim(y, u)$ if $y=\varepsilon x$ and $u=\varepsilon t$ with $\varepsilon=\pm 1$. Then E'' is the total space of a quasi-vector bundle over $P_1(\mathbb{R})$ called the *infinite Moebius band.* By identifying $P_1(\mathbb{R})$ with S^1 by the map $z\mapsto z^2$, we see easily that E'' is also the quotient of $I\times\mathbb{R}$ by the equivalence relation which identifies $(0, u)$ with $(1, -u)$. If we restrict u to have norm less than 1, we obtain the classical Moebius band.

We claim that the bundles E' and E'' over S^1 are not isomorphic. Suppose there exists an isomorphism $g\colon E'\to E''$; then we must have $E'-X'$ homeomorphic to $E''-X''$ where X' (and X'') denote the set of points of the form $(x, 0)$ with $x\in S^1$ (note that $X''\approx X'$). But $E''-X''$ is connected and $E'-X'$ is not.

1.10. Let V be a finite dimensional vector space (as always over k). The preceding examples show the importance of quasi-vector bundles of the form $E=X\times V$, as models. To be more precise, $E_x=V$ and the total space may be identified with $X\times V$ provided with the product topology. Such bundles are called trivial quasi-vector bundles or simply *trivial vector bundles.*

1.11. Let $E=X\times V$ and $E'=X\times V'$ be trivial vector bundles with base X. We want to explicitly describe the morphisms from E to E' (again in the category defined in 1.7). Since the diagram

$$\begin{array}{ccc} X\times V & \longrightarrow & X\times V' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

is commutative, for each point x of X, g induces a linear map $g_x\colon V\to V'$. Let $\check{g}\colon X\to \mathscr{L}(V, V')$ be the map defined by $\check{g}(x)=g_x$.

1.12. Theorem. *The map $\check{g}: X \to \mathscr{L}(V, V')$ is continuous relative to the natural topology of $\mathscr{L}(V, V')$. Conversely, let $h: X \to \mathscr{L}(V, V')$ be a continuous map, and let $\hat{h}: E \to E'$ be the map which induces $h(x)$ on each fiber. Then $\hat{h}$ is a morphism of quasi-vector bundles.*

Proof. To prove this theorem we choose a basis $e_1, \ldots, e_n$ of V and a basis $\varepsilon_1, \ldots, \varepsilon_p$ of V'. With respect to this basis, g_x may be regarded as the matrix $(\alpha_{ij}(x))$ where $\alpha_{ij}(x)$ is the i^{th} coordinate of the vector $g_x(e_j)$. Hence the function $x \mapsto \alpha_{ij}(x)$ is obtained from the composition of the following continuous maps.

$$X \xrightarrow{\beta_j} X \times V \xrightarrow{g} X \times V' \xrightarrow{\gamma} V' \xrightarrow{p_i} k,$$

where $\beta_j(x) = (x, e_j)$, $\gamma(x, v') = v'$, and p_i is the i^{th} projection of $V' \cong k^p$ on k. Since the functions $\alpha_{ij}(x)$ are continuous, the map $\check{g}$ which they induce is also continuous according to the definition of the topology of $\mathscr{L}(V, V')$.

Conversely, let $h: X \to \mathscr{L}(V, V')$ be a continuous map. Then $\hat{h}$ is obtained from the composition of the continuous maps

$$X \times V \xrightarrow{\delta} X \times \mathscr{L}(V, V') \times V \xrightarrow{\varepsilon} X \times V',$$

where $\delta(x, v) = (x, h(x), v)$ and $\varepsilon(x, u, v) = (x, u(v))$. Hence $\hat{h}$ is continuous and defines a morphism of quasi-vector bundles. □

1.13. *Remark.* Clearly we have the identities $\hat{\check{g}} = g$ and $\check{\hat{h}} = h$.
∗ The reader may also note that the second part of the theorem can be generalized to Banach bundles (see Lang [2]), but not the first part.∗

1.14. *Remark.* As we have seen in Example 1.9, it is not obvious whether or not a given quasi-vector bundle is isomorphic to a trivial bundle. Let TS^n denote the quasi-vector bundle considered in 1.3 (this is the "tangent bundle" of the sphere). Then it is only at the end of this book that we are able to show that TS^n is *not* isomorphic to a trivial bundle unless $n = 1$, 3, or 7 (cf. Section V.2).

1.15. Let $\xi = (E, \pi, X)$ be a quasi-vector bundle, and let X' be a subspace of X. The triple $(\pi^{-1}(X'), \pi|_{\pi^{-1}(X')}, X')$ defines a quasi-vector bundle ξ' which is called the *restriction* of ξ to X'. We denote it by $\xi|_{X'}$, $E|_{X'}$, or even simply $E_{X'}$. The fibers of ξ' are just the fibers of ξ over the subspace X'. If $X'' \subset X' \subset X$, we have $(\xi|_{X'})|_{X''} = \xi|_{X''}$.

1.16. More generally, let $f: X' \to X$ be any continuous map (X' is not necessarily a subspace of X). For every point x' of X', let $E'_{x'} = E_{f(x')}$. Then the set $E' = \coprod_{x' \in X'} E'_{x'}$ may be identified with the *fiber product* $X' \times_X E$, i.e. with the subset of $X' \times E$ formed by the pairs (x', e) such that $f(x') = \pi(e)$. If $\pi': E' \to X'$ is defined by $\pi'(x', e) = x'$, it is clear that the triple (E', π', X') defines a quasi-vector bundle over X', when we provide E' with the topology induced by $X' \times E$. We write ξ' as

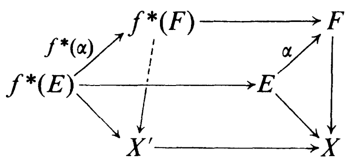
$f^*(\xi)$ or $f^*(E)$: this is the *inverse image* of ξ by f. We have $f^*(\xi)=\xi$ for $f=\mathrm{Id}_X$, and also $(f\cdot f')^*(\xi)=f'^*(f^*(\xi))$ if $f'\colon X''\to X'$ is another continuous map. If $X'\subset X$ and f is the inclusion map, then $f^*(\xi)=\xi|_{X'}$.

1.17. Let $(f,g)\colon E_1'\to E$ be a general morphism of quasi-vector bundles with $f\colon X'\to X$ (1.6). This general morphism induces a morphism $h_1\colon E_1'\to E'=f^*(E)$ as shown in the diagram

$$\begin{array}{ccccc} E_1' & \xrightarrow{h_1} & E'\approx X'\times_X E & \xrightarrow{h} & E \\ \downarrow{\scriptstyle \pi_1'} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\mathrm{Id}_{X'}} & X' & \xrightarrow{\quad f \quad} & X, \end{array}$$

where h is induced by the projection of $X'\times E$ on its second factor. The general morphism (f,g) is called *strict* if h_1 is an isomorphism.

1.18. Let us now consider two quasi-vector bundles over X and a morphism $\alpha\colon E\to F$. If we let $E'=f^*(E)$ as in 1.16 and $F'=f^*(F)$, we can also define a morphism $\alpha'=f^*(\alpha)$ from E' to F' by the formula $\alpha'_{x'}=\alpha_{f(x')}$. If we identify E' with $X'\times_X E$ and F' with $X'\times_X F$, then α' is identified with $\mathrm{Id}_{X'}\times_X\alpha$, which proves the continuity of the map α'.



In particular, if $X'\subset X$ and if f is the inclusion map, then $f^*(\alpha)$ is the restriction of α. We denote it by $\alpha|_{X'}$ or simply $\alpha_{X'}$. The proof of the next proposition is easy and is left as an exercise for the reader:

1.19. Proposition. *Let $f\colon X'\to X$ be a continuous map. Then the correspondence $E\mapsto f^*(E)$ and $\alpha\mapsto f^*(\alpha)$ induces a functor between the category of quasi-vector bundles over X and the category of quasi-vector bundles over X'.*

Exercises (Section I.9) 1–4 and 6.

2. Vector Bundles

A vector bundle is a quasi-vector bundle which is locally isomorphic to a trivial vector bundle. The next definition will make this idea more precise.

2.1. Definition. Let $\xi=(E, \pi, X)$ be a quasi-vector bundle. Then ξ is said to be "*locally trivial*" or a "*vector bundle*" if for every point x in X, there exists a neighbourhood U of x such that $\xi|_U$ is isomorphic to a trivial bundle.

2.2. The last condition may be expressed in the following way: there exists a finite dimensional vector space V and a *homeomorphism* $\varphi\colon U\times V\to \pi^{-1}(U)$ such that the diagram

$$\begin{array}{ccc} U\times V & \xrightarrow{\varphi} & \pi^{-1}(U) \\ & {\scriptstyle \mathrm{pr}_1}\searrow \quad \swarrow{\scriptstyle \pi|_{\pi^{-1}(U)}} & \\ & U & \end{array}$$

commutes, and such that for every point y in U, the map $\varphi_y\colon V\to E_y$ is k-linear. We call U a *trivialization domain* of the vector bundle ξ. A cover (U_i) of X is called a *trivialization cover* if each U_i is a trivialization domain.

Of course, there exist quasi-vector bundles which are not locally trivial (1.4).

2.3. Example. Let us prove that Example 1.3, where $E=TS^n$, is in fact a vector bundle. Let $x\in S^n$ and let U be the neighbourhood of x defined by $U=\{y\in S^n \mid \langle y, x\rangle \neq 0$

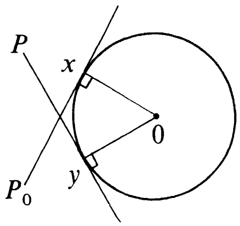


Fig. 1

where $\langle\ ,\ \rangle$ denotes the usual scalar product in $\mathbb{R}^{n+1}$. Let P_0 be the subspace of $\mathbb{R}^{n+1}$ which is orthogonal to x, and let $\varphi\colon TS^n|_U\to U\times P_0$ be the map taking the pair (y, v) to the pair (y, w), where w is the orthogonal projection of v on P_0. Explicitly $w=v-\langle x, v\rangle x$. Conversely, v may be obtained from w by the formula $v=w-\dfrac{\langle y, w\rangle}{\langle x, y\rangle}x$, showing that φ is a homeomorphism, and hence that TS^n is locally trivial.

2.4. Example. Let V be a finite dimensional vector space over k, and let $P(V)$ be the associated projective space (provided with the quotient topology). The subspace E of $P(V)\times V$ which consists of pairs (D, e) where $D\in P(V)$ and $e\in D$, is fibered over $P(V)$ by the first projection. More precisely, the fiber E_D, where $D\in P(V)$, is the one-dimensional vector space whose elements are the vectors e such that $e\in D$. We prove now that E is actually a vector bundle. If we provide V with a positive Hermitian form when $k=\mathbb{C}$, or a positive quadratic form when

$k = \mathbb{R}$, for each line D we can consider the neighbourhood U_D which consists of the lines Δ which are not orthogonal to D. Now a trivialization of $E|_{U_D}$ is given by the map $\varphi: E|_{U_D} \to U_D \times D$ defined by $\varphi(\Delta, e) = (\Delta, e')$, where e' is the orthogonal projection of e on D. By exhibiting explicit formulas for these projections as in 2.3, one shows that φ is a homeomorphism. This bundle E is called the *canonical line bundle* over $P(V)$.

2.5. There are other ways to deal with Example 2.4. For the real case it is well known that $P(V) \sim S^n/\mathbb{Z}_2$, where the dimension of $V = n+1$ (explicitly $P(V)$ is the quotient of S^n by the equivalence relation $x \sim \pm x$). Let F be the quotient of $S^n \times \mathbb{R}$ by the equivalence relation $(x, t) \sim (x', t') \Leftrightarrow (x', t') = (\varepsilon x, \varepsilon t)$ where $\varepsilon = \pm 1$. Then F is a quasi-vector bundle over $P(V)$, and thus we can define a morphism $f: F \to E$ by the formula $f(x, t) = (\pi(x), tx)$ where $\pi: S^n \to P(V)$ is the natural projection, and $tx \in \pi(x)$. One can also define a morphism $g: E \to F$ by the formula $g(D, v) = (x, t)$ where $x \in D \cap S^n$ and t is the scalar such that $tx = v$. (Of course in these formulas (x, t) represents the class of the pair (x, t) in $S^n \times \mathbb{R}/\sim$.) Then f and g are isomorphisms, with $f = g^{-1}$.

In the complex case, $P(V) \approx S^{2n+1}/U$ where the dimension of $V = n+1$, and where U is the group of complex numbers of norm 1 (explicitly $P(V)$ is the quotient of S^{2n+1} by the equivalence relation $x \sim \lambda x$ if $|\lambda| = 1$). The vector bundle E may be identified in a similar fashion with the quotient of $S^{2n+1} \times \mathbb{C}$ by the equivalence relation $(x, t) \sim (x', t') \Leftrightarrow (x', t') = (\varepsilon x, \bar{\varepsilon} t)$ for $\varepsilon \in U$.

2.6. Now for some terminology. When $k = \mathbb{R}$ (resp. $k = \mathbb{C}$) a vector bundle will be called *real* (resp. *complex*). By abuse of our definitions, a trivial vector bundle will mean a vector bundle which is *isomorphic* to a bundle $E = X \times V$ as defined in 1.10. Vector bundles are in fact the objects of a full subcategory of the category of quasi-vector bundles considered in 1.7. We will denote this category by $\mathscr{E}(X)$, or by $\mathscr{E}_k(X)$ when we want to make the basic field k explicit. If $f: X' \to X$ is a continuous map, the functor f^* defined in 1.19 induces a functor from $\mathscr{E}(X)$ to $\mathscr{E}(X')$. To see this it suffices to show that $f^*(\xi)$ is locally trivial whenever ξ is locally trivial over X. Let $x' \in X'$ and let U be a neighbourhood of $f(x')$ such that $\eta = \xi|_U$ is trivial. Then $\xi'|_{U'} = g^*(\eta)$ where $U' = f^{-1}(U)$ and $g: U' \to U$ is the map induced by f. Hence we have $\eta \approx U \times V$ and $g^*(\eta) \approx U' \times_U (U \times V) \approx U' \times V$ which is trivial over U'. In particular, if X' is a subspace of X, then $\xi|_{X'}$ is a vector bundle.

2.7. Proposition. *Let E and F be two vector bundles over X and let $g: E \to F$ be a morphism of vector bundles such that $g_x: E_x \to F_x$ is bijective for each point x in X. Then g is an isomorphism in the category $\mathscr{E}(X)$.*

Proof. Let $h: F \to E$ be the map defined by $h(v) = g_x^{-1}(v)$ for $v \in F_x$. It suffices to prove that h is continuous. Consider a neighbourhood U of x and isomorphisms $\beta: E_U \to U \times M$ and $\gamma: F_U \to U \times N$. If we let $g_1 = \gamma \cdot g_U \cdot \beta^{-1}$ we have $h_U = \beta^{-1} \cdot h_1 \cdot \gamma$ where h_1 is defined by $\check{h}_1(x) = (\check{g}_1(x))^{-1}$ (cf. 1.12). Since the map from $\mathrm{Iso}(M, N)$ to $\mathrm{Iso}(N, M)$ defined by $\alpha \mapsto \alpha^{-1}$ is continuous, h_1 is continuous. Thus

h is continuous on a neighbourhood of each point of F; hence h is continuous on all of F. □

2.8. Let $\xi=(E, \pi, X)$ be a vector bundle. We define two maps (where $E\times_X E$ is the fiber product)

$$s: E\times_X E \to E \quad \text{and} \quad p: k\times E \to E$$

by the formulas $s(e, e')=e+e'$ and $p(\lambda, e)=\lambda e$, where e and e' are vectors of the same fiber. These maps are continuous. To see this, it is enough to consider the case where $E=X\times V$, since continuity is a local condition as before. In this case, $E\times_X E\approx X\times V\times V$ and under this isomorphism s becomes the map from $X\times V\times V$ to $X\times V$ defined by $(x, v, v')\mapsto(x, v+v')$ which is clearly continuous. The continuity of p is proved in the same way.

2.9. We define the *rank* of a vector bundle $\xi=(E, \pi, X)$ to be the locally constant function $r: X\to \mathbb{N}$ given by $r(x)=\mathrm{Dim}(E_x)$. The rank of ξ is equal to an integer n if $r(x)=n$ for each point x of X. When the base is connected the rank is constant.

Exercise (I.9.5).

3. Clutching Theorems

In the preceding section we defined vector bundles as locally trivial quasi-vector bundles. Now we would like to construct vector bundles using their restrictions to suitable subsets.

3.1. Theorem ("clutching of morphisms"). *Let $\xi=(E, \pi, X)$ and $\xi'=(E', \pi', X)$ be two vector bundles on the same base X. Let us consider also*

a) *A cover of X consisting of open subsets U_i (resp. a locally finite cover of X of closed subsets U_i).*

b) *A collection of morphisms $\alpha_i: \xi|_{U_i}\to \xi'|_{U_i}$ such that $\alpha_i|_{U_i\cap U_j}=\alpha_j|_{U_i\cap U_j}$.*

Then there exists a unique morphism $\alpha: \xi\to\xi'$ such that $\alpha_{U_i}=\alpha_i$.

Proof. The proof naturally breaks into two parts:

(i) *Uniqueness.* Let e be a point of E. Since (U_i) is a cover of X, the point e belongs to some E_{U_i}. Hence $\alpha(e)=r_i'(\alpha_i(e))$ where $r_i': E'_{U_i}\to E'$ is the inclusion map.

(ii) *Existence.* To simplify the notation, let us identify E_{U_i} and E'_{U_i} with subsets of E and E' respectively. For $e\in E$, let $\alpha(e)=\alpha_i(e)$ for $e\in E_{U_i}$. It follows from b) that this definition is independent of the choice of i. The subsets $E_{U_i}=\pi^{-1}(U_i)$ form an open cover (resp. a locally finite cover of closed subsets) of the space E; hence α is continuous. Since $\alpha_x: E_x\to E_x'$ is linear, the map α defines a morphism of vector bundles. □

3.2. Theorem ("clutching of bundles"). *Let (U_i) be an open cover of a space X (resp. a locally finite closed cover of a paracompact space X). Let $\xi_i=(E_i, \pi_i, U_i)$ be a vector bundle over each U_i, and let $g_{ji}: \xi_i|_{U_i\cap U_j} \to \xi_j|_{U_i\cap U_j}$ be isomorphisms which satisfy the compatibility condition $g_{ki}|_{U_i\cap U_j\cap U_k}=g'_{kj}\cdot g'_{ji}$, where $g'_{kj}=g_{kj}|_{U_i\cap U_j\cap U_k}$ and $g'_{ji}=g_{ji}|_{U_i\cap U_j\cap U_k}$. Then there exists a vector bundle ξ over X and isomorphisms $g_i: \xi_i \to \xi|_{U_i}$ such that the diagram*

$$\begin{array}{ccc} \xi_i|_{U_i\cap U_j} & \xrightarrow{g_{ji}} & \xi_j|_{U_i\cap U_j} \\ {\scriptstyle g_i|_{U_i\cap U_j}}\searrow & & \swarrow{\scriptstyle g_j|_{U_i\cap U_j}} \\ & \xi|_{U_i\cap U_j} & \end{array} \qquad \textit{(Diagram 1)}$$

is commutative.

Proof. For simplicity we use the same letter to denote a morphism and its restriction to a subspace. In the topologically disjoint union $\bigsqcup E_i$, consider the equivalence relation $e_i \sim e_j \Leftrightarrow g_{ji}(e_i)=e_j$, and let $E=\bigsqcup E_i/\sim$ be given the quotient topology. The continuous map $\bigsqcup E_i \to X$ induced by the π_i defines a continuous map $\pi: E \to X$. For $x \in U_i$, the structure of the vector space $E_x=\pi^{-1}(\{x\})$, which is induced by the isomorphism $E_x \approx E_i|_{\{x\}}$, does not depend on the choice of i since g_{ji} is linear on each fiber. Let $g_i: E_i \to \pi^{-1}(U_i)$ be the map defined by $g_i(e)=\bar{e}$, where $\bar{e}$ is the class of e in E. Then g_i is continuous, bijective, open, and induces a linear isomorphism on each fiber. Therefore g_i defines an isomorphism between the quasi-vector bundles (E_i, π_i, U_i) and $(E_{U_i}, \pi|_{E_{U_i}}, U_i)$, where $E_{U_i}=\pi^{-1}(U_i)$.

Suppose that (U_i) is an *open* cover of X. Let x be a point of U_i, and let V be a neighbourhood of x contained in U_i such that $\xi_i|_V$ is trivial. If ξ is the quasi-vector bundle (E, π, X) as defined above, we have $\xi_{U_i} \approx \xi_i$. Hence $\xi|_V \approx \xi_i|_V$ is trivial, which proves that ξ is locally trivial.

Let us assume now that X is paracompact and that (U_i) is a closed cover which is locally finite. Let x_0 be a point of X. Since the cover (U_i) is locally finite, there exists a closed neighbourhood V of x_0 which meets only a finite number of subsets $U_{i_1}, \ldots, U_{i_p}$, and such that the bundles $\xi_j|_{V_j}$ are trivial, where $V_j=U_{i_j}\cap V$ for $j=1, \ldots, p$. Without loss of generality we may assume that $x_0 \in V_1$ and that $\xi_j|_{V_j} \approx V_j \times k^n$. Starting with an arbitrary isomorphism $\alpha_1: \xi|_{V_1} \approx V_1 \times k^n$ we are going to define by induction on r, a morphism α_r between $\xi|_{V_1\cup\cdots\cup V_r}$ and the trivial bundle $(V_1 \cup \cdots \cup V_r) \times k^n$. Since $\xi|_{V_r}$ is trivial, this is equivalent to defining a continuous map $\beta_r: V_r \to \mathscr{L}(k^n, k^n)$ which extends $\tilde{\gamma}_r$ with $\gamma_r=\alpha_{r-1}|_{V_r\cap(V_1\cup\cdots\cup V_{r-1})}$. This extension is possible due to the Tietze extension theorem (Kelley [1], Bourbaki [1]). Let $\alpha: \xi|_V \to V \times k^n$ be the morphism thus obtained. Since $\mathrm{Iso}(k^n, k^n)$ is open in $\mathscr{L}(k^n, k^n)$, Theorem 1.12 shows that the set of points x of V_s, such that α_x is an isomorphism, is an open subset of V_s. Since the sets V_s are finite in number, the set of points x of V such that α_x is an isomorphism is a neighbourhood W of x_0. The map $\alpha_W: E|_W \to W \times k^n$ induces a homeomorphism $E|_{V_s\cap W} \to (V_s \cap W) \times k^n$ for each s. Hence α_W is a homeomorphism itself. Since this holds for every point x_0 in X, we see that ξ is locally trivial in this case also. □

3.3. *Remark.* Moreover one may say that the bundle ξ which we just constructed is "unique" in the following sense. Let ξ' be another vector bundle, and let $g'_i\colon \xi_i \to \xi'|_{U_i}$ be isomorphisms which make the diagram

$$\begin{array}{ccc} \xi_i|_{U_i\cap U_j} & \xrightarrow{\ g_{ji}\ } & \xi_j|_{U_i\cap U_j} \\ & {\scriptstyle g'_i|_{U_i\cap U_j}}\searrow \qquad \swarrow{\scriptstyle g'_j|_{U_i\cap U_j}} & \\ & \xi'|_{U_i\cap U_j} & \end{array} \qquad (\textit{Diagram 2})$$

commutative. Then there exists a unique isomorphism $\alpha\colon \xi \to \xi'$ which makes the following diagram commutative.

$$\begin{array}{ccc} & \xi_i & \\ & {\scriptstyle g_i}\swarrow \qquad \searrow{\scriptstyle g'_i} & \\ \xi|_{U_i} & \xrightarrow{\ \alpha|_{U_i}\ } & \xi'|_{U_i} \end{array}$$

In fact, one may construct α in the following way. The morphism $\alpha_i = g'_i \cdot g_i^{-1}$ is an isomorphism from ξ_{U_i} to ξ'_{U_i}, and over $U_i \cap U_j$, we have the identity $g_{ji} = g_j^{-1} \cdot g_i = g_j'^{-1} \cdot g'_i$ according to diagrams (1) and (2). Therefore, over $U_i \cap U_j$ we have $\alpha_i = g'_i \cdot g_i^{-1} = g'_j \cdot g_j^{-1} = \alpha_j$. The existence of α is then guaranteed by Theorem 3.1. Its uniqueness is obvious.

3.4. Example. Let S^n be the sphere of $\mathbb{R}^{n+1}$, i.e. the set of points $x = (x_1, \ldots, x_{n+1})$ such that $\|x\|^2 = \sum_{i=1}^{n+1} (x_i)^2 = 1$. Let S^n_+ (resp. S^n_-) be the subset of S^n whose points x satisfy $x_{n+1} \geqslant 0$ (resp. $x_{n+1} \leqslant 0$). Then S^n is compact, S^n_+ and S^n_- are closed subsets, and $S^n_+ \cap S^n_- = S^{n-1}$.

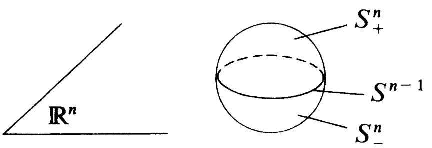


Fig. 2

Let $f\colon S^{n-1} \to \mathrm{GL}_p(k)$ be a continuous map. According to Theorem 3.2, there is a bundle E_f over S^n which is naturally associated with f. It is obtained from the clutching of the trivial bundles $E_1 = S^n_+ \times k^p$ and $E_2 = S^n_- \times k^p$ by the "transition function" $g_{21} = \hat{f}\colon S^{n-1} \times k^p \to S^{n-1} \times k^p$ (g_{11} and g_{22} are the identity map). We see later (7.6) that all bundles over S^n are isomorphic to bundles of this type.

3.5. Theorem 3.2 is related to the problem of classification of "*G-principal bundles*", where G is the topological group $\mathrm{GL}_n(k)$. To be more precise, let us consider an arbitrary topological group G and a topological space X. A *G-cocycle*