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Problem Solving in Mathematics Instruction and Teacher Professional Development

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Research in Mathematics Education

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Problem Solving in Mathematics Instruction and Teacher Professional Development

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Introduction

Problem solving is of fundamental importance in mathematics knowledge construction from the very beginning of our human history. As such, it is widely agreed that problem solving should be a fundamental activity in mathematics classrooms. The ideas around this statement were put into the educational discussion more than 70 years ago by George Pólya with the publication of his famous book *How to Solve It* (1945). Since then, an increasing number of researchers have been developing theories and programs and setting up experiments for the introduction of problem solving in classrooms as part of regular mathematics activity. In parallel, many countries started to include problem solving in their national curricula, in many cases, putting it at the center of the national mathematics education agenda. Despite these enormous efforts and achievements, there is still a long way to go before problem solving would be considered as a regular activity in a regular classroom – so that school children would study mathematics through practicing problem solving. This book provides a contribution with a variety of approaches to move forward in this direction.

The origin of this book is a conference held at the end of 2017 in the Chilean city of Punta Arenas, located in Patagonia. The city, together with the University of Magallanes, offered the participants a wonderful academic environment in which to share their ideas. The magnificent natural scenery of the surrounded areas and the rich history of the city and of the region were perfect complements for the week-long academic gathering. Since its discovery by Portuguese explorer Ferdinand of Magellan in 1520, until the construction of the Panama Canal, the Magellan Canal was for many centuries the only route connecting Atlantic and Pacific Ocean. Settled on the banks of the Canal, Punta Arenas has become a well-known city for sailor of many nations. Before the city started to grow with the settlement of immigrants from many countries, the region had been inhabited by many indigenous tribes of fishermen. These tribes were eventually exterminated by the outrageous commercial greed of the new settlers. Punta Arenas is also known as a city where Charles Darwin, during his famous trip on board of Beagle, began to develop his ideas about evolution. The natural surroundings of Punta Arenas offer vast sea and tundra, impressive mountains, strong wind, and quickly changing weather, together with an exotic fauna and flora, whose most popular representatives are penguins, condors, upland gooses, pumas, south Andean deer, and whales, among the animals, and box-leaf barberry, southern beech, fuchsia, and tuft grasses, among the plants.

The conference *Problem Solving in Patagonia* took place between 27 November and 1 December 2017, which is early summer in Patagonia. The conference was devoted to the recent advances in research in mathematics education with special focus on mathematical problem solving. The conference was organized in the context of a Chilean project to enhance collaboration in mathematics education between Chile and Canada, specifically Simon Fraser University. With additional support from the Center for Advanced Research in Education (CIAE) and the Center for Mathematical Modeling (CMM) from the University of Chile and with the local support of the University of Magallanes, 25 researchers gathered for a week in Punta Arenas. We were fortunate to have the participation of many renowned researchers in the mathematics education area as well as a number of young researchers. Alphabetically, Miriam Amit, Sergio Celis, Eugenio Chandía, Lisa Darragh, Danyal Farsani, Patricio Felmer, Frédéric Gourdeau, Patricio Herbst, Gabriele Kaiser, Boris Koichu, Richard Lagos, Roza Leikin, Peter Liljedahl, John Mason, Vilma Mesa, Carmen Oval, Cristián Reyes, Annette Rouleau, Natalia Ruiz, Farzaneh Saadati, Manuel Santos-Trig, Jorge Soto, Peter Taylor, Luz Valoyes, and Claudia Vargas took part in the conference.

After the conference, the participants were invited to prepare a chapter for the current book. We also invited some researchers who did not participate in the conference but were willing to share their research on mathematical problem solving. A particularly salient feature of this book is that it fosters the much needed dialogue between mathematicians and mathematics education researchers by including authors from these two strongly related but separate fields.

We have organized the book in five parts, putting together chapters addressing similar themes. In Part I, we gathered four chapters addressing problem solving in mathematics instruction from theoretical and practical perspectives. In Part II, the design of problem-solving situations is addressed in four chapters. Part III is devoted to the effects of engagement with problem solving in four chapters, and Part IV is dedicated to the role of teachers in problem-solving classrooms also with four chapters. The last part addresses, in three chapters, issues of teacher professional development and problem solving. In what follows, we briefly discuss the content of each part.

Part I. Theoretical and practical perspectives on problem solving in mathematics instruction. The chapter by Peter Taylor opens the book by analyzing some existing curricular constructs and proposing a new one in which problem solving provides students with true mathematical experiences. Then, Frédéric Gourdeau addressed the ongoing dialogue between mathematics and mathematics education through his own experience in problem solving as both a subject and a pedagogical approach. In the next chapter, a discursively oriented conceptualization of mathematical problem solving is offered by Boris Koichu, providing reanalysis of two past studies for illustrating this conceptualization. Finally, Jorge Soto-Andrade and Alexandra Yáñez-Aburto discuss an enactivist and metaphoric approach to problem posing and problem solving which is based on Valera's theory of knowledge.

Part II. Design of powerful problem-solving situations. The opening of the second part of this book is a chapter by John Mason in which he challenges the distinction between play and exploration while enhancing the former. It is followed by Erkki Pehkonen's chapter, proposing an alternative method to promote pupils' mathematical understanding via problem solving. Next, Patricio Herbst revisits the ranking triangles task as a way to both analyze geometric modeling tasks and provide opportunities to learn geometry. Manuel Santos-Trigo, Daniel Aguilar-Magallón, and Isaid Reyes-Martínez offer a chapter on a problem-solving approach based on digital technology and how this provides affordances to represent, explore, and solve problems via geometric reasoning. Part II ends with a chapter by Roza Leikin, discussing varying mathematical challenges related to teaching mathematics in a heterogeneous classroom and how "stepped tasks" can be used for the sake of students' self-regulated variation of mathematical challenge.

Part III. Effects of engagement with problem solving. Farzaneh Saadati and Cristián Reyes open the third part of the book with a chapter on the use of collaborative learning to improve problem-solving skills and how this is related to students' attitudes toward mathematics. The next two chapters, authored by Annette Rouleau, Natalia Ruiz, Cristián Reyes, and Peter Liljedahl, address teacher beliefs and student self-efficacy and how these change within the context of whole-class problem solving. Finally, Roberta and Jodie Hunter report on the use of culturally embedded problem-solving tasks to promote equity within mathematical inquiry communities.

Part IV. On the role of teachers in problem-solving classrooms. Sergio Celis, Carlos Quiroz, and Valentina Toro-Vidal open the fourth part of the book with a chapter on the influence of teacher-student interactions on group problem-solving capabilities. Markus Hähkiöniemi and John Francisco address teacher guidance in mathematical problem-solving lessons in the context of professional development programs. The chapter authored by José Carrillo, Nuria Climent, Luis Contreras, and Miguel Montes deals with mathematics teachers' specialized knowledge (MTSK) in managing problem-solving classroom tasks. The last chapter by Angeliki Mali, Saba Gerami, Amin Ullah, and Vilma Mesa is on teacher questioning as a means of supporting problem solving within community college algebra classrooms.

Part V. Teacher professional development and problem solving. The first chapter of this part is authored by Lisa Darragh and Darinka Radovic, who address success and sustainability of professional development programs in which teachers study how to enhance problem solving in their own classrooms. Next, Josefa Perdomo-Díaz, Patricio Felmer, and Cristóbal Rojas present a study on teachers' mathematical tensions surfacing at the beginning of a problem-solving professional development workshop. Finally, Luz Valoyes-Chávez closes the book with a chapter on stereotypes and the education of in-service mathematics teachers in urban schools.

Last but not least, we would like to thank many peoples and organizations for their help in making the conference and this book a reality. We are especially grateful to the reviewers who helped us to improve the chapters in the book. We thank the generous support from the International Cooperation Program (PCI) of

CONICYT, the Center for Mathematical Modeling, and the Center for Advanced Research in Education from the University of Chile. We also thank the Universidad de Magallanes for their hospitality and support. Special thanks go to Richard Lagos for the organization of the conference. But above all, we thank Gladys Cavallone for her work on the practical aspect of organizing of the conference and her wonderful job coordinating the work on this book.

Resilient tree in southern extreme weather conditions (Shutterstock)

Santiago, Chile — Patricio Felmer
Rehovot, Israel — Boris Koichu
Burnaby, BC, Canada — Peter Liljedahl

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Part V Teacher Professional Development and Problem Solving

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Part I
Theoretical and Practical Perspectives on Problem Solving in Mathematics Instruction

Chapter 1
Reforming School Mathematics: Two Levels of Structure

Peter Taylor

Abstract Many articles and papers over the past 100 years have suggested that mathematics education has lost its way in a number of critical respects. One indication of this is certainly the hugely destructive debate between discovery and drill, a consequence of which is an emphasis, throughout the school curriculum, on technical routines.

For me, mathematics is the abstract study of structure. The structures that mathematicians choose to work with have sophistication and beauty, and it is remarkable that these same structures arise in art, in nature, and in the physical and even social sciences. So often, it is by following the beauty that we are led to the truth, and mathematics is a powerful showcase for this delightful principle. But in spite of a century-long call that school math attend to this vital aspect of mathematics, an emphasis on structure and beauty, for example, in the choice of material, is notably absent from realized curricula.

My view is that such a curriculum change cannot happen without a change in the very structure of the curriculum. Quite simply, we must put aside our technical wish list and select for our students activities and problems that give them a true mathematical experience, and then build the curriculum from there. Thus this article is about structure at two different levels: The first is the structural richness of the mathematical activities I want to see in the classroom, and the second is a new structure for the curriculum itself.

Keywords Secondary school · Curriculum · History · Projects · Papert · Whitehead · Dewey

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1.1 Some Brief Historical Comments

It has been 100 years since the end of the First Great War. The history of secondary school math education reform during that century has been a tangled tale, and I will begin with a summary of some of the main episodes. In this, I will be following two articles, one by Jeremy Kilpatrick (1997) and another by Kate Raymond (2018). The tangled nature of the tale comes from the fact that there were always two forces at work, one narrow and the other wide, but at different times and in different movements, these forces locked horns along different axes. Along one such axis, the narrow view focused on the preparation of students for college and university and ultimately for their participation in technology and the STEM disciplines, while the wider view emphasized the more general humanistic development of informed citizens for a full rich life. Along another axis, the narrow view tended to focus on procedural fluency (back to basics), and the wider on creativity, discovery, and conceptual understanding. As a general rule, as we will see that the wider view tended to have less effect on classroom practice than the narrow view. That's not surprising—narrow, more focused objectives tend to be easier to grasp and implement.

In general, both views make good sense to me, and one would think they could happily coexist. Indeed, the oscillations that appear in the historical record often seem to me to be overreactions to positions that were not as far apart as many seem to have thought. Indeed my, perhaps idealistic, objective in this article is to outline a curriculum structure, one that was long ago elegantly articulated in the philosophical record, which would support both of these viewpoints and be true to the nature of the subject.

Following the first war, there was definitely a flowering of a wide view of "math education for all." Philosophically this can be seen in the writings of both Whitehead and Dewey (and more on this later), but as both Kilpatrick (1997) and Raymond (2018) observe, it was also explicit in the 1923 report of the MAA National Committee on Mathematical Requirements (1923). The report argued that

> …the practical aims of school mathematics should be secondary to the mental training and development of skills necessary to the discipline of mathematics and the development of an appreciation for the beauty, power, and logic in mathematics and geometric objects. By focusing on these aims, scholars hoped to avoid school mathematics becoming "a collection of isolated and unrelated details" and instead make mathematics more appealing to a broader range of students. (cited by Raymond, 2018 p. 3.)

Raymond goes on to suggest that these ideas appear to have had little effect on classroom practice. The technological growth emerging from the Second Great War, along with the 1957 "Sputnik" wake-up call, promoted along one axis a narrowing emphasis on student preparedness for future scientific and engineering challenges and along another axis, a widening view of the nature of mathematics, away from procedural fluency toward conceptual understanding (Raymond, 2018 p. 4). A dominant idea was that to succeed, students would need a "proper" treatment of mathematics, often interpreted to mean pure math and abstract structures, and this became known as the "new math."

Of course, there was swift reaction, and Morris Kline's, 1973 book *Why Johnny Can't Add: The Failure of the New Math* became in many ways the face of the reaction. Kilpatrick (1997 p. 956) notes that "Kline ended the book by arguing that the appropriate direction for any reform 'should be diametrically opposite to that taken by the new mathematics' (1973, p. 144), toward mathematics as an integral part of a liberal education, with connections to culture, history, science, and other subjects." But that component of Kline's message did not catch on, and the "back-to-basics" reaction to the new math won the day. Both Kilpatrick (1997 p. 956–957) and Raymond (2018 page 5) argue that the new math movement was far more diverse than is commonly realized and was never properly tested.

In the 1980s, the reform movement returned but this time under the formidable banner of the National Council of Teachers of Mathematics (NCTM) *Standards* (1989) which advocated "mathematics for all'—the intention of which was to empower all students with the skills and abilities that would enable them to be active, engaged, and critical members of democratic society. After decades of narrowing the focus of school mathematics to prepare students for technological careers, these documents were the first to push back against the limited view of school mathematics and insist on a broader conceptualization (Raymond, 2018, p. 6).

Of course, there was again strong reaction, strong enough that the term "math wars" was used. The main target of the reaction was the "discovery" approach to learning which, at the elementary level, diverted students from the important task of learning multiplication tables and adding fractions and at the secondary level, with its use of heuristics and diagrams, prepared students badly for a rigorous course in university calculus. Indeed, the debate had an echo at the university level in the reform calculus movement, which in itself has had a huge effect on first-year university calculus courses today. In the early 1980s, there was a suggestion that the coming world of computer technology might be better served by a course in discrete math or linear algebra rather than calculus and, led by Andy Gleason and others, there was a response to make calculus more relevant and mainstream. That movement was successful in that calculus remains today the default (and often required) first-year university math course. Interestingly enough, in a somewhat altered form, the idea, that calculus might not be the best default course, is now coming back, though in altered form, one that features areas of math and stats that are closer to data analysis.

A central figure in the traditionalist camp was H. Wu of Stanford University. To get a sense of the state of the argument at the close of the twentieth century, it is interesting to look at a pair of papers of Kilpatrick (1997) and Wu (1997) which appeared side by side in the *American Math Monthly*, and in fact the last part of Kilpatrick's remarks focused on the Wu paper. Wu makes a number of interesting points—interesting in that they are well worth discussing. He does accept the appropriateness of reform calculus for the typical science and engineering student but fears that it will not well serve the student who is destined for serious university mathematics. Such students "need rigorous mathematical training, and would not be satisfied with a steady diet of persuasive heuristics, graphic displays, and nothing else" (Wu, 1997 p. 947). I go most of the way with this but would phrase it differently. Students who are destined to study serious mathematics need to be able to make rigorous arguments, but I believe that the opportunity to understand and practice these can be given to them in a course that features persuasive heuristics and graphic displays.

1.2 My Own Half-Century

For the past 50 years, I have been constructing "discovery" problems for high school students. But over that period, there have been a few ways in which my work has changed. At the beginning, I regarded these problems as "after-school" enrichment for motivated students. That possibility still exists, but, for me, the main stage is now the regular classroom. That objective requires tasks that provide a low mathematical floor (requiring minimal prerequisite knowledge) and a high mathematical ceiling (offering opportunities to explore more complex concepts and relationships and more varied representations) (Gadanidis, Borba, Hughes, & Lacerda, 2016 p. 236, Boaler, 2016 p. 115). As I pursue that objective, I find to my surprise that many high-ceiling problems, such as those found in university mathematics, can be engineered to have an invitingly low floor and can work beautifully in high school.

Over the past few years, I have made a deliberate effort to tie my problems to the mandated curriculum, and this has affected my choice of subject matter. For example, for the first few decades, I chose problems that were fun, enticing, and mysterious and worked with areas such as geometry, probability, combinatorics, logic, games, and puzzles. But in Ontario, fully half of the entire high school math curriculum works with properties of functions, and while I believe that this is unbalanced, my basket of activities has moved somewhat in the direction of functions. But here's an interesting anecdote. In my third-year undergraduate course for future math teachers, I take my problems/activities from a balanced set of areas including the analysis of functions. Toward the end of the course, I have group projects, and students can choose the problems they want to work with. In 20 years with that course, no student has ever chosen to work with functions. What that tells me is that their own school experience with functions has hardly ever engaged them in play, in design and construction, or in mathematical thinking.

I have always had an eye on the preparation of our secondary school students for university, but only recently has that become my main focus. I watch carefully to see what my first-year university students struggle with. That can be hard to perceive, but my feeling is that their struggles seem to be more connected with the focus and clarity of their thinking rather than the execution of what are called "the basics." A related aspect of these struggles is their handling of problems with a complex structure. Complexity can be contrived, and I find that to be often the case in problems that the students are given, but there are also complexities that are organic to the structure of the problem. These are more important, in part because they arise naturally and are thereby closely related to structural complexities that the students will encounter in their own future lives, both professional and personal. In university, students frequently encounter structures with this level of sophistication, but I find almost no problems of this kind in high school mathematics.

What do I do with my ever-growing collection of problems? I show them to the teachers that I know or might meet and ask if they want to use them, or if they would invite me into their classroom to try them out, or better still, let me come and watch while *they* work with them. I do get offers, but the teachers that I talk with are often wary. There could be many reasons for that, but the one typically stated is that they are running short of time. They have after all a curriculum to cover, and it can easily require the full 110 hours that the Ministry allocates. Of course, my "wonderful" problems are designed to *be* the curriculum, such that nothing else is needed. If the students can do those, they will surely be ready for my first-year calculus and linear algebra courses. But I can't yet promise that because the problems are a long way from being organized into a complete, coherent, well-supported package. So I certainly understand the teachers' hesitation and am grateful to those wonderful colleagues who have been happy to work with me.

But this brings up the question of the nature and the structure of the curriculum. Certainly the curriculum of problems has quite a different structure from the one we currently find in school mathematics. Is it apt to work? Is there anything to be said for such a curriculum? In fact, the ideas of some of the greatest thinkers of the past 100 years interact well with this question of curriculum structure. I have three of these in mind: Alfred North Whitehead, John Dewey, and Seymour Papert. Having mentioned these, a reviewer suggested I look at C. S. Peirce (1939–1914), an American philosopher and scientist, who is said to have influenced Dewey. Indeed, he held many of Dewey's views on the nature and centrality of experience in education (Strand, 2011) and the pedagogical significance of surprise (Gadanidis et al., 2016) (Fig. 1.1).

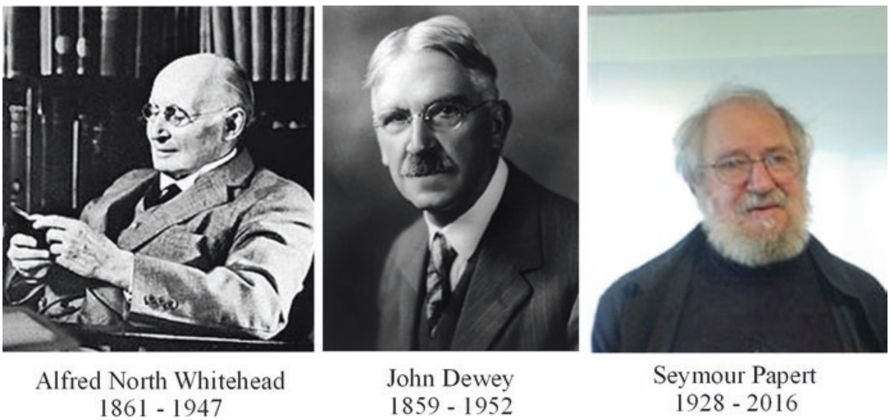

Fig. 1.1 My three intellectual heroes

1.3 The Search for a Curriculum Structure

Whitehead's power and beauty of ideas and Dewey's experience of the artist both emphasize the richness of the learning experience and the importance of the training of the mind. I have argued (Taylor, 2018) that the writings of both these philosophers have a lot to offer us today. Raymond (2018) agrees with this but suggests that these ideas might have had little effect on classroom practice.

I start with Whitehead. His *Rhythm of Education* (1929, Chap. 2) effectively provides a structure for the curricula of all disciplines. Here he identifies three stages of learning: Romance, Precision, and Generalization. To some extent, our learning proceeds through these three stages in order, such that, roughly speaking, the child is dominated by Romance, the youth by Precision, and the adult by Generalization. In practice, however, the stages cycle continuously like eddies in the fast-flowing stream of life (and indeed at different times, we can all be children or adults).

The first stage, of Romance, is one of ferment, novelty, and mystery, of hidden possibilities and barely justifiable leaps. This stage, in its fullness, motivates the second stage, of Precision, in which we strive for comprehension and mastery—ideas must be tamed and organized, requiring care, honesty, and restraint. Finally, the third stage, of Generalization, is essentially a return to Romance, but now with the technique acquired at stage two. Our ideas have new power because we have harnessed them. The great fruit of this ultimate stage of learning is wisdom: the capacity to handle knowledge. The central point that Whitehead makes is that the discipline of stage two must not be imposed until the fullness of stage one has properly prepared the student. Failing that, the knowledge that is obtained will be inert and ineffective.

This "rhythm" sets a structure for the entire 12 years of schooling, one which will hopefully sustain us for the remaining years of our learning. For each particular course and indeed for each learning hour, it provides a ritual that we too often fail to observe. I find that it makes a great difference if, when planning a lecture, I remind myself of the precedence of Romance. Certainly, Whitehead's rhythm lays to rest that ridiculous conflict between discovery and basics; the first most often provides the Romance, the second the Precision.

Moving on to John Dewey, his search for a structure is encapsulated in the title "The Need of a Theory of Experience" of Chap. 2 of his 1938 essay *Experience & Education*:

> I assume that amid all uncertainties there is one permanent frame of reference: namely, the organic connection between education and personal experience." (1938, page 8)

That "frame of reference" is what defines the structure of Dewey's encounter with education. He had of course already, in 1934, developed that theory in the powerful context of the aesthetic. There, his attention was on the audience much more than on the performer, particularly in his insistence that the heart of the aesthetic experience is found in the response of the viewer.

> The word "aesthetic" refers, as we have already noted, to experience as appreciative, perceiving and enjoying. It denotes the consumer's rather than the producer's standpoint. It is Gusto, taste; and, as with cooking, overt skillful action is on the side of the cook who prepares, while taste is on the side of the consumer, as in gardening there is a distinction between the gardener who plants and tills and the householder who enjoys the finished product. (Dewey, 1934, p. 37)

In fact, too much emphasis on the "finished product" can detract from the experience. The opening paragraph of *Art as Experience* emphasizes this:

> In common conception, the work of art is often identified with the building, book, painting, or statue in its existence apart from human experience. Since the actual work of art is what the product does with and in experience, the result is not favorable to understanding. In addition, the very perfection of some of these products, the prestige they possess because of a long history of unquestioned admiration, creates conventions that get in the way of fresh insight. When an art product once attains classic status, it somehow becomes isolated from the human conditions under which it was brought into being and from the human consequences it engenders in actual life-experience. (Dewey, 1934, p. 1)

Some time ago it was not uncommon to hear teachers proudly proclaim: "I don't teach math; I teach students." I thought at the time that this was a bit silly because of course, we do both. But I'm guessing that the purpose of the phrase was effectively to reinforce Dewey's important insight.

This then brings us to what Dewey calls the central problem of an education based upon experience: "to select the kind of present experiences that can live fruitfully and creatively in subsequent experiences" (Dewey, 1938, p. 9).

The conclusions he draws from that are, on the whole, well understood today, for example, that meaning comes only from the present experience of the student, and that subject matter earned in isolation, put, as it were, in a watertight compartment to be opened only at the time of the exam contributes nothing to the student's future life. But although these conclusions are well understood, they are widely ignored. When I am working in a high school classroom, I put the students in groups either at tables or (preferably) standing at white- or blackboards, and I evaluate the quality of the problem in part on signs of an engaging and even intense experience.

Finally, I add one more layer to this search for the right structure, and that emerges from Seymour Papert's idea of a project as a significant activity that provides meaning to the student's life.

> The important difference between the work of a child in an elementary mathematics class and that of a mathematician is not in the subject matter (old fashioned numbers versus groups or categories or whatever) but in the fact that the mathematician is creatively engaged in the pursuit of a personally meaningful project. In this respect a child's work in an art class is often close to that of a grown-up artist. (Papert, 1972, p. 249)

More recently, Jo Boaler makes the same point comparing mathematics to language studies:

> When we ask students what math is, they will typically give descriptions that are very different from those given by experts in the field. Students will typically say it is a subject of calculations, procedures, or rules. But when we ask mathematician what math is, they will say it is the study of patterns that is an aesthetic, creative, and beautiful subject. Why are

> these descriptions so different? When we ask students of English literature what the subject is, they do not give descriptions that are markedly different from what professors of English literature would say. (Boaler, 2016, p. 21–22)

In effect this is an argument by analogy that at the school level, we should be teaching the mathematics that mathematicians do (Taylor, 2018). I draw from that idea when I find myself constructing a new high school problem. If, when I am writing it up, I, as a mathematician, feel the life and energy waning, that's a signal the problem might not after all be right. On the other hand, if the excitement builds, I feel I must be on the right track.

A "project" for Papert is necessarily a sustained endeavor, and that has a number of consequences:

> This *project-oriented* approach contrasts with the *problem* approach of most mathematics teaching: a bad feature of the typical problem is that the child does not stay with it long enough to benefit much from success or from failure. Along with time-scale goes structure. A project is long enough to have recognizable phases—such as planning, choosing a strategy of attempting a very simple case first, finding the simple solution, *debugging it* and so on. And if the time scale is long enough, and the structures are clear enough, the child can develop a vocabulary for articulate discussion of the process of working towards his goals. (Papert, 1972, p. 251)

The last idea of this remarkable paragraph is worth highlighting. Math students often have trouble talking about the subject they are studying; they lose the big picture, if they ever had it, and they get lost in the details. Papert suggests that a habit of sustained engagement can foster discussion at the structural level—if the structure is rich, there is more to talk about.

Barabe and Proulx (2017 p. 26) make the important point that Papert's projects emphasize *doing* more than *knowing* and thereby give the students something much more powerful than mathematical knowledge, and that is what Papert calls "mathematical ways of thinking." That's really another way of saying that we should be teaching the mathematics that mathematicians do.

For me this project structure has the power to give us a natural realization of the structures put forward by Whitehead and Dewey. When our curriculum planning is on the level of the project, we seldom need to search for Romance; it is typically already in place as an organic component of the process. In the same way, Dewey's "experience" is typically an integral part of the activity generated by the problem. I find that when I am considering whether or not a problem passes the bar of admission to my classroom, I pay early attention to the student experience (Dewey's "consumer"), looking for aspects such as surprise (Gadanidis et al., 2016), wonder (Sinclair & Watson, 2001), flow (Liljedahl, 2017), beauty (Sinclair, 2006), low floor, or high ceiling (Gadanidis et al., 2016 p. 236, Boaler, 2016 p. 115).

And of course, a project-oriented curriculum structure is much more creative, challenging, and even "humanizing" for the teacher; it can nurture her development as an artist.

Time to sum up and put things together. The more I reflect on the present reality of high school math, the more of a disaster it seems. That's strong language, but it's what comes to mind when I think of the students. Quite simply, they deserve better—they deserve the real thing. That simple truth strikes me most forcefully when I go into the classroom and work with them. For the most part, they are ready to work, and more importantly, they are ready to play.

Of course, as things stand at present, most of them feel that what they are getting in the classroom is what mathematics is; indeed, they simply don't know what they are missing. More than once, after 75 minutes in the classroom, I get the comment, "Why isn't math always like this?" I do note that back in the 1950s we did at least encounter the grandeur of the subject, as in Grade 10 we had a full-year course in Euclidean geometry.

So what *are* they missing?—the best way to answer that is to observe that mathematics is the study of structure and that high school math currently offers no identifiable structures of any sophistication. Papert's projects offer us a way toward a curriculum with genuine mathematics. But how do we get there?

There are difficulties. First of all, projects are harder to work with and often require a level of mathematical and pedagogical experience that many teachers do not yet have. And there is the question of time. The activities take time and patience, and teachers often feel that the job of building a proper technical foundation for their students already takes almost all of the available class hours. And finally, because my visit is effectively an intervention, the activities can seem disconnected and even contrived. I will discuss each of these factors.

1.3.1 The Technical Skills

They are important; we can't do mathematics without them. But if we assemble ahead of time all the ones we think we might need, for example, to do calculus, the basket will be too heavy and will divert us from the real goal. To work and play effectively, we need to travel light, and that requires putting that basket aside and having the simple faith that the activities we choose will be comprehensive enough to look after the student's future technical needs. Those who worry that the students might miss some critical skills should spend some time in a first-year university calculus course and find out that many of the skills that were "taught" in high school were not in fact learned in any effective way. Skills need meaningful context; the more powerful the context, the more solid the skill.

What *is* important is that students learn how to master skills. That's well understood by students who play guitar or basketball; they simply have to realize that the same principles apply to mathematics. This idea works so seamlessly in music and sports because they in fact have that powerful context. Well, mathematics has an equally powerful context to offer, but it's one that few students have ever encountered.

The other thing to notice is that universities, professional programs, and employers are increasingly emphasizing a new level of what are often called "secondary" skills, sometimes called the "C-words"—care, creativity, critical thinking, communication, and collaboration. A project-based curriculum can often relate more naturally to these.

1.3.2 Teacher Preparation

Even experienced teachers find it a challenge to work with investigative activities. First of all, there are usually different ways tackle the problems, and it helps to be able to anticipate these. That takes more in the way of preparation time and, often, mathematical knowledge as well. And there are balances to be struck—between giving the students ideas and letting them find avenues on their own, between keeping the class together and giving the faster students questions on the side, and between individual work and collaboration within groups.

A project-oriented curriculum can be an enormous challenge for teacher candidates. My colleagues in Faculties of Education well realize that this is an increasingly important part of their job, but there is only so much they can do. The simple fact is that most of our learning about how to teach happens when we ourselves are being taught, and most of today's fledgling teachers have spent too little time in their own mathematics learning exploring and investigating. I will mention three phases of that experience. One of these is their school experience, and that's not surprising as that is of course exactly what we are working to change. Another is the time they spend out of school, and there is evidence that the technological and media explosion has seduced many of them away from much of that experience. The third is their undergraduate learning, and that is an experience that many of the readers of this volume have some control over. I am definitely not happy with the nature of most of the teaching in undergraduate math courses in North American universities, particularly in the "service" courses, and those are often the courses taken by future math teachers. These courses need to purvey less in the way of mathematical knowledge and put much more emphasis on inquiry and mathematical thinking. Students who might actually need considerable mathematical knowledge typically already know that this is the case and respond accordingly.

1.3.3 What Mathematics?

I want to briefly return to this question of the dominant place the study of functions plays in the senior school curriculum, certainly in North America. I have observed that the cause of this is almost certainly the role of calculus as the default math course in first-year university and college. Now whether that remains the case or not, my belief is that the current introductory calculus course offered in the senior school curriculum is not the right preparation. It is technical in nature and is very much oriented toward the transfer of mathematical knowledge, with little attention given to mathematical thinking. It also gives the students the misleading impression that they have already covered much of the first semester of university calculus. I would prefer a course with a theme of modeling and optimization, using many different approaches, analytical, geometric, and graphical. It would not follow the logical technical development of the subject, leaving that for university, but would still remain true to the *ideas* of calculus. The few technical pieces such as the arithmetic laws of the derivative could be quickly covered and then employed "in action," thus remaining true to Whitehead's Romance and Dewey's present experience.

I illustrate these remarks with two examples taken from my own body of work (Taylor, 2016). Example 1 is a model for the speed at which a car should be driven to minimize the cost of gas.

1.3.4 Example 1: Gas Consumption for Optimal Driving Speed

We need to start with a graph of gas consumption against speed, and there are some simple mainstream kinetic energy principles that lead to a simple equation for this. A senior class that has some acquaintance with Newton's Laws of Motion will enjoy the challenge of finding the algebraic form of the gas consumption graph found in Fig. 1.2a. It gives rise to some interesting questions such as why is it expected to be concave-up. For the various components of the problem, we have the choice of working with the formula we have derived and using algebra or even calculus, working with the geometric form of the graph, or of course both. I will highlight the graphical argument.

To begin we ask for the velocity that minimizes the cost of making a trip of a fixed distance. Now the vertical axis z has units in liters consumed per hour at any fixed speed v. But to use least gas over a given distance, we want to minimize liters per km (z/v), and that requires us to minimize the slope of a secant line drawn from the origin to the graph. This occurs when the secant is tangent to the graph, and the optimal speed in this case (Fig. 1.2b) is 50 km/h. This is considerably slower than we typically drive on the highway, and the reason for this of course is that we put a value on our time; to account for that, what we really need to minimize is the sum of gas cost and the effective wage we are paying ourselves. This sum is minimized with an elegant generalization of the secant construction of Fig. 1.2b. Putting the cost of 6 liters of gas as the value of an hour of our time (thus with a gas cost of \$1.50/L, this would be \$9/h), Fig. 1.2c gives us the reasonable optimal speed of 90 km/h. This is a rich, multifaceted problem that can be tuned and extended in different ways at different grade levels. It certainly earns the status of a Papert project.

1.3.5 Example 2: Counting Trains

Some branches of mathematics lend themselves more readily than others to investigation and what is called "mathematical thinking." In my experience projects involving discrete structures, geometry, simple probability, and strategic thinking (games) are more accessible to students and more naturally investigative than is the