Saas-Fee Advanced Course 48
Swiss Society for Astrophysics and Astronomy

Tiziana Di Matteo
Andrew King
Neil J. Cornish

Black Hole Formation and Growth

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Edited by Roland Walter, Philippe Jetzer,
Lucio Mayer and Nicolas Produit

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Cover illustration: Scientists have obtained the first image of a black hole, using Event Horizon Telescope observations of the center of the galaxy M87. The image shows a bright ring formed as light bends in the intense gravity around a black hole that is 6.5 billion times more massive than the Sun. Credit: Event Horizon Telescope Collaboration

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Preface

The 48th "Saas-Fee Advanced Course" of the Swiss Society for Astrophysics and Astronomy (SSAA) was held from 28 January to 3 February 2018 in Saas-Fee, in the Swiss Alps. It was very timely devoted to:

Black Hole Formation and Growth

and attended by 119 participants. The Saas-Fee courses are intended mainly for postgraduate, Ph.D. students, astronomers and physicists who wish to broaden their knowledge. The lectures were organised in the morning and late afternoon leaving free time for informal discussions, studies and outdoor activities in the afternoons.

This advanced course provided three comprehensive and up-to-date reviews covering the gravitational wave breakthrough, our understanding of accretion and feedback in supermassive black hole and the relevance of black hole to the Universe structure since the Big Bang. The lectures were given by three world experts in the field:

Prof. Tiziana Di Matteo (Carnegie Mellon University, USA)
Tiziana Di Matteo is a Professor in the McWilliams Center for Cosmology of the Physics Department at Carnegie Mellon University, USA. She received her Ph.D. in 1998 at Cambridge University, UK. She was a Chandra Fellow at Harvard and a junior faculty member at the Max Planck Institute for Astrophysics in Germany. She is a theorist with expertise in both high energy astrophysics and cosmology. Her interests focus on state-of-the-art cosmological simulations of galaxy formation with special emphasis on modelling the impact of black holes on structure formation in the Universe.

Prof. Andrew King (University of Leicester, UK)
Andrew King is Professor of Theoretical Astrophysics at the University of Leicester and holds visiting appointments at the Universities of Amsterdam and Leiden. During his career, he has been awarded a PPARC Senior Fellowship, the Gauss

Professorship at the University of Goettingen, a Royal Society Wolfson Merit Award, and the RAS Eddington Medal. He is an author and co-author of several books, including Stars, a Very Short Introduction, and Accretion Power in Astrophysics. His research interests include accretion disc structure, supermassive black hole growth and feedback, active galactic nuclei, compact binary evolution, and ultraluminous X-ray sources.

Prof. Neil J. Cornish (Montana State University, USA)
Neil J. Cornish is Regents Professor of Physics and Director of the eXtreme Gravity Institute at Montana State University. He completed his Ph.D. at the University of Toronto, followed by postdoctoral fellowships in Steven Hawking's group at the University of Cambridge and in David Spergel's group at Princeton University. He is a multiwavelength gravitational-wave astronomer, and he is a member of the Laser Interferometer Gravitational-Wave Observatory (LIGO) Scientific Collaboration, the North American Nanohertz Observatory for Gravitational Waves (NANOGrav) Collaboration and the NASA Laser Interferometer Space Antenna (LISA) Science Study team.

Exactly 100 years before this Saas-Fee course, on 31 January 1918, Einstein submitted his paper entitled "Gravitational waves" to be presented at the Prussian Academy meeting held on 14 February. Actually, Einstein started to think on gravitational waves (at least as far it is documented) in 1913 when he was still Professor at ETH in Zürich. At the 85th Congress of the German Natural Scientists and Physicists (9 September 1913) in Vienna, Max Born asked Einstein about the speed of propagation of gravitation, in particular, whether it would be that of light. Einstein replied that it is extremely simple to write down the equations for the case in which the disturbance in the field is extremely small and this is what Einstein then did in his 1916 paper "Approximate Integration of the Field Equations of Gravitation", with some mistake, and in a more correct way in 1918.

It took a century from Einstein's papers to the actual detection of a gravitational wave (and the ultimate proof that black holes exist). A pretty important step, demonstrating that the content of this book is closer to reality than it would have been just a few years ago.

We are very grateful to the lecturers for their enthusiasm in communicating their deep knowledge, their brilliant lectures, as well as for writing the rich manuscripts composing this book. We extend our warmest thanks to the course secretaries, Martine Logossou and Marie-Claude Dunand, for their effective administration and organisational help during the course. We also would like to thank our students and collaborators helping in Saas-Fee and finalising the manuscripts, in particular, V. Sliusar, C. Panagiotou, M. Balbo, E. Lyard, T. Bernasconi and M. Kole.

Saas-Fee provided a very pleasant environment with two metres of fresh snow and an entirely sunny week. We enjoyed the first century birthday of gravitational wave on 31 January 2018, with a concert of the Swiss ethno-electronic music band "Vouipe", who composed the track "Black Hole in Saas-Fee"[1] based on the

[1]Available at http://vouipe.com/.

space-time chirp of GW150914! We also enjoyed a concert from Moncef Genoud and Ernie Odoom. These were magical evenings, and we would like to thank again the outstanding performers for their delighting music.

Finally, this course would not have been possible without the financial support of the Swiss Society for Astrophysics and Astronomy, the Société Académique de Genève and the Universities of Geneva and Zürich. We are very grateful to these organisations for their contributions, which allowed the participants to attend a very interesting and successful course.

The course organisers

Geneva, Switzerland — Roland Walter, Nicolas Produit
Zürich, Switzerland — Philippe Jetzer, Lucio Mayer


LISA science highlights
Seed Black Hole mergers to z > 20
Extremely high SNR for SMBH mergers at z < 5 - EM counterparts
100's of Extreme Mass Ratio Captures informing galactic cusp structure
10,000's Galactic White Dwarf binaries - EM counterparts
100's of LIGO progenitors (NS, BH) and precursors (BH-BH)
Phase transitions at the TeV - PeV scale
High precision tests of GR (orders of magnitude beyond LIGO)

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Black Hole Merging and Gravitational Waves

Neil J. Cornish

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Abstract I was tasked with covering a wide swath of gravitational wave astronomy—including theory, observation, and data analysis—and to describe the detection techniques used to span the gravitational wave spectrum—pulsar timing, ground based interferometers and their future space based counterparts. For good measure, I was also asked to include an introduction to general relativity and black holes. Distilling all this material into nine lectures was quite a challenge. The end result is a highly condensed set of lecture notes that can be consumed in a few hours, but may take weeks to digest.

1 Introduction

In writing up these lecture notes I have mostly followed the order in which the material was presented in Saas Fee, with the exception of the discussion of the detectors, which have been grouped here in a single section. My goal is not to write a textbook on each topic—many excellent texts and review articles on general relativity and gravitational wave astronomy already exist (see e.g. [11, 16, 37, 40, 50]). Rather, I try to highlight the key concepts and techniques that underpin each topic. I also strive to provide a unified picture that emphasizes the similarities between pulsar timing, ground base detectors and space based detectors, and commonalities in how the data is analyzed across the spectrum and across source types.

2 General Relativity

The historical course that lead Einstein to develop the general theory of relativity had many twists and turns, but as Einstein reflected in 1922, one of his primary goals was to understand the equivalence between inertial mass and gravitational mass "It was most unsatisfactory to me that, although the relation between inertia and energy is so beautifully derived [in Special Relativity], there is no relation between inertia and weight. I suspected that this relationship was inexplicable by means of Special Relativity" [43]. Einstein found the resolution to this conundrum by adopting a geometrical picture that generalized Minkowski's description of special relativity to allow for spacetime curvature.

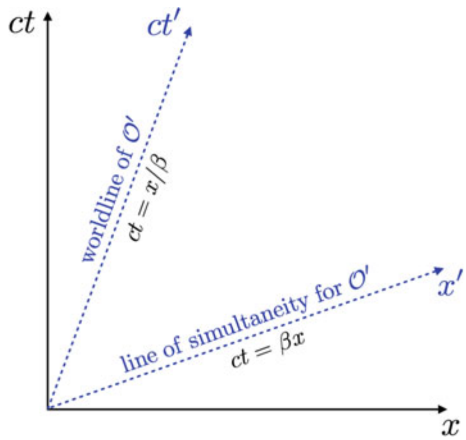

Fig. 1 A spacetime diagram shown in the the rest frame of observer $\mathcal{O}$. Observer $\mathcal{O}'$ is moving at velocity v with respect to $\mathcal{O}$ in the x direction

2.1 Special Relativity

Minkowski showed that phenomena such as time dilation and length contraction followed naturally as a consequence of space and time being combined into a single spacetime geometry with distances measured by the invariant interval

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \,. \tag{1}$$

The $t = \text{const.}$ spatial section of this geometry is ordinary three dimensional Euclidean space, which is invariant under translations and rotations, and can be described by the special Euclidean group E(3) = SO(3) × T(3). The full Minkowski spacetime is invariant under the Poincaré group, which is made up of translations in time and space $T(1, 3)$, rotations in time and space SO(1, 3), otherwise known as the Lorentz group. The Lorentz group includes regular spatial rotations SO(3), and boosts, which can be thought of as a hyperbolic rotation in a plane that includes a time-like direction.

The key results of special relativity can be derived by considering motion that is restricted to the $(1 + 1)$ dimensional sub-manifold spanned by coordinates (t, x) with invariant interval $ds^2 = -c^2dt^2 + dx^2$. Rotations in this two-dimensional Minkowski space leave fixed hyperbolae, $x^2 - c^2t^2 = \pm a^2$, just as rotations in two dimensional Euclidean space leave fixed circles, $x^2 + y^2 = a^2$. The coordinates (t, x) define a frame of reference $\mathcal{O}$. An observer at rest in this coordinate system will follow the trajectory $x = \text{const.}$: in other words, a line parallel to the t-axis. A particle moving at velocity v in the positive x-direction will follow the trajectory (worldline) $x = vt + \text{const.}$ We can perform a boost to a new reference frame $\mathcal{O}'$, with coordinates (t', x'), where the particle is a rest: $x' = \text{const.}$ This implies that the two coordinate systems are related: $x' = \gamma(x - vt)$, where γ is a constant. Objects at rest in frame $\mathcal{O}$ will be moving at velocity $-v$ in the x' direction in frame $\mathcal{O}'$, so it follows that $x = \gamma(x' + vt')$. Solving for t' we find $t' = \gamma(t + (1-\gamma^2)/(v\gamma^2)x)$. Invariance of the interval $x'^2 - c^2t'^2 = x^2 - c^2t^2$ fixes the constant to be $\gamma = 1/\sqrt{1-\beta^2}$, where $\beta = v/c$. Thus we have derived the following coordinate transformation for a boost with velocity v in the positive x direction:

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - vt)\,. \end{aligned} \tag{2}$$

The transformation can be viewed as a hyperbolic rotation with $\cosh\eta = \gamma$ and $\sinh\eta = \beta\gamma$, were the "angle" $\eta = \text{arctanh}\,\beta$ is called the rapidity. Lines of simultaneity in $\mathcal{O}'$ have $t' = 0$, and lie parallel to the line $x = ct/\beta$ in frame $\mathcal{O}$. Figure 1 displays a spacetime diagram illustrating the relationship between the two reference frames.

Classic results such as time dilation and length contraction follow directly from the spacetime geometry of Minkowski space. For example, consider two events A, B along the worldline of observer $\mathcal{O}'$. The proper time elapsed as measured by a clock carried by observer $\mathcal{O}'$ is $T' = \Delta t'$, while the time elapsed as measured by a clock carried by observer $\mathcal{O}$ is $T = \Delta t$. The invariant interval is

$$\Delta s_{\rm AB} = -(c\Delta t')^2 = -(c\Delta t)^2 + \Delta x^2\,. \tag{3}$$

Since $\Delta x = v\Delta t$ we find that

$$T = \gamma T'\,, \tag{4}$$

so that the time elapsed in the static frame is greater than the time elapsed in the moving frame. Next consider a rod of proper length L' moving at velocity v relative to observer $\mathcal{O}$. At a given instant, the ends of the rod at at events D, F in frame $\mathcal{O}$ and at events D, E in frame $\mathcal{O}'$. Thus the length of the rod in frame $\mathcal{O}$ is $L = \Delta s_{\rm DF}$, while the length of the rod in frame $\mathcal{O}$ is $L' = \Delta s_{\rm DE}$. Using the invariance of the interval we have

$$\begin{aligned} \Delta s^2_{\rm DF} &= L^2 = \Delta x^2 = \Delta x'^2 - (c\Delta t')^2 \\ \Delta s^2_{\rm DF} &= L'^2 = \Delta x'^2 = (\Delta x + v\Delta t)^2 - (c\Delta t)^2\,. \end{aligned} \tag{5}$$

Incorporating the time dilation found earlier, $\Delta t = \gamma\Delta t'$, we find the lengths to be related:

$$L = L'/\gamma\,, \tag{6}$$

so that the rod appears shorter in the static frame. The spacetime diagram makes it clear that this discrepancy is due to the two frames having different lines of simultaneity (Fig. 2).

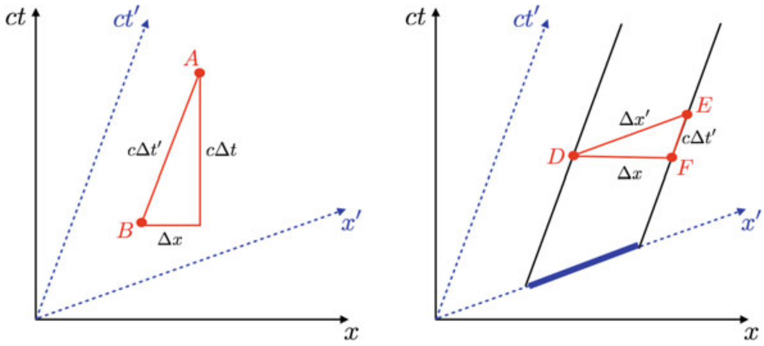

Fig. 2 Spacetime diagrams illustrating time dilation (left) and length contraction (right)

2.2 The Equivalence Principle

Einstein, like Newton before him, was struck by the equivalence of the inertial mass that appears in the relation between force and acceleration $\mathbf{F} = m_I \mathbf{a}$ and the gravitational charge, or mass, that appears in Newton's gravitational force law $\mathbf{F}_G = -Gm_G M\mathbf{r}/r^3$. He also noted that inertial frames of reference play a special role in both Newtonian mechanics and special relativity. In Newtonian mechanics, objects in a non-inertial frame that is uniformly accelerating and rotating experience "pseudo-forces" of the form

$$\mathbf{F}_\mathrm{P} = -m_I\mathbf{a} - 2m_I\omega \times \mathbf{v} - m_I\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})\,. \tag{7}$$

Since these forces are a coordinate effect, they must scale with the inertial mass. The first term in the above expression is referred to as a rectilinear force, the second term is called the Coriolis force, while the third term is called the centrifugal force. The "pseudo" moniker is perhaps a little misplaced—hurricanes that get stirred up by the Coriolis force due to the rotation of the Earth are real enough. Einstein suggested that the rectilinear term be identified with a uniform gravitational field. "I was sitting in a chair in the patent office in Bern when all of a sudden a thought occurred to me: if a person falls freely he will not feel his own weight. I was startled. This simple thought made a deep impression upon me. It impelled me towards a theory of gravitation" [43]. In other words, the acceleration $g \simeq 9.8\,\mathrm{m\,s^{-2}}$ that we experience while sitting or standing on the surface of the Earth is due to us being in a non-inertial frame of reference. Take away the ground, say by jumping into a mineshaft, and the "force of gravity" goes away. The equivalence of inertial and gravitational mass follows naturally from the equivalence of uniform gravitational fields and uniform accelerations.

Einstein set out to incorporate this insight into a modification of special relativity that could account for gravitational effects. The connection to coordinate transformations suggested a geometrical approach, which caused Einstein to pay more at-

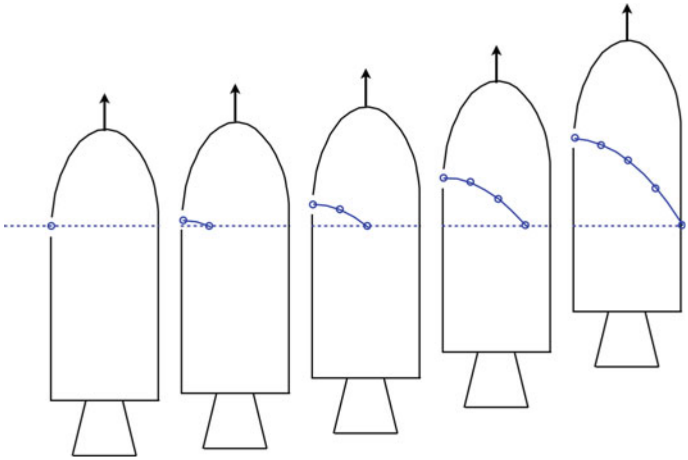

Fig. 3 Path of light as seen in a uniformly accelerated reference frame. In an inertial frame the light follows the straight-line dotted path, while in the accelerated frame of the rocket the path appears to follow a curved path shown here as a solid line

tention to Minkowski's geometrical formulation of special relativity. Einstein began by showing that the path of light seen by a uniformly accelerated observer could be interpreted in terms of spacetime geometry where the speed of light depends on position.

Consider the picture in Fig. 3 where a rocket accelerates from rest with uniform acceleration a in the positive z direction, and a photon traveling in the positive x direction enters a window in the rocket at time t. The photon follows the path $x = ct$, $z = 0$, shown as a horizontal dotted line, in the inertial frame where the rocket was originally at rest. A first the velocity of the rocket $v_z = at$ is much less than the speed of light, and the coordinates in the two frames are related such that $t' = t$, $x' = x$ and $z' = z - \frac{1}{2}at^2$. Thus, in the non-inertial frame of the rocket, the photon follows the parabola $z' = -ax'^2/(2c^2)$. Einstein showed that this "bending of light" could be derived from the line element for a uniformly accelerated observer, which to leading order in a has the form [18]

$$ds^2 = -c^2\left(1 + \frac{2az'}{c^2}\right)dt'^2 + dx'^2 + dy'^2 + dz'^2 \,. \tag{8}$$

To confirm that light paths in this geometry are indeed parabolas, we need to derive the geodesic equation, which describes the shortest/straightest paths in spacetime.

Using the short-hand notation $\mathbf{x} \rightarrow \{x^\mu\} = \{ct, x, y, z\}$ and $ds^2 = g_{\mu\nu}(\mathbf{x})$ $dx^\mu dx^\nu$, with summation implied on repeated indices, the path length between events A, B, is given by

$$S = \int_A^B \sqrt{-ds^2} = \int_{\lambda_A}^{\lambda_B} \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} d\lambda \equiv \int_{\lambda_A}^{\lambda_B} L\left(x^\mu, \frac{dx^\mu}{d\lambda}\right) d\lambda. \quad (9)$$

Holding the end points fixed and extremizing the path length yields the Euler–Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial(dx^\alpha/d\lambda)}\right) = \frac{\partial L}{\partial x^\alpha} \quad (10)$$

which evaluate to

$$\frac{d^2x^\alpha}{d\lambda^2} = -\frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \equiv -\Gamma^\alpha_{\mu\nu}\frac{dx^\nu}{d\lambda}\frac{dx^\nu}{d\lambda}\,. \quad (11)$$

Here commas denote partial derivatives $h_{,\mu} = \partial h/\partial x^\mu$, and the collection of metric derivatives appearing on the right-hand side of the above equation are referred to as the Christoffel symbol $\Gamma^\alpha_{\mu\nu}$. Note that $g^{\alpha\beta}$ denotes the components of the inverse metric tensor, so that $g^{\alpha\beta}g_{\alpha\kappa} = \delta^\alpha_\kappa$. The geodesic equation can be simplified by introducing the notation $u^\alpha = dx^\alpha/d\lambda$ for the 4-velocity and $\nabla_\beta u^\alpha = u^\alpha_{\ ,\beta} + u^\nu\, \Gamma^\alpha_{\beta\nu}$ for the covariant derivative. With these definitions we have $d/d\lambda = u^\alpha\nabla_\alpha$, and the geodesic equation takes the more compact form

$$u^\beta \nabla_\beta u^\alpha = 0\,. \quad (12)$$

Returning to the metric for a uniformly accelerated observer, we find $\Gamma^z_{tt} = a$ and all others zero. For a photon with initial velocity $\mathbf{U} \to (1, c, 0, 0)$ the geodesic equations integrate to give $t' = \lambda$, $x' = ct'$ and $z' = -\frac{1}{2}at'^2$. This confirms that photons do indeed follow parabolic paths in the $x' - z'$ spacetime with line element given in Eq. (8). The result can be generalized to describe uniform acceleration in any direction and uniform rotation about any axis. The line element for this non-inertial frame is given by (dropping the primes to simplify the notation)

$$ds^2 = -((c + \mathbf{a}\cdot\mathbf{x})^2 - (\boldsymbol{\omega}\times\mathbf{x})^2)dt^2 + 2c(\boldsymbol{\omega}\times\mathbf{x})_i dx^i dt + dx^2 + dy^2 + dz^2\,. \quad (13)$$

The convention being used here is that Roman indices run over spatial coordinates, while Greek indices run over time and space coordinates. To leading order in **a** and $\boldsymbol{\omega}$ the non-vanishing Christoffel symbols are:

$$\begin{aligned} \Gamma^i_{tt} &\simeq -g_{tt,i} = a^i + (\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{x}))^i \\ \Gamma^i_{tj} &\simeq \frac{1}{2}(g_{ti,j} - g_{tj,i}) = -c\,\epsilon_{ijk}\omega^k\,. \end{aligned} \quad (14)$$

and the geodesic equations yield

$$\frac{d^2\mathbf{x}}{dt^2} = -\mathbf{a} - 2\,\omega\times\mathbf{v} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{x})\,, \quad (15)$$

which recovers the form of the acceleration attributed to pseudo forces in a non-inertial frame. Turning this around, it is always possible to find a coordinate transformation to an inertial frame where the pseudo forces vanish. But this does not mean that we can simply transform gravity away.

2.3 Tides and Curvature

Einstein's trick for making gravity vanish only works in a small region of spacetime. It is impossible to remove the tidal forces that manifest over larger regions.

Suppose we do an experiment in a laboratory on the surface of the Earth as shown in Fig. 4. We can set up a coordinate system where the z axis points in the outward radial direction at the center of the lab, and the x, y directions span the floor of the lab. Now suppose that we release two masses from near the ceiling of the lab, the first with position vector relative to the center of the Earth given by $\mathbf{r}_1 = \mathbf{R} + \mathbf{x}_1$, where $\mathbf{R}$ is a vector connecting the center of the Earth to the floor of the lab, and the second with $\mathbf{r}_2 = \mathbf{R} + \mathbf{x}_2$. Initially the distance between the two masses is $L = |\Delta x| = |\mathbf{x}_2 - \mathbf{x}_1| = y_2 - y_1$. To leading order in $L/R_\oplus$ Newton's theory of gravity predicts a constant acceleration in the $-z$ direction:

$$\frac{d^2\mathbf{x_{1,2}}}{dt^2} = -\frac{GM_\oplus}{R_\oplus^2}\hat{z} = -g\hat{z}\,. \tag{16}$$

This part of the gravitational field can be transformed away by adopting a freely falling reference frame. Continuing the expansion of the Newtonian equations of motion to next order we encounter tidal forces in the y direction:

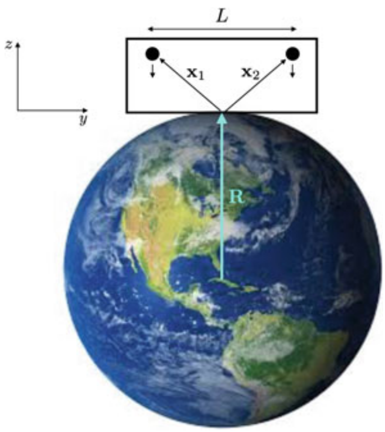

Fig. 4 Tidal forces in an Earth bound laboratory

$$\frac{d^2\mathbf{\Delta x}}{dt^2} = -\frac{GM_\oplus L}{R_\oplus^3}\hat{y}\,. \tag{17}$$

These non-uniform accelerations cannot be transformed away. Similarly, if one were to consider the motion of a single mass over an extended period of the time the local value of the acceleration g would change with time, and this change in acceleration can not be transformed away. In summary, the effects of gravity can only be transformed away across small regions of space for a short period of time.

Looking at the geodesic equation (11), we see that transforming away local acceleration terms is equivalent to setting the first derivatives of the metric equal to zero. It turns out that we always have enough coordinate freedom to set the components of the metric in the neighborhood of an event equal to the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, *and* to set the first derivatives equal to zero. However, there is not enough coordinate freedom to remove the second and higher derivatives. As shown by Riemann, the second derivatives of the metric describe the curvature of the spacetime. The components of the Riemann curvature tensor are given by

$$R^\kappa_{\mu\lambda\nu} = \Gamma^\kappa_{\mu\lambda,\nu} - \Gamma^\kappa_{\mu\nu,\lambda} + \Gamma^\alpha_{\mu\nu}\Gamma^\kappa_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda}\Gamma^\kappa_{\alpha\nu}\,. \tag{18}$$

The Christoffel symbols $\Gamma^\alpha_{\mu\nu}$, defined in Eq. (11), involve first derivatives of the metric and can be made to vanish at any point in spacetime, however their derivatives can not. In local free fall coordinates the metric components take the form

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}\Delta x^\alpha \Delta x^\beta + \cdots \tag{19}$$

This is nothing other than a Taylor series expansion about a point P, where the coordinates have been chosen such that $g_{\mu\nu,\lambda}|_P = 0$. These are variously refers to as free fall, locally Lorentzian or Riemann normal coordinates. These coordinates can be extended along the worldline of a particle to yield a locally non-rotating inertial frame defined by a set of four orthogonal basis vectors $\mathbf{e}_{(\gamma)}$. Here γ labels the basis vectors and should not be confused with the components of the basis vector which are labelled by a superscript: $e^\alpha_{(\gamma)}$. The locally non-rotating frame is carried along the worldline of the particle by Fermi–Walker transport:

$$u^\beta \nabla_\beta e^\alpha_{(k)} = e^\alpha_{(t)} g_{\mu\nu} e^\mu_{(k)} u^\beta \nabla_\beta e^\nu_{(t)}\,. \tag{20}$$

Fermi–walker transport keeps the basis vector $\mathbf{e}_{(t)}$ tangent to the worldline, as shown in Fig. 5.

While Fermi–Walker transport can eliminate "pseudo forces" along the worldline of a single particle, spacetime curvature prevents us from extending this inertial coordinate system globally. Spacetime curvature manifest as a tidal force that causes initially parallel geodesics to converge, diverge or twist about one another. Considering the geodesic equation for two nearby geodesics with separation vector $\boldsymbol{\xi}$. We find that spacetime curvature causes a relative acceleration: