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Ilijas Farah

Combinatorial Set Theory of C^* -algebras



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Combinatorial Set Theory of C^* -algebras



Springer

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*To the memory of my mother, Vera Gajić
and my uncle, Radovan Gajić.
Chapter 17 is dedicated to Colette.
Another smart, generous, and witty soul
gone too soon.*

Reader,

I Here put into thy Hands, what has been the diversion of some of my idle and heavy Hours: If it has the good luck to prove so of any of thine, and thou hast but half so much Pleasure in reading, as I had in writing it, thou wilt as little think thy Money, as I do my Pains, ill bestowed. Mistake not this, for a Commendation of my Work; nor conclude, because I was pleased with the doing of it, that therefore I am fondly taken with it now it is done.

John Locke, The Epistle to the Reader
An Essay Concerning Human Understanding, 1660

Well, that's about it for tonight ladies and gentlemen, but remember if you've enjoyed watching the show just half as much as we've enjoyed doing it, then we've enjoyed it twice as much as you. Ha, ha, ha.

Monty Python's Flying Circus, Episode 23, 'Scott of the Antarctic', 1970

Preface

This book is shorter than *In Search of Lost Time*, is easier to read than *Principia Mathematica*, and has more mathematical content than *War and Peace*.^{1,2} It provides an introduction to set-theoretic methods in the field of C^* -algebras, functional analysis, and general large metric algebraic structures. The main objects of the study are the two classes of C^* -algebras: (1) nonseparable but usually nuclear, and even approximately finite, C^* -algebras and (2) properties of massive quotient C^* -algebras such as coronas, ultraproducts, and relative commutants of separable subalgebras of massive algebras.

While writing this book, I had in mind four types of readers:

1. Graduate students who had already taken an introductory course in C^* -algebras and would like to learn set-theoretic methods
2. Graduate students who had already taken an introductory course in combinatorial set theory and would like to apply their knowledge to C^* -algebras
3. Graduate students who had taken a first course in functional analysis, and possibly a first course in mathematical logic (the latter can be replaced by ‘sufficient mathematical maturity’), and are interested in learning about set-theoretic methods in functional analysis, and C^* -algebras in particular
4. Mature mathematicians interested in learning about applications of set theory to C^* -algebras

This book can be used as a text for an advanced two-semester graduate course. Alternatively, one can use Chapters 1–8, Section 9.2, and Chapters 10 and 11 for a one-semester course on constructions of nonseparable C^* -algebras.

¹If you thought there wasn’t much mathematical content in *War and Peace* then you haven’t made it as far as the second epilogue, where the following sentence can be found: ‘Arriving at infinitesimals, mathematics, the most exact of sciences, abandons the process of analysis and enters on the new process of the integration of unknown, infinitely small, quantities’. Tolstoy proceeded to speculate on applications of calculus to history. This was written in the 1860s, barely 10 years after the birth of Riemann’s integral and full 80 years before Asimov’s ‘Foundation’.

²... and some of the jokes were not stolen from Douglas Adams.

Another alternative is to use Chapters 1 and 2, and all of Part III (except Section 12.5, which relies on Chapter 5) for a one-semester course on set-theoretic aspects of the Calkin algebra, other coronas, and ultraproducts.

Yet another possibility for a minicourse on representations of C^* -algebras would be to use only Chapters 1–5. This option involves no set theory, but it covers aspects of the representation theory of C^* -algebras not covered elsewhere.

If a course is given to students with a solid background in set theory, then all of Chapter 6 and parts of Chapters 7 and 8 should be omitted.

In the dual situation, when the audience consists of students with a solid background in C^* -algebras, Chapters 1–3 can be omitted.

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First of all, I should thank Paul Szeptycki and Ray Jayawardhana for kindly arranging a half-course teaching reduction in the fall 2016 semester that greatly helped in the preparation of this book. I would also like to thank Bruce Blackadar, Sarah L. Browne, George A. Elliott, Saeed Ghasemi, Eusebio Gardella, Bradd Hart, Se-Jin Sam Kim, Akitaka Kishimoto, Boriša Kuzeljević, Paul Larson, Mikkel Munkholm, Narutaka Ozawa, N. Christopher Phillips, Assaf Rinot, Ralf Schindler, Hannes Thiel, Andrea Vaccaro, Alessandro Vignati, and Beatriz Zamora–Aviles for their critical remarks on the early drafts. I am indebted to Bruce Blackadar, Se-Jin Sam Kim, and Narutaka Ozawa who provided a significant mathematical input, including simple proofs of Lemma 2.3.11, Example 2.4.5, and Lemma 3.1.13 (B.B.) and Lemma 1.10.7, Theorem 1.10.8, Lemma 3.2.10, and Theorem 3.2.9, as well as Lemma 3.4.3 and its proof (N.O.). Special thanks to Chris Schafhauser for the occasional illuminating remark. I am indebted to my two wonderful editors: Eugene Ha, who made me start this project (he is forgiven), and Elizabeth Loew, for expertly and patiently navigating me throughout this endeavour. While we are at the editors, many thanks to Assaf Rinot for suggesting that I try using Texpad; it made writing the last few sections of this text feel even more drastically different than Richard Strauss’s writing ‘An Alpine Symphony’. Most of this book had been written using certain well-known $\text{\LaTeX}$ editor that shall remain nameless.³ Last but not least, I owe special thanks to my daughter, Gala, for her impeccable and generous linguistic support.⁴

Toronto, ON, Canada
July 4, 2019

Ilijas Farah

³This occasionally did feel like ‘a job that, when all’s said and done, amuses me even less than chasing cockroaches’—which is how Strauss described the process of writing the said piece.

⁴I claim credit for the remaining mistakes, obscurities, and all missing or misplaced articles in particular.

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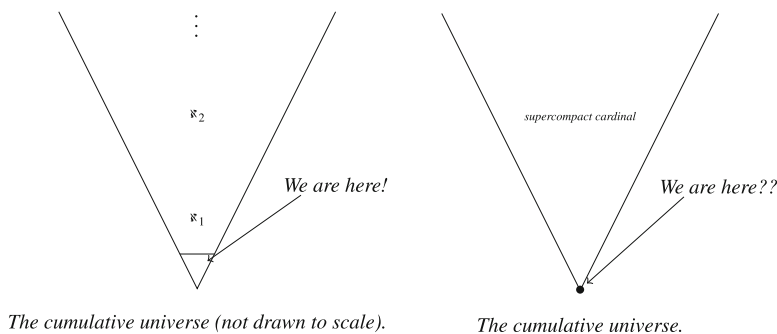
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Why Bother?

In his 1925 PhD thesis, John von Neumann defined the cumulative hierarchy by transfinite recursion on the ordinals. The 0th level is the empty set, the successor of a given level is its power set, and a limit level is equal to the union of all preceding levels (see Section A.5). Virtually, all of mathematics as we know it (and then some) takes place within the first ten infinite levels of this hierarchy.



So why do we need all these sets?

The use of abstract set theory in mathematics can be compared to the analytic number theory, where analytic methods are applied to prove statements about natural numbers. Or think of the definition of cohomology, using uncountable free groups. The only difference is that the gap between the cardinalities of objects considered and tools used can be much larger and that it is known that in many situations, this is necessarily so. Here are a few examples.

Take $n \geq 1$ and a Borel subset X of $\mathbb{R}^{n+2}$. Let Y be the projection of X to $\mathbb{R}^{n+1}$. Let Z be the projection of $\mathbb{R}^{n+1} \setminus Y$ to $\mathbb{R}^n$. Can one prove that Z is Lebesgue measurable for every choice of the Borel set X ? In Gödel's constructible universe L , the answer is negative, and with $n = 2$, one can even choose X so that Z is a

well-ordering of the reals. However, the *existence* of a very large cardinal called measurable cardinal⁵ implies a positive answer. This is the question of Lebesgue measurability of Σ_2^1 sets of reals (see [148, 216]).

The assertion that there are uncountably many infinite cardinals cannot be proved in Zermelo's original axiomatization of set theory Z . This can be proved by using the replacement axiom, added by Fraenkel, to obtain the Zermelo–Fraenkel set theory (ZF). *Borel Determinacy* asserts that natural two-player games of perfect information with a Borel payoff are determined. Borel Determinacy cannot be proved in Z [108]. It was proved in ZF by Martin (see [151]) using transfinite iteration of the power set operation. The interplay between the Axiom of Determinacy and large cardinals that dwarf measurables provides one of the most fascinating justifications of the higher set theory (see [148] and [265]).

Laver used one of the strongest large cardinal assumptions not known to lead to a contradiction (the existence of a nontrivial elementary embedding of a rank-initial segment of von Neumann's universe into itself) to solve a problem about algebras satisfying the left-distributive law $a(bc) = (ab)(ac)$. This assumption has been removed, but large cardinals provided a natural route towards the solution (see [53]).

Last but not least, some questions about operator algebras on a (separable!) Hilbert space can be answered only by using abstract set theory. Read on.

⁵The extent of largeness of measurable cardinals is the subject of Exercise 16.8.32.

Introduction for Experts

Some papers should be seen as territorial claims, not instruments of instruction.

A.R.D. Mathias

A C^* -algebra is a subalgebra of $\mathcal{B}(H)$, the algebra of all bounded linear operators on a complex Hilbert space H , closed under the formation of adjoints and the norm topology. A *von Neumann algebra* is a unital subalgebra of $\mathcal{B}(H)$ which is closed in the weak operator topology. The study of von Neumann algebras, under the name of ‘rings of operators’, was initiated in the 1930s and 1940s in the work of Murray and von Neumann. The study of C^* -algebras began in the 1940s by a result of Gelfand and Naimark stating that a complex Banach algebra with an involution is isomorphic to a C^* -algebra if and only if it satisfies the C^* -equality $\|aa^*\| = \|a\|^2$. It has since expanded to touch much of modern mathematics, including number theory, geometry, ergodic theory, mathematical physics, and topology.

Set theory is an area of mathematical logic concerned with the foundational aspect of mathematics and to some extent (but by no means exclusively) independence results. Gödel’s Incompleteness Theorem implies that no consistent and recursive set of axioms that extends the theory of natural numbers can decide every statement expressible in its language. Therefore, some statements of number theory can be neither proved nor refuted on the basis of the standard Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC). Gödel’s statements code metamathematical statements and are unnatural from the point of view of number theory. In spite of Gödel’s Theorem, parts of core mathematics are generally considered immune to set-theoretic independence, and such independent statements in the field of operator algebras were found only recently. The answers to some prominent and long-standing open problems on operator algebras are independent from ZFC, and this is one of the main themes of this text.

In the past century, the connection between logic and operator algebras was sparse albeit fruitful. In this text, we present one aspect of this progress that brought two subjects closer together. Our main goal is to give a self-contained and as elementary as possible ‘instrument of instruction’ for set-theoretic methods used in the past 15 years to resolve several long-standing problems in the theory of

C^* -algebras. Along the way, we provide an introduction to several loosely related topics, such as the representation theory of C^* -algebras, massive C^* -algebras such as coronas, asymptotic sequence algebras (reduced products), and ultraproducts, as well as infinitary combinatorics and its applications to functional analysis.⁶

The problems from the field of operator algebras that were resolved using set theory generally fall into one of the two following categories.

Nonseparable Examples

The problems in the first group ask whether all operator algebras with a certain property P also satisfy a related property, Q . Two textbook examples of such problems are Dixmier's 1967 problems, asking is every unital inductive limit of full matrix algebras isomorphic to a tensor product, and is every C^* -algebra that locally looks like a full matrix algebra isomorphic to an inductive limit of full matrix algebras? (In technical terms, is every unital approximately matricial (AM) C^* -algebra is uniformly hyperfinite (UHF), and is every unital locally matricial (LM, or matroid) C^* -algebra approximately matricial?) Yet another example is Naimark's problem, asking whether a C^* -algebra all of whose irreducible representations are unitarily equivalent is isomorphic to the algebra of compact operators on some Hilbert space. A problem closely related to Naimark's is whether Glimm's Dichotomy, asserting that the number of unitary equivalence classes of irreducible representations of a separable and simple C^* -algebras is either 1 or not smaller than $\mathfrak{c}$ ⁷ holds for all simple C^* -algebras. Another problem asks whether every amenable norm-closed algebra of operators on a Hilbert space is isomorphic to a C^* -algebra. All of these problems, except possibly the last one, have positive solutions when restricted to separable C^* -algebras.

In the simplest form of a set-theoretic resolution to a problem of this sort, an example is *defined* from concrete parameters such as cardinals or real numbers. Both Dixmier's problems fall into this class, with counterexamples of minimal density characters⁸ equal to the first two uncountable cardinals, $\aleph_1$ and $\aleph_2$, respectively.

Some other problems in the first group are solved by a recursive construction of transfinite length. An example was provided by Weaver's construction of a prime, but not primitive, C^* -algebra. Crabb and Katsura later simplified Weaver's construction and provided a definition of such an algebra, thus 'upgrading' (or perhaps downgrading?) the solution of this problem to the first class. The problem of the existence of an amenable norm-closed algebra of operators on a Hilbert space

⁶Students and non-experts, please proceed to 'Annotated Contents' on page xx or even straight to 'Prerequisites and Appendices' on page xxv and later use the earlier pages of this introduction as a reference.

⁷ $\mathfrak{c} := 2^{\aleph_0}$, the cardinality of the real line.

⁸The *density character* of a topological space is the smallest cardinality of a dense subset.

not isomorphic to a C^* -algebra is resolved by *defining* a counterexample, but the definition uses as a parameter a so-called Luzin family of subsets of $\mathbb{N}$, constructed by transfinite recursion.

In the third subgroup of examples, the algebras are again constructed by a transfinite recursive construction, but the construction is facilitated by a diagonalization principle independent from the standard ZFC axioms of set theory such as the Continuum Hypothesis or its strengthening, Jensen's diamond $\diamond_{\aleph_1}$. The latter is used to construct counterexamples to Naimark's problem and Glimm's Dichotomy for C^* -algebras of density character $\aleph_1$. More precisely, for every $1 \leq n \leq \aleph_0$, we construct a simple C^* -algebra with exactly n irreducible representations (up to the unitary equivalence) that is not isomorphic to the algebra of compact operators on a Hilbert space. This algebra can be chosen to be nuclear, stably finite, and even approximately finite.

Properties of Massive Quotients

Another, even more exciting and more fundamental, group of questions is concerned with properties of familiar, canonical, examples of C^* -algebras. Explicitly defined C^* -algebras, some of whose essential properties may depend on set theory, are unlikely to be separable.⁹ The massive quotient algebras are the most likely candidates for having a 'set-theoretically malleable' theory. This unruliness is closely related to the fact observed in both noncommutative geometry and descriptive set theory, that the quotient spaces are frequently intractable. The simplest, and most established, example of a massive quotient C^* -algebra is the quotient of the algebra of bounded linear operators on $H = \ell_2(\mathbb{N})$ over the ideal of compact operators. This is the Calkin algebra, $\mathcal{Q}(H)$.

Motivated by their seminal work on extensions of separable C^* -algebras by the algebra of compact operators, Brown, Douglas, and Fillmore asked whether $\mathcal{Q}(H)$ has an outer automorphism. By results of Phillips–Weaver and the author, the answer to this question is independent from ZFC.

Separable C^* -algebras and the General Theory

While the main goal of this text is to present the theory of nonseparable C^* -algebras, a substantial amount of space and effort is devoted to the theory of C^* -algebras that does not explicitly involve set theory. In addition, a fair portion of this theory applies only to separable C^* -algebras.

⁹This is essentially a consequence of Shoenfield's Absoluteness Theorem and standard descriptive set-theoretic coding arguments (see Theorem B.2.12).

This choice was guided by two rationales. First, students may appreciate having all the required information in a single volume. The second one is more substantial. Many of the standard results needed in the latter parts of this text cannot be easily found elsewhere, and in at least one case, a complete proof hasn't been available at all until now. A list of these results follows.

The strong homogeneity of the pure state space of every separable C^* -algebra is Theorem 5.6.1. A rough sketch of a proof of this theorem was given by Akemann and Weaver in [6], combining techniques of [110, 161], and [121].¹⁰ We give a complete proof, including Kishimoto's rather elementary proof [159] of Haagerup's result on the existence of an approximate diagonal of an irreducible representation of a C^* -algebra (Theorem 5.1.2). The Wolf–Winter–Zacharias Structure Theorem for completely positive maps of order zero is Theorem 3.2.9. Theorem 12.3.2 is the Johnson–Parrott Theorem about derivations of C^* -algebras (or rather its consequence that the image of any masa in $\mathcal{B}(H)$ under the quotient map is a masa in the Calkin algebra). The Akemann–Anderson–Pedersen Theorem on excision of pure states is proved in Theorem 5.2.1. We use excision to provide a proof of Kirchberg's Slice Lemma.¹¹ Excision also forms the basis for the theory of noncommutative analogs of ultrafilters, known as maximal quantum filters (see Section 5.3). They are used to facilitate the use of set-theoretic methods in the study of pure states on C^* -algebras. A proof of Ulam stability of ε -representations of compact groups and ε -*-homomorphisms between finite-dimensional C^* -algebras is in Section 17.2.

Many of the proofs have been taken apart, have had all of their gears oiled, and were then reassembled. One example is Theorem 3.7.2, Glimm's Theorem that every non-type I C^* -algebra has a subalgebra with a quotient isomorphic to the CAR algebra.¹²

Annotated Contents

Part I is about C^* -algebras.

In Chapter 1, we introduce the abstract C^* -algebras and work towards the Gelfand–Naimark–Segal Theorem (Theorem 1.10.1). Along the way, we discuss abelian C^* -algebras and Gelfand–Naimark and Stone dualities, continuous functional calculus, positivity in C^* -algebras, approximate units, and quasi-central approximate units.

¹⁰As I was writing these lines, a more detailed exposition of this proof became available in [242].

¹¹This proof is adapted from [206], where a proof of the excision of pure states in the case of purely infinite and simple C^* -algebras can be found.

¹²This result can be found, for example, in the classic [192], but the proof given here is somewhat different.

In Chapter 2, we give basic examples of constructions C^* -algebras: direct sums and products, inductive limits, stabilization, suspension, cone, hereditary C^* -subalgebras, quotients, tensor products, full and reduced group C^* -algebras, and full and reduced crossed products. Finite-dimensional C^* -algebras and $*$ -homomorphisms between them are classified by Bratteli diagrams. Universal C^* -algebras given by generators and relations are studied in some detail. After a discussion of automorphisms of C^* -algebras, we conclude with a section on C^* -algebras of real rank zero.

Chapter 3 starts with a telegraphic introduction to von Neumann algebras. We also prove the Stinespring and Wolf–Winter–Zacharias Structure Theorems for completely positive maps and completely positive maps of order zero, respectively, and introduce averaging techniques and conditional expectations. We prove the Kadison Transitivity Theorem and its generalization due to Glimm–Kadison. After studying pure states and equivalence relations on the space of pure states of a C^* -algebra (unitary/spatial equivalence and conjugacy by an automorphism), we conclude with a study of the second dual of a C^* -algebra.

In Chapter 4, we adapt a technique, borrowed from the theory of II_1 factors, of juxtaposing the GNS Hilbert space structure associated with a tracial state and the C^* -algebra structure to study reduced group C^* -algebras. An emphasis is given to the C^* -algebras associated to free products of groups. We give basic norm estimates for the elements of a group algebra and present basics of Powers groups and criteria for simplicity of reduced group C^* -algebras. The chapter concludes with a study of normalizers of diffuse masas.

Chapter 5 starts with Kishimoto’s construction of approximate diagonals. We then prove the Akemann–Anderson–Pedersen Theorem on excision of pure states and apply it to prove Glimm’s Lemma and Kirchberg’s Slice Lemma. Excision is also used to exhibit a bijection between ‘maximal quantum filters’ and pure states of a C^* -algebra. The maximal quantum filters are used to study extensions of pure states. The chapter concludes with a proof of the Kishimoto–Ozawa–Sakai Theorem on the homogeneity of the pure state space of separable C^* -algebras.

In Part II, we introduce set-theoretic tools and apply them to C^* -algebras.

Chapter 6 is devoted to infinitary combinatorics: club filter, nonstationary ideal, the pressing down lemma, variants of the Δ -system lemma, and Kueker’s Structure Theorem for clubs in $[X]^{\aleph_0}$.

In Chapter 7, we introduce continuous variants of the results from Chapter 6 in which $[X]^{\aleph_0}$, the family of countably infinite subsets of a fixed uncountable set X , is replaced with $\text{Sep}(A)$, the family of all separable substructures of a nonseparable metric structure A . The latter directed set is σ -directed, but not concretely represented, and in some cases, only the approximate versions of results studied in Chapter 6 hold. These approximate versions are used to reflect properties of large C^* -algebras to their separable subalgebras and prove that algebras indistinguishable by any of the standard K-theoretic invariants are not isomorphic. The spaces of models—both discrete and metric—are studied in Section 7.1. Proposition 7.2.9 is a metric variant of the Pressing Down Lemma.

Chapter 8 is devoted to the additional set-theoretic axioms used in this text. They are literally treated as axioms—no attempt has been made to prove their relative consistency with ZFC. We start with rather standard, brief, and self-contained treatments of the Continuum Hypothesis and Jensen’s $\diamond_{\aleph_1}$ and use $\diamond_{\aleph_1}$ to construct a Suslin tree. The σ -complete directed systems of isomorphisms between separable substructures are studied in some detail. The axiom asserting that a Polish space cannot be covered by fewer than $\mathfrak{c}$ meager subsets is used to construct a selective ultrafilter on $\mathbb{N}$. The chapter ends with a discussion of the Ramseyan axiom OCA_T and some of its applications.

Several themes picked up later on in this text originate in Chapter 9. The first one is the structure of the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{Fin}$ and related quotient structures. The interplay between separability of $\mathcal{P}(\mathbb{N})$ (identified with the Cantor space) and countable saturation of $\mathcal{P}(\mathbb{N})/\text{Fin}$ is used to construct several objects witnessing the incompleteness of $\aleph_1$, such as the independent families, almost disjoint families, and gaps in $\mathcal{P}(\mathbb{N})/\text{Fin}$. In the latter sections, this Boolean algebra is injected into massive corona C^* -algebras. This is used to construct subalgebras of $\mathcal{B}(H)$ with unexpected properties, such as an amenable norm-closed algebra of operators on a Hilbert space not isomorphic to a C^* -algebra (Section 15.5), and Kadison–Kastler near, but not isomorphic, C^* -algebras (Section 14.4). We introduce the Rudin–Keisler ordering on the ultrafilters and construct Rudin–Keisler incomparable nonprincipal ultrafilters on $\mathbb{N}$. The basics of the Tukey ordering of directed sets are presented in Section 9.6. We prove that two directed sets are cofinally equivalent if and only if they are isomorphic to cofinal subsets of some directed set. We study the directed set $\mathbb{N}^{\mathbb{N}}$, the associated small cardinals $\mathfrak{b}$ and $\mathfrak{d}$, and two directed sets cofinally equivalent to $\mathbb{N}^{\mathbb{N}}$ used to stratify the Calkin algebra, $\text{Part}_{\mathbb{N}}$ and Part_{ℓ_2} . This chapter ends with a convenient and well-known structure result for comeager subsets of products of finite spaces.

In Chapter 10, infinitary combinatorics is applied to study graph CCR algebras. These ‘twisted’ reduced group C^* -algebras associated to a Boolean group and a cocycle given by a graph are AF (approximately finite) and even AM (approximately matricial) if they are simple (i.e. if they have no proper norm-closed, two-sided ideals). After developing structure theory, we recast results of the author and Katsura and show that in spite of their simplicity (here ‘simplicity’ stands for ‘lack of complexity’), graph CCR algebras provide counterexamples to several conjectures about the structure of simple, nuclear C^* -algebras. We construct an AM C^* algebra that is not UHF, but it has a faithful representation on a separable Hilbert space. In every uncountable density character κ , there are 2^κ nonisomorphic graph CCR algebras with the same K-theoretic invariants as the CAR algebra. By using an independent family of subsets of $\mathbb{N}$, we construct a simple graph CCR algebra that has irreducible representations on both separable and nonseparable Hilbert spaces.

Other examples of nonseparable C^* -algebras are constructed in Chapter 11. We start with Akemann’s C^* -algebra with no abelian approximate unit. This is followed by a strengthening of a result of Akemann and Weaver due to the author and Hirshberg. It gives a counterexample to Glimm’s Dichotomy for nonseparable C^* -algebras: for every $n \leq \aleph_0$, there exists a simple C^* -algebra of density character $\aleph_1$

with exactly n unitarily inequivalent irreducible representations. The case $n = 1$ is a counterexample to Naimark's problem. These results use Jensen's $\diamond_{\aleph_1}$, and it is not known whether they can be proved in ZFC. The chapter concludes with a study of $C_r^*(F_\kappa)$, the reduced group algebra of the free group with κ generators. For every κ , this C^* -algebra has only separable abelian C^* -subalgebras (Popa), and every two pure states are conjugate by an automorphism (Akemann–Wassermann–Weaver). Both results apply to $C_r^*(\Gamma)$, where Γ is the free product of any family of nontrivial countable groups.

Part III of this text is devoted to the Calkin algebra and other massive quotient structures—coronas, ultraproducts, asymptotic sequence algebras, and relative commutants of their separable C^* -subalgebras. A mix of set-theoretic, model-theoretic, and operator-algebraic techniques provides means for a unified treatment of these C^* -algebras.

In Chapter 12, we introduce the Calkin algebra $\mathcal{Q}(H)$ and establish the parallel between the poset of its projections and the quotient Boolean algebra $\mathcal{P}(\mathbb{N})/\text{Fin}$. We prove the Weyl–von Neumann–Berg–Sikonia Theorem on the existence of diagonalized liftings of singly generated abelian C^* -subalgebras of $\mathcal{Q}(H)$. In the separable case, the only obstructions to the existence of diagonalized liftings are K-theoretic. In the nonseparable case, examples can be provided by a simple counting argument or by a more profound Luzin-type construction of a ‘twist’ of projections in $\mathcal{Q}(H)$. We prove the Johnson–Parrot Theorem, that the image of a masa in $\mathcal{B}(H)$ under the quotient map is a masa in $\mathcal{Q}(H)$. Pure states on $\mathcal{Q}(H)$ are the noncommutative analogs of nonprincipal ultrafilters on $\mathbb{N}$, and this connection is brought forth by the language of maximal quantum filters. A recursive construction of maximal quantum filters facilitated by the Continuum Hypothesis is used to construct non-diagonalizable pure states on $\mathcal{Q}(H)$, refuting a conjecture of Anderson and giving a negative answer to a problem of Kadison and Singer; this is a theorem of Akemann and Weaver. We offer additional proofs of this theorem from two weakenings of the Continuum Hypothesis ($\text{cov}(\mathcal{M}) = \mathfrak{c}$ and $\mathfrak{d} \leq \mathfrak{t}^*$) that between them cover many ‘common’ models of ZFC. No ZFC construction of a nondiagonalizable pure state on $\mathcal{Q}(H)$ is presently known.

In Chapter 13, we study multiplier algebras and coronas of non-unital C^* -algebras. These are the noncommutative analogs of the Čech–Stone compactification and the Čech–Stone remainder, respectively, of a locally compact Hausdorff space.

Gaps in coronas are studied in Chapter 14. We show that the rich and well-studied gap spectrum of $\mathcal{P}(\mathbb{N})/\text{Fin}$ embeds into the corona of every σ -unital, non-unital C^* -algebra. This is used to prove two incompleteness results. The Choi–Christensen construction of Kadison–Kastler near, but non-isomorphic, C^* -algebras is recast in terms of gaps: every gap in the Calkin algebra can be used to produce a family of examples of this sort. Every uniformly bounded representation of a countable, amenable group in the Calkin algebra is unitarizable. Using a Luzin family, one defines a uniformly bounded, non-unitarizable representation of $\bigoplus_{\aleph_1} \mathbb{Z}/2\mathbb{Z}$ in the Calkin algebra. This example yields an amenable operator algebra not isomorphic to a C^* -algebra. This is a result of the author, Choi and Ozawa.

In Chapter 15, we study the overarching concept of countable degree-1 saturation and prove a theorem of the author and Hart that that all massive quotient C^* -algebras (coronas of σ -unital, non-unital C^* -algebras, ultraproducts associated with a nonprincipal ultrafilter on $\mathbb{N}$, and relative commutants of separable C^* -subalgebras of countably degree-1 saturated C^* -algebras) have this property. Countable degree-1 saturation subsumes several separation properties of massive C^* -algebras with neat acronyms, such as Pedersen’s SAW^* , $CRISP$, and $AA-CRISP$, also sub-Stonean and Kirchberg’s σ -sub-Stonean C^* -algebras, and those C^* -algebras satisfying the conclusion of Kasparov’s Technical Lemma. Many of these properties are analogs of the absence of gaps with countable sides in $\mathcal{P}(\mathbb{N})/\text{Fin}$. Among other applications, we prove that countably degree-1 saturated C^* -algebras are essentially non-factorizable (Section 15.4.3), that every uniformly bounded representation of a countable amenable group into such C^* -algebra is unitarizable (Section 15.4.1; this fails for uncountable groups, Section 14.5), and that such C^* -algebras admit a poor man’s version of Borel functional calculus that generalizes the Brown–Douglas–Fillmore ‘Second Splitting Lemma’ (Section 15.4.2).

In Chapter 16, we use continuous model theory to study ultraproducts and asymptotic sequence algebras (i.e. reduced products associated with the Fréchet filter). The Fundamental Theorem of Ultraproducts (Łoś’s Theorem) and the corresponding result for reduced products, Ghasemi’s Feferman–Vaught Theorem, are proved for arbitrary metric theories. These theorems are used to prove the countable saturation of ultraproducts associated with countably incomplete ultrafilters and the countable saturation of reduced products associated with the Fréchet filter. The σ -complete back-and-forth systems of partial isomorphisms between metric structures (introduced in Section 8.2) are used in Section 16.7 to prove that the Continuum Hypothesis implies all ultrapowers and all relative commutants of a separable C^* -algebra associated with nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic.¹³ The chapter ends with theorems due to Ge–Hadwin and the author, Hart, and Sherman, in which a large number of (outer) automorphisms of ultrapowers, asymptotic sequence algebras, and related massive C^* -algebras are constructed using the Continuum Hypothesis.

Chapter 17 begins with a rather elementary proof of the Phillips–Weaver Theorem: the Continuum Hypothesis implies that the Calkin algebra has outer automorphisms. The analogous result, due to Coskey and the author, is proved for the coronas of all stable, σ -unital C^* -algebras. This is followed by a proof of the Burger–Ozawa–Thom Theorem on Ulam stability of ε -homomorphisms, used to prove an Ulam stability result for ε -*-homomorphisms whose domain is a finite-dimensional C^* -algebra. The final sections of this chapter (and the text) are devoted to a complete proof that OCA_T implies that all automorphisms of the Calkin algebra are inner.

¹³This assertion is equivalent to the Continuum Hypothesis. In order to keep the cardinality of the set of pages of this text within reasonable limits, the proof of the converse is only outlined.

Prerequisites and Appendices

The reader is assumed to have taken a standard one-semester first course in functional analysis. This subsumes some familiarity with the basic point-set topology: compactness, Hausdorffness, nets, and Cauchy nets. Paracompactness is used exactly once in this text. The reader is also assumed to be familiar with rudimentary axiomatic and naive set theory. Some acquaintance with model theory and logic of metric structures is helpful, but not necessary (except in Chapter 16, but this chapter is an end in itself). To be specific, most uses of model theory outside of Chapter 16 can be summarized in three words, ‘Löwenheim–Skolem Theorem’, and the readers who would rather call it ‘Blackadar’s method’ are by all means welcome to do so. In addition, all uses of Łoś’s Theorem for ultraproducts relevant to us can be proved with one’s bare hands. However, one can also solve any cubic equation using Tartaglia’s original algorithm.

The appendices contain brief reviews of the axiomatic and naive set theory (Appendix A), descriptive set theory (Appendix B), functional analysis (Appendix C), and model theory (Appendix D).¹⁴

Notation

Our notation is mostly standard is what I wish I could say at this point. Alas! This text attempts to bridge the gap between two sophisticated areas of mathematics, each of which has its own (often idiosyncratic) notation and terminology. The best that I can do is provide a list of notational conventions and conflicts.

The issue of choosing the right fonts and symbols cannot be overestimated—one of the most serious criticisms of [52] that I am aware of is that ‘the author used all the wrong fonts’. I will mostly refrain from using ω (see Exercise 1 and Exercise 2), with two exceptions. The vector state associated to a unit vector ξ will be denoted ω_ξ . I will write $\mathbb{N}$ instead of ω almost everywhere; in transfinite constructions, ω will denote the least infinite ordinal.

Greek letters ξ , η , and ζ , denote vectors in a Hilbert space except in the appendix, where they denote ordinals. Thus, ω_ξ stands for the vector state associated with vector ξ throughout this text, except in Appendix A (see page xxviii, Exercise 2). The letters ε and δ will stand for small but positive real numbers, with one exception. The standard δ symbol is defined by $\delta_{x,y} = 1$ if $x = y$ and $\delta_{x,y} = 0$ otherwise.

¹⁴An early draft of this text contained an extensive appendix on absoluteness. I am convinced that understanding absoluteness is necessary for understanding the role (and the limitations) of set theory as we presently know it. Nevertheless, the insane idea of cramming this sensitive material into an appendix to a 500 page book that already contained diverse, and sometimes technically demanding, material has been abandoned. I owe special thanks to Matt Foreman for providing the voice of reason.

The symbol π (sometimes embellished with subscripts) is used to denote representations of C^* -algebras, projections from a Cartesian product to its components, various quotient maps, and, last but not least, the area of the unit disk. Symbols m, n, i, j, k , and l stand for natural numbers, with an occasional $i = \sqrt{-1}$.

The asterisk is even more overused than ω or π (at least in operator algebras). In addition to having a^* denote the adjoint operation, X^* denote the dual of a Banach space, and assorted C^* s and W^* s denote self-adjoint operator algebras, I will use the standard set-theoretic notation $X \subseteq^* Y$ for ‘the difference $X \setminus Y$ is finite’.

We shall be using $|a|$ only to denote the absolute value of a scalar, the absolute value of a function or an operator, and the cardinality of a set.¹⁵

Some of the rules for the assignment of fonts to data types are described in the following lines.

Ultrafilters are assumed to be nonprincipal (or free) ultrafilters on $\mathbb{N}$, and they will be denoted $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$.

Blackboard bold font is used for sets of numbers, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{T}$, and $\mathbb{D}$, where

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\} \text{ and } \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

The remaining symbols, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, are I believe standard enough (but note that $0 \in \mathbb{N}$ and $\omega = \mathbb{N}$ when convenient).

For abstract sets, I use the sans-serif font: A , X , Y , s , and t . This font is also used to denote distinguished sets of operators or functionals associated to a C^* -algebra A : $S(A)$ is the set of states on A , $P(A)$ is the space of pure states in A , and $U(A)$ is the unitary group of A . In order to avoid confusion with the power set $\mathcal{P}(A)$ and the set of pure states $P(A)$, the poset of projections of a C^* -algebra A is denoted $\text{Proj}(A)$.

Capital letters A , B , C , and D will usually denote C^* -algebras, and capital letters M and N will usually denote von Neumann algebras or multiplier algebras of non-unital C^* -algebras. Operators are denoted by lower case letters, mostly a , b , c , and d , but in some of the more complex arguments, we use up a fair portion of the alphabet. When venturing into operator theory and talking about concrete operators on a Hilbert space that do not belong to any given operator algebras, we denote them with capital Roman letters R , S , T , $\dots$. Capital letters in fraktur font are used to denote structures (in the model-theoretic sense), both discrete and metric: $\mathfrak{A}$, and $\mathfrak{B}$. The domains of these structures are denoted by the corresponding letters in sans serif font, so that the domain of $\mathfrak{A}$ is denoted A , the domain of $\mathfrak{B}$ is denoted B , and so on. This convention is used only until the distinction has been made very clear. Small fraktur font is reserved for small cardinals associated with the continuum.

And now, for the good news, the fruitful interaction between operator algebras and descriptive set theory is reflected in some common terminology. In both

¹⁵Indeed, N.C. Phillips pointed out that the absolute value signs are even more overused in mathematics than the asterisk.

subjects, *analytic* sets are continuous images of Borel sets, and an equivalence relation is *smooth* if the quotient Borel space is standard. Unlike ‘normal’, the word ‘compact’ has, to the best of my knowledge, the same meaning throughout all of mathematics. ‘Weakly compact’ will mean ‘compact in some weak topology’ or ‘compact in the weak operator topology’; weakly compact cardinals haven’t been used in the field of C^* -algebras, yet.

The ε -ball centred at x in a metric or normed space is denoted

$$B_\varepsilon(x) := \{y : d(x, y) < \varepsilon\}.$$

If X and Y are subsets of a metric space, I write $X \subseteq_\varepsilon Y$ if $\inf_{y \in Y} d(x, y) < \varepsilon$ for all $x \in X$. For elements x and y of a metric space, $x \approx_\varepsilon y$ stands for $d(x, y) < \varepsilon$. (In spite of the suggestive notation, this is certainly not an equivalence relation.)

We write $F \Subset A$ for ‘ F is a finite subset of A ’.

It will be convenient to use the following two quantifiers:

$(\forall^\infty n)$ stands for $(\exists m \in \mathbb{N})(\forall n \geq m)$

$(\exists^\infty n)$ stands for $(\forall m \in \mathbb{N})(\exists n \geq m)$.

Following a convention going back to von Neumann, an ordinal is identified with the set of smaller ordinals, and natural numbers are identified with finite ordinals: $0 := \emptyset$, $1 := \{\emptyset\}$, and $n = \{0, \dots, n-1\}$ for all $n \in \mathbb{N}$.¹⁶ This is but one reason why it is important to distinguish between $f(\mathbf{X})$ and

$$f[\mathbf{X}] := \{f(x) : x \in \mathbf{X}\}.$$

The characteristic function of a set $\mathbf{X}$ (considered as a subset of some fixed set clear from the context) is denoted $\chi_{\mathbf{X}}$. (Some authors, and functional analysts in particular, use $1_{\mathbf{X}}$, but in this text, 1_A is reserved for the unit of a C^* -algebra A .)

Symbols for index sets are omitted whenever this is convenient both for myself and—to the best of my knowledge—for the reader. I will interchangeably write $(b_j : j \in \mathbb{J})$, $(b_j)_j$, or even (b_j) when the index-set is clear from the context. The same remark applies to standard abbreviations such as $\prod_{\mathcal{A}} A_j$ for the ultraproducts and $\bigotimes_{\mathbb{N}} A_n$ or $\prod_n A_n$ for products. The symbol $\lim_n$ stands for $\lim_{n \rightarrow \infty}$, and $\lim_\lambda$ stands for $\lim_{\lambda \rightarrow \Lambda}$ if Λ is a net clear from the context.

Apart from the hopefully innocuous conventions described in the previous paragraph, between redundancy and confusion, I systematically choose redundancy.¹⁷ Every ordered set is therefore either ‘linearly ordered’ (this is synonymous to ‘totally ordered’) or ‘partially ordered’. A relation is a *quasi-ordering* if it is transitive but not necessarily antisymmetric.

¹⁶The line has to be drawn somewhere; I avoid writing $(\forall j \in n)$ in place of $(\forall j < n)$.

¹⁷With apologies to George Elliott.

Exercises

As an ice-breaker, I provide a multiple-choice quiz.

Exercise 1 What does R^ω stand for?

1. An ultrapower of the hyperfinite II_1 factor R associated to a free ultrafilter ω on $\mathbb{N}$
2. The space of all sequences $(r_n : n \in \omega)$ of real numbers, where ω denotes the least infinite ordinal, identified with $\mathbb{N}$ (and yes, zero is a natural number)

Exercise 2 What does ω_ξ stand for?

1. The vector state on $\mathcal{B}(H)$ associated with a vector ξ in the Hilbert space H , in symbols $\omega_\xi(a) := (a\xi|\xi)$.
2. The ξ th infinite cardinal, also denoted $\aleph_\xi$, where ξ is an ordinal and counting starts at 0.

Exercise 3 What is the meaning of ‘ φ is a contraction’?

1. $\|\varphi(x) - \varphi(y)\| \leq \|x - y\|$ for all x and y in the domain of φ .
2. $\|\varphi(x) - \varphi(y)\| < \|x - y\|$ for all x and y in the domain of φ .
3. Well, it’s the shortening of a word or a group of words by omission of a sound or letter.

Hint Apparently, there is not much use for Banach’s fixed point theorem in operator algebras. *Contraction* is a 1-Lipshitz function, i.e. function f between metric spaces such that $d(x, y) \geq d(f(x), f(y))$ for all x, y . (A notable special case of a contraction is a linear operator of norm ≤ 1 .)

Exercise 4 What does $|A|$ stand for?

1. $(A^*A)^{1/2}$.
2. The smallest ordinal equinumerous with A , assuming the Axiom of Choice. Otherwise, this is the equivalence class of all sets equinumerous with A .

Exercise 5 A C^* -algebra A is finite if

1. For every partial isometry $v \in A$ such that the projections $p := vv^*$ and $q := v^*v$ satisfy $pq = p$ one has $p = q$
2. $|A| < \aleph_0$

Exercise 6 What are the elements of L_2 ?

1. Equivalence classes of square-integrable functions.
2. They are $\emptyset$ and $\{\emptyset\}$.

If you answered (1) to more than half of the questions, you are an operator algebraist. If you answered (2) to more than half of the questions, you are a set

theorist.¹⁸ If you answered (3), then you may be a linguist or a ‘Weird Al’ Yankovich fan.

Exercises

A fair number of the exercises form an integral part of the text. They are chosen to widen and deepen the material from the corresponding chapters. Some other exercises serve as a warm-up for the latter chapters, either by preparing the technical grounds or by putting a bug into the reader’s ear. Every single instance of the dreaded ‘it is easy to see’ phrase has been (at least) repackaged as a timely exercise, invoked later on, sometimes several chapters later. Every exercise used in some proof has been clearly marked. Finally, the entire subsections work as mini Moore-style courses, enticing and cajoling the reader to learn more about C^* -algebras and set theory.¹⁹

Silliness

Every effort has been made to relieve and reward the reader’s efforts. In addition to providing the best available proofs and unearthing analogies and connections previously unknown to humanity, the text contains a variety of quips of varying relevance (and, regrettably, of varying funniness degrees). All of them serve the purpose of putting the reader’s mind at ease before hitting them with complex (no pun intended) mathematics. All quotations are related to the section they precede in some, frequently unobvious, way.

What Has Been Omitted

In order to maintain an elementary level and a (relatively) slow pace, it was necessary to omit numerous results. Descriptive set theory and abstract classification are mentioned only in passing, and model theory plays a nontrivial role only in Chapter 16 (cf. [83]). We do not prove the relative consistency of set-theoretic axioms, such as Jensen’s $\diamond_{\aleph_1}$ or OCA_T , with ZFC. The reader can either take this on faith or read any of the existing excellent presentations (e.g. [165, §III.7], also [258]).

¹⁸Bonus question: What is M_n ?

¹⁹Or so I like to think.

The original (abandoned) plan for this text included a section on selective ultrafilters, but see [98].

The most recent general book on C^* -algebras that omits K-theory that comes to my mind is the (recently reissued) 40 years old [192]. The K-theory, K-homology, and Ext haven't yet been applied to C^* -algebras in conjunction with set theory, but this may be only a matter of time.

As I am writing these lines, exciting new developments are taking place in direct applications of forcing to C^* -algebras and set-theoretic analysis of the uniform Roe algebras, but it is now too late to start another elephant (see [183, p. 155]).

All C^* -algebras in Part I are separable unless otherwise specified. All C^* -algebras in Part II and Part III are nonseparable unless they are obviously separable.