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# Two Algebraic Byways from Differential Equations: Gröbner Bases and Quivers

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# Two Algebraic Byways from Differential Equations: Gröbner Bases and Quivers

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# Preface

This volume is a collection of several works focusing on differential equations from viewpoints of formal calculus and geometry through applications of quiver theory. This book consists of two parts. The first one introduces the theory of Gröbner bases in their commutative and noncommutative contexts. In particular, the lectures will focus on algorithmic aspects and applications of Gröbner bases to analysis on systems of partial differential equations, effective analysis on rings of differential operators, and homological algebra. The second part constitutes an introduction to representations of quivers, quiver varieties, and their applications to the moduli spaces of meromorphic connections on the complex projective line $\mathbb{P}^1$. All the contributions are presented without assuming any particular background, and the authors have done their best to make the chapters suitable for graduate students.

**Gröbner bases and quivers in algebra and geometry**. Gröbner bases and more generally linear rewriting systems constitute models for computation in algebras of various types (associative, commutative, Lie…). One of the applications of the theory is to compute normal forms, bases, and more generally Hilbert or Poincaré series. Another important application is a generalization of Gaussian elimination to polynomial systems in various types of algebras (commutative, Weyl algebra…). The theory of Gröbner bases was developed in the twentieth century. Several works had led to the development of computational methods in algebra well before the introduction of algebraic structures such as ideals and algebras and the modern algebraic language. Chapter 1 explains the long and rich developments from the work of M. Janet in 1920 on partial differential equations, elimination theory with seminal works of E. Noether in 1921, and the computational methods in algebraic geometry with the theory of Gröbner bases for commutative algebras developed by B. Buchberger in 1965. In recent years, new algorithms of the theory of Gröbner bases were developed in rings of differentials operators by Oaku–Takayama. In the meanwhile, decision problems in semigroups and groups by A. Thue in 1914 and M. Dehn in 1910 motivate a new combinatorial theory of equivalence relations, the rewriting theory. This theory was expended throughout the twentieth century, in particular with seminal results on confluence by M. Newman in 1942, on completion by Knuth–Bendix in 1970. Rewriting theory had been applied to algebra with works of A. I. Shirshof in 1962 for computing bases in Lie algebras and L. A. Bokut and G. Bergman independently in 1976–1978 for associative algebras. More recently at the end of 1980s, rewriting methods were applied in homological algebra by several authors such as D. J. Anick, C. Squier, K. Brown, and Y. Kobayashi.

Graphical methods in representation theory are rather new in comparison to the theory of Gröbner bases. Nevertheless, many applications are developed in the last decade. In 1934, H. Coxeter classified the finite real reflection groups and represented their fundamental relations in terms of graphs which was applied by E. Witt in 1941 to study the structure of semisimple Lie algebras. H. Weyl, in 1925–1926, and B. L. van der Waerden in 1933 simplified the classification of simple Lie algebras after W. Killing in 1888–1890, but it was E. B. Dynkin in 1946 who used the graphical expression to classify simple Lie algebras, where the name (Coxeter-) Dynkin diagram came from. In 1972, the Dynkin diagrams of type ADE have re-appeared by the work of P. Gabriel in view of the classification of the algebras with finitely many isomorphic classes of simple modules, see Chap. 6. It was only in the 1990s when the so-called quiver varieties were introduced by G. Lusztig for his study on quantum groups and H. Nakajima for his study on gauge theory, see Chap. 7. Their geometric approaches have big impacts not only on representation theory but also on algebraic geometry, for example, the moduli spaces of meromorphic connections on compact Riemann surfaces.

**Gröbner bases and applications**. The aim of the first part of the volume is to focus on various aspects of the theory of Gröbner bases and of the mathematical problems at the origin of the theory. Chapter 1 briefly reviews the seminal works on constructive methods for computing in ideals by M. Janet in 1920 motivated by integration of partial equation differential systems by C. Riquier and É. Cartan. The main tool introduced by M. Janet is the notion of involutive bases which are particular cases of Gröbner bases. Another domain in application that will be treated is the effective analysis on rings of differential operators. In particular, integral transformations and restriction functors on $D$-modules will be presented using noncommutative Gröbner bases. Chapters 2 and 3 present algorithmic aspects on $D$-modules. In particular, Chap. 2 deals with the notion of Gröbner bases in $D$-modules and their applications to Bernstein–Sato polynomials. An introduction to algorithms for $D$-modules with Quiver $D$-modules is also given in Chap. 3. Another aspect of Gröbner bases theory for noncommutative associative algebras is given in Chap. 4. A generalization of noncommutative Gröbner bases without a monomial order and a link between the theory of Gröbner bases and rewriting theory will be also explained. Finally, an application of Gröbner bases to the computation of free resolutions for associative algebras will be given. Chapter 5 will conclude this part with applications of the theory Gröbner bases to computational algebraic statistics.

**Quivers and applications**. The lectures of this part will be devoted to a geometric application of quivers. In particular, the geometry of the moduli spaces of meromorphic connections on $\mathbb{P}^1$ with irregular singularities is one of the subjects which has been developed recently, and this is the main theme of this part. Chapters 6 and 7 will provide introduction to representations of quivers and quiver varieties. There are some results known by the experts but never explained in the literature. In Chap. 8, the so-called additive Deligne–Simpson problem will be presented including some background materials. Some known results due to Crawley-Boevey and the author himself will also be explained. Geometric aspects of this problem with some recent development will be given in the final chapter (Chap. 9), where the author will recall necessary backgrounds from quiver varieties and symplectic geometry.

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# Part I
# First Algebraic Byway: Gröbner Bases

# Chapter 1
# From Analytical Mechanics Problems to Rewriting Theory Through M. Janet's Work

**Kenji Iohara and Philippe Malbos**

## 1 Introduction

This chapter is devoted to a survey of the historical background of Gröbner bases for $D$-modules and linear rewriting theory largely developed in algebra throughout the twentieth century and to present deep relationships between them. Completion methods are the main streams for these computational theories. In the theory of Gröbner bases, they were motivated by algorithmic problems in elimination theory such as computations in quotient polynomial rings modulo an ideal, manipulating algebraic equations, and computing Hilbert series. In rewriting theory, they were motivated by computation of normal forms and linear bases for algebras and computational problems in homological algebra.

In this chapter, we present the seminal ideas of the French mathematician M. Janet on the algebraic formulation of completion methods for polynomial systems. Indeed, the problem of completion already appears in Janet's 1920 thesis [47], which proposed an original approach by formal methods in the study of systems of linear partial differential equations, PDE systems for short. The corresponding constructions were formulated in terms of polynomial systems, but without the notions of ideal and Noetherian induction. These two notions were introduced by Noether in 1921 [68] for commutative rings.

The work of M. Janet was forgotten for about half of a century. It was rediscovered by Schwarz in 1992 in [81]. Our exposition in this chapter does not follow the historical order. The first section deals with the problems that motivate the PDE study undertaken by M. Janet. In Sect. 3, we present completion for monomial PDE

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systems as introduced by Janet in his monograph [51]. This completion used an original division procedure on monomials. In Sect. 4, we present an axiomatization of this Janet notion of division, called *involutive division*, due to V. P. Gerdt. The last two sections concern the case of polynomial PDE systems, with M. Janet's completion method used to reduce a linear PDE system to a canonical form and the axiomatization of the reductions involved in terms of rewriting theory.

## 1.1 From Analytical Mechanics Problems to Involutive Division

**1.1.1 From Lagrange to Janet.** The analysis of linear PDE systems was mainly motivated in eighteenth century by the desire to solve problems of analytical mechanics. The seminal work of J.-L. Lagrange gave the first systematic study of PDE systems launched by such problems. The case of PDE of one unknown function of several variables has been treated by J. F. Pfaff. The Pfaff problem will be recalled in Sect. 2.1. This theory was developed in two different directions: toward the general theory of differential invariants and the existence of solutions under given initial conditions. The differential invariants approach will be discussed in Sects. 2.1 and 2.1.4. The question of the existence of solution satisfying some initial conditions was formulated in the Cauchy–Kowalevsky theorem recalled in Sect. 2.1.3.

**1.1.2 Exterior Differential Systems.** Following the work of H. Grassmann in 1844 which did set up the rules of exterior algebra computations, É. Cartan introduced exterior differential calculus in 1899. This algebraic calculus allowed him to describe a PDE system by an exterior differential system that is independent of the choice of coordinates. This did lead to the so-called Cartan–Kähler theory, reviewed in Sect. 2.2. We will present a geometrical property of involutivity on exterior differential systems in Sect. 2.2.6, which motivates the formal methods introduced by M. Janet for the analysis of linear PDE systems.

**1.1.3 Generalizations of the Cauchy–Kowalevsky Theorem.** Another origin of the work of M. Janet is the Cauchy–Kowalevsky theorem that gives the initial conditions of solvability of a family of PDE systems that we describe in Sect. 2.1.3. É. Delassus, C. Riquier, and M. Janet attempted to generalize this result to a wider class of linear PDE systems which in turn led them to introduce the computation of a notion of normal form for such systems.

**1.1.4 The Janet Monograph.** Section 3 presents the historical work that motivated M. Janet to introduce an algebraic algorithm in order to compute normal form of linear PDE systems. In particular, we recall the problem of computation of *inversion of differentiation* introduced by M. Janet in his monograph ≪ *Leçons sur les systèmes d'équations aux dérivées partielles* ≫ on the analysis of linear PDE systems, published in 1929 [51]. Therein, M. Janet introduced formal methods based on polynomial computations for analysis of linear PDE systems. He developed an algorithmic approach for analyzing ideals in the polynomial ring $\mathbb{K}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$ of differential operators with constant coefficients. Having the ring isomorphism between this ring and the ring $\mathbb{K}[x_1, \ldots, x_n]$ of polynomials with $n$ variables in mind, M. Janet gave its algorithmic construction in this latter ring. He began by introducing some remarkable properties of monomial ideals. In particular, he recovered Dickson's Lemma [17], assertion that monomial ideals are finitely generated. This property is essential for the Noetherian properties on the set of monomials. Note that M. Janet was not familiar with the axiomatization of the algebraic structure of ideals and the property of Noetherianity already introduced by Noether in [68] and [69]. Note also that the Dickson Lemma was published in 1913 in a paper on number theory in an American journal. Due to the First World War, it took a long time before these works became accessible to the French mathematical community. Janet's algebraic constructions given in his monograph will be recalled in Sect. 3 for monomial systems and in Sect. 5 for polynomial systems.

**1.1.5 Janet's Multiplicative Variables.** The computations on monomial and polynomial ideals carried out by M. Janet are based on the notion of *multiplicative variable* that he introduced in his thesis [47]. Given an ideal generated by a set of monomials, he distinguished the monomials contained in the ideal and those contained in the complement of the ideal. The notions of multiplicative and non-multiplicative variables appear in order to stratify these two families of monomials. We will recall this notion of multiplicativity of variables in Sect. 3.1.9. This leads to a refinement of the classical division on monomials, nowadays called *Janet's division*.

**1.1.6 Involutive Division and Janet's Completion Procedure.** The notion of multiplicative variable is local, in the sense that it is defined with respect to a subset $\mathcal{U}$ of the set of all monomials. A monomial $u$ in $\mathcal{U}$ is said to be a Janet divisor of a monomial $w$ with respect to $\mathcal{U}$, if $w = uv$ and all variables occurring in $v$ are multiplicative with respect to $\mathcal{U}$. In this way, we distinguish the set $\mathrm{cone}_{\mathcal{J}}(\mathcal{U})$ of monomials having a Janet divisor in $\mathcal{U}$, called *multiplicative* or *involutive cone* of $\mathcal{U}$, and the set $\mathrm{cone}(\mathcal{U})$ of multiple of monomials in $\mathcal{U}$ for the classical division. The Janet division being a refinement of the classical division, the set $\mathrm{cone}_{\mathcal{J}}(\mathcal{U})$ is a subset of $\mathrm{cone}(\mathcal{U})$. M. Janet called a set of monomials $\mathcal{U}$ *complete* when this inclusion is an equality.

To a monomial PDE system $(\Sigma)$ of the form

$$\frac{\partial^{\alpha_1+\ldots+\alpha_n}\varphi}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} = f_\alpha(x_1, x_2, \ldots, x_n),$$

where $(\alpha_1, \ldots, \alpha_n)$ belongs to a subset $I$ of $\mathbb{N}^n$, M. Janet associated the set of monomials

$$\mathrm{lm}(\Sigma) = \{x_1^{\alpha_1}\ldots x_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in I\}.$$

The compatibility conditions of the system $(\Sigma)$ correspond to the factorizations of the monomials $ux$ in $\mathrm{cone}_{\mathcal{J}}(\mathrm{lm}(\Sigma))$, where $u$ is in $\mathrm{lm}(\Sigma)$ and $x$ is a non-multiplicative variable of $u$ with respect to $\mathrm{lm}(\Sigma)$, as explained in Sect. 3.3.1. By definition, for any monomial $u$ in $\mathrm{lm}(\Sigma)$ and $x$ non-multiplicative variable of $u$ with respect to $\mathrm{lm}(\Sigma)$, the monomial $ux$ admits such a factorization if and only if $\mathrm{lm}(\Sigma)$ is complete, see Proposition 3.2.5.

The main procedure presented in Janet's monograph [51] completes in a finite number of operations a finite set of monomials $\mathcal{U}$ to a complete set of monomials $\widetilde{\mathcal{U}}$ that contains $\mathcal{U}$. This procedure consists in analyzing all the local defects of completeness, by adding all the monomials $ux$ where $u$ in $\mathcal{U}$ and $x$ is a non-multiplicative variable for $u$ with respect to $\mathcal{U}$. This procedure will be recalled in Sect. 3.2.9. A generalization of this procedure to any involutive division was given by Gerdt in [25], and is recalled in Sect. 4.2.12.

Extending this procedure to a set of polynomials, M. Janet applied it to linear PDE systems, giving a procedure that transforms a linear PDE system into a complete PDE system with the same set of solutions. This construction is given in Sect. 5.6. In Sect. 6, we present such a procedure for an arbitrary involutive division given by V. P. Gerdt and Blinkov in [27] and its relation to the Buchberger completion procedure in commutative polynomial rings, [7].

**1.1.7 The Space of Initial Conditions.** In order to stratify the complement of the involutive cone $\mathrm{cone}_{\mathcal{J}}(\mathcal{U})$, M. Janet introduced the notion of *complementary monomial*, see Sect. 3.1.13. With this notion, the monomials that generate this complement in a such a way that the involutive cone of $\mathcal{U}$ and the involutive cone of the set $\mathcal{U}^{\complement}$ of complementary monomials form a partition of the set of all monomials, see Proposition 3.2.2.

For each complementary monomial $v$ in $\mathrm{lm}(\Sigma)^{\complement}$, each analytic function in the multiplicative variables of $v$ with respect to $\mathrm{lm}(\Sigma)^{\complement}$ provides an initial condition of the PDE system $(\Sigma)$ as stated by Theorem 3.3.3.

**1.1.8 Polynomial Partial Differential Equations Systems.** In Sect. 5, we present the analysis of polynomial PDE systems as Janet [51]. To deal with polynomials, he defined some total orders on the set of derivatives, corresponding to total orders on the set of monomials. We recall them in Sect. 5.1. The definitions on monomial orders given by M. Janet clarified the same notion introduced previously by Riquier in [74]. In particular, he made more explicit the notion of parametric and principal derivatives in order to distinguish the leading derivative in a polynomial PDE. In this way, he extended the algorithms for monomial PDE systems to the case of polynomial PDE systems. In particular, using these notions, he defined the property of completeness for polynomial PDE systems. Namely, a polynomial PDE system is complete if the associated set of monomials corresponding to leading derivatives of the system is complete. Moreover, M. Janet extended the notion of complementary monomials to define the notion of *initial condition* for a polynomial PDE system as in the monomial case.

**1.1.9 Initial Conditions.** In this way, the notion of completeness provides a suitable framework to discuss the existence and the uniqueness of the initial conditions for a linear PDE system. M. Janet proved that if a linear polynomial PDE system of the form

$$D_i \varphi = \sum_j a_{i,j} D_{i,j} \varphi, \quad i \in I,$$

with one unknown function $\varphi$ is such that all the functions $a_{i,j}$ are analytic in a neighborhood of a point $P$ in $\mathbb{C}^n$ and if it is complete with respect to some total order, then it admits at most one analytic solution satisfying the initial condition formulated in terms of complementary monomials, see Theorems 5.3.4 and 5.3.6.

**1.1.10 Integrability Conditions.** A linear polynomial PDE system of the above form is said to be *completely integrable* if it admits an analytic solution for any given initial condition. M. Janet gave an algebraic characterization of complete integrability by introducing integrability conditions formulated in terms of factorization of leading derivatives of the PDE by non-multiplicative variables. These integrability conditions are stated explicitly in Sect. 5.4.4 as generalization to the polynomial situation of the integrability conditions formulated above for monomial PDE systems in Sect. 3.3. M. Janet proved that a linear polynomial PDE system is completely integrable if and only if every integrability condition is trivially satisfied, as stated in Theorem 5.4.7.

**1.1.11 Janet's Procedure of Reduction of Linear PDE Systems to a Canonical Form.** In order to extend algorithmically the Cauchy–Kowalevsky theorem on the existence and uniqueness of solutions of initial value problems as presented in Sect. 2.1.3, M. Janet considered normal forms of linear PDE systems with respect to a suitable total order on derivatives, satisfying some analytic conditions on coefficients and a complete integrability condition on the system, as defined in Sect. 5.5.2. Such normal forms of PDE systems are called *canonical* by M. Janet.

Procedure 7 is *Janet's method* for deciding if a linear PDE system can be transformed into a completely integrable system. If the system cannot be reduced to a canonical form, the procedure returns the obstructions to such a reduction. Janet's procedure depends on a total order on derivatives of unknown functions of the PDE system. For this purpose, M. Janet introduced a general method to define a total order on derivatives using a parametrization of a weight order on variables and unknown functions, as explained in Sect. 5.1.5. The Janet procedure uses a specific weight order called canonical and defined in Sect. 5.6.2.

The first step of Janet's method consists in applying *autoreduction procedure*, defined in Sect. 5.6.4, in order to reduce any PDE of the system with respect to the total order on derivatives. Namely, two PDE of the system cannot have the same leading derivative, and any PDE of the system is reduced with respect to the leading derivatives of the others PDE, as specified in Procedure 5.

The second step is the *completion procedure*, Procedure 6. In it, the set of leading derivatives of the system defines a complete set of monomials in the sense given in Sect. 5.3.2.

Having transformed the PDE system to an autoreduced and complete system, one can look at its integrability conditions. M. Janet showed that this set of integrability conditions is a finite set of relations that do not contain principal derivatives, as explained in Sect. 5.4.4. Hence, these integrability conditions are $\mathcal{J}$-normal forms and uniquely defined. By Theorem 5.4.7, if all of these normal forms are trivial, then the system is completely integrable. Otherwise, any nontrivial condition in the set of integrability conditions that contains only unknown functions and variables imposes a relation on the initial conditions of the system. If there is no such relation, the procedure is applied again on the PDE system completed by all the integrability conditions. Note that this procedure depends on the Janet division and on a total order on the set of derivatives.

By this algorithmic method, M. Janet did generalize in certain cases the Cauchy–Kowalevsky theorem at the time where the algebraic structures have not been introduced to perform computations with polynomial ideals. This is pioneering work in the field of formal approaches to analysis of PDE systems. Algorithmic methods for dealing with polynomial ideals were developed throughout the twentieth century and extended to a wide range of algebraic structures. In the next subsection, we present some milestones on these formal themes in mathematics.

## 1.2 Constructive Methods and Rewriting in Algebra Through the Twentieth Century

The constructions developed by M. Janet in his formal theory of linear partial differential equation systems are based on the structure of ideals that he called *module of forms*. This notion corresponds to those introduced previously by Hilbert in [43] with the terminology of *algebraic form*. Notice that Gunther studied such a structure in [39]. The axiomatization of the notion of ideal in an arbitrary ring is due to Noether [68]. As we will explain in this chapter, M. Janet introduced algorithmic methods to compute a family of generators of an ideal having the involutive property and called an *involutive basis*. This property is used to obtain a normal form of linear partial differential equation systems.

Janet's computation of involutive bases is based on a refinement of classical polynomial division, called *involutive division*. He defined a division that is suitable for reduction of linear partial differential equation systems. Thereafter, other involutive divisions were studied, in particular, by Thomas [86] and by Pommaret [72]; we refer to Sect. 4.3 for a discussion on these divisions.

The main purpose is to complete a generating family of an ideal to an involutive basis with respect to a given involutive division. This completion process is quite similar to those introduced by means of the classical division in the theory of Gröbner bases. In fact, involutive bases appear to be particular cases of Gröbner bases. The principle of completion has been developed independently in rewriting theory, which proposes a combinatorial approach to equivalence relations motivated by several computational and decision problems in algebra, computer science, and logic.

**1.2.1 Some Milestones in Algebraic Rewriting and Constructive Algebra.** The main results in the work of M. Janet rely on constructive methods in linear algebra using the principle of computing normal forms by rewriting and the principle of completion of a generating set of an ideal. These two principles have been developed through the twentieth century in many algebraic contexts with different formulations and in several instances. We review below some important milestones in this long and rich history from T. Seki to the more recent developments.

**1683.** Seki introduced the notion of resultant and developed the notion of determinant to express the resultant. He also made progress in elimination theory in the Kai-fukudai-no-hō, see, e.g., [94].

**1840.** Sylvester studied the resultant of two polynomials in [85] and gave an example for two quadratic polynomials.

**1882.** Kronecker [54] gave the first result in elimination theory using this notion.

**1886.** Weierstrass proved a fundamental result called *preparation theorem* on the factorization of analytic functions by polynomials. As an application, he obtained a division theorem for rings of convergent series [93].

**1890.** Hilbert proved that any ideal in a ring of commutative polynomials in a finite set of variables over a field or over the ring of integers is finitely generated [43]. This is the first formulation of the Hilbert basis theorem, which states that every polynomial ring over a Noetherian ring is Noetherian.

**1913.** In a paper on number theory, L. E. Dickson proved a monomial version of the Hilbert basis theorem by a combinatorial method [17, Lemma A].

**1913.** In a series of forgotten papers, N. Günther developed algorithmic approaches for polynomials rings [38–40]. A review of Günther's theory can be found in [41].

**1914.** Dehn described the word problem for finitely presented groups [16]. Using systems of transformations rules, A. Thue studied the problem for finitely presented semigroups [87]. It was only much later, in 1947, that the problem for finitely presented monoids was shown to be undecidable, independently by Post [73] and Markov [64, 65].

**1916.** Macaulay was one of the pioneers in commutative algebra. In his book *The algebraic theory of modular systems* [59], following the fundamental Hilbert basis theorem, he initiated an algorithmic approach to treat generators of polynomial ideals. In particular, he introduced the notion of *H-basis* corresponding to a monomial version of Gröbner bases.

**1920.** Janet defended his doctoral thesis [47], which presents a formal study of systems of partial differential equations following works of Ch. Riquier and É. Delassus. In particular, he analyzed completely integrable systems and Hilbert functions of polynomial ideals.

**1921.** In her seminal paper, *Idealtheorie in Ringbereichen* [68], Noether laid the foundation of general commutative ring theory, and gave one of the first general definitions of a commutative ring. She also formulated the Finite Chain Theorem [68, Satz I, *Satz von der endlichen Kette*].

**1923.** Noether formulated in [69, 70] concepts of elimination theory in the language of ideals that she had introduced in [68].

**1926.** Hermann, a student of Noether [42], initiated purely algorithmic approaches to ideals, such as the ideal membership problem and primary decomposition ideals. This work is a fundamental contribution to the emergence of computer algebra.

**1927.** Macaulay showed in [60] that the Hilbert function of a polynomial ideal $I$ is equal to the Hilbert function of the monomial ideal generated by the set of leading monomials of the elements in $I$ with respect a monomial order. As a consequence, the coefficients of the Hilbert function of a polynomial ideal are polynomial for sufficiently big degree.

**1937.** Based on early works by Ch. Riquier and Janet, in [86] J. M. Thomas reformulated in the algebraic language of B. L. van der Waerden, *Moderne Algebra* [89, 90], the theory of normal forms of systems of partial differential equations.

**1937.** In [32], W. Gröbner exhibited the isomorphism between the ring of polynomials with coefficients in an arbitrary field and the ring of differential operators with constant coefficients, see Proposition 3.1.2. The identification of these two rings was used before in the algebraic study of systems of partial differential equations, but without being explicit.

**1942.** In a seminal paper on rewriting theory, M. Newman presented rewriting as a combinatorial approach to study equivalence relations [66]. He proved a fundamental rewriting result stating that under a termination hypothesis, the confluence property is equivalent to local confluence.

**1949.** In his monograph *Moderne algebraische Geometrie. Die idealtheoretischen Grundlagen* [33], W. Gröbner surveyed algebraic computation on ideal theory with applications to algebraic geometry.

**1962.** Shirshov introduced in [83] an algorithmic method to compute normal forms in a free Lie algebra with respect to a family of elements of the Lie algebra satisfying a confluence property. The method is based on a completion procedure. He also proved a version of Newman's lemma for Lie algebras, called *composition lemma*, and deduced a constructive proof of the Poincaré–Birkhoff–Witt theorem.

**1964.** Hironaka introduced in [44] a division algorithm and proposed the notion of *standard basis*, analogous to the notion of Gröbner basis, for rings of power series in order to solve problems of resolution of singularities in algebraic geometry.

**1965.** Under the supervision of W. Gröbner, B. Buchberger developed in his Ph.D. thesis an algorithmic theory of Gröbner bases for commutative polynomial algebras [7, 8, 10]. Buchberger gave a characterization of Gröbner bases in terms of *S-polynomials* as well as an algorithm to compute such bases, with a complete implementation in the assembler language of the computer ZUSE Z 23 V.

**1967.** Knuth and Bendix defined in [53] a completion procedure that completes with respect to a termination a set of equations in an algebraic theory into a confluent term rewriting system. The procedure is similar to Buchberger's completion procedure. We refer the reader to [9] for a historical account of critical pair/completion procedures.

**1972.** Grauert introduced in [30] a generalization of Weierstrass's preparation division theorem in the language of Banach algebras.

**1973.** Nivat formulated a critical pair lemma for string rewriting systems and proved that for a terminating rewriting system, the local confluence is decidable [67].

**1976, 1978.** Bokut in [5] and Bergman in [4] extended the Gröbner bases and Buchberger's algorithm to associative algebras. They obtained the confluence Newman Lemma for rewriting systems in free associative algebras compatible with a monomial order, called, respectively, Diamond Lemma for ring theory and Composition Lemma.

**1978.** Pommaret introduced in [72] a global involutive division simpler than those introduced by M. Janet.

**1980.** Schreyer in his Ph.D. thesis [80] gave a method that computes syzygies in commutative multivariate polynomial rings using the division algorithm, see [18, Theorem 15.10].

**1980.** Huet [45] gave a proof of Newman's lemma using a Noetherian well-founded induction method.

**1985.** Gröbner basis theory was extended to Weyl algebras by A. Galligo in [24], see also [79].

**1997.** Gerdt and Blinkov [25, 27] introduced the notion of involutive monomial division and its axiomatization.

**1999, 2002.** Faugère developed efficient algorithms for computing Gröbner bases, algorithm F4 [20], then an algorithm F5 [21].

**2005.** Gerdt [26] presented and analyzed an efficient involutive algorithm for computing Gröbner bases.

**2012.** Bächler, Gerdt, Lange-Hegermann, and Robertz algorithmized in [2] the Thomas decomposition of algebraic and differential systems.

## 1.3 Conventions and Notations

The set of nonnegative integers is denoted by $\mathbb{N}$. In this chapter, $\mathbb{K}[x_1, \dots, x_n]$ denotes the polynomial ring on the variables $x_1, \dots, x_n$ over a field $\mathbb{K}$ of characteristic zero. For a subset $G$ of $\mathbb{K}[x_1, \dots, x_n]$, we will denote by $\mathrm{Id}(G)$ the ideal of $\mathbb{K}[x_1, \dots, x_n]$ generated by $G$. A polynomial is either zero or it can be written as a finite sum of nonzero *terms*, each term being the product of a scalar in $\mathbb{K}$ by a *monomial*.

**1.3.1 Monomials.** We denote by $\mathcal{M}(x_1, \dots, x_n)$ the set of monomials in the ring $\mathbb{K}[x_1, \dots, x_n]$. For a subset $I$ of $\{x_1, \dots, x_n\}$ we will denote by $\mathcal{M}(I)$ the set of monomials in $\mathcal{M}(x_1, \dots, x_n)$ whose variables lie in $I$. A monomial $u$ in $\mathcal{M}(x_1, \dots, x_n)$ is written as $u = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, were the $\alpha_i$ are nonnegative integers. The integer $\alpha_i$ is called the *degree* of the variable $x_i$ in $u$, it will be also denoted by $\deg_i(u)$. For $\alpha = (\alpha_1, \dots, \alpha_n)$ in $\mathbb{N}^n$, we denote $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

For a finite subset $\mathcal{U}$ of $\mathcal{M}(x_1, \dots, x_n)$ and $1 \leqslant i \leqslant n$, we denote by $\deg_i(\mathcal{U})$ the largest degree in the variable $x_i$ of the monomials in $\mathcal{U}$, that is

$$\deg_i(\mathcal{U}) = \max\big(\deg_i(u) \mid u \in \mathcal{U}\big).$$

We call the *cone* of a subset $\mathcal{U}$ of $\mathcal{M}(x_1, \dots, x_n)$ the set of all multiples of monomials in $\mathcal{U}$, defined by

$$\mathrm{cone}(\mathcal{U}) = \bigcup_{u \in \mathcal{U}} u\mathcal{M}(x_1, \dots, x_n) = \{\, uv \mid u \in \mathcal{U},\ v \in \mathcal{M}(x_1, \dots, x_n)\,\}.$$

**1.3.2 Homogeneous Polynomials.** A *homogenous polynomial* in $\mathbb{K}[x_1, \dots, x_n]$ is a polynomial for which all nonzero terms have the same degree. A homogenous polynomial is of *degree* $p$ if all its nonzero terms have degree $p$. We denote by $\mathbb{K}[x_1, \dots, x_n]_p$ the space of homogeneous polynomials of degree $p$. The dimension of this space is given by the formula:

$$\Gamma_n^p := \dim\big(\mathbb{K}[x_1, \dots, x_n]_p\big) = \frac{(p+1)(p+2)\cdots(p+n-1)}{1 \cdot 2 \cdot \cdots \cdot (n-1)}.$$

**1.3.3 Monomial Order.** Recall that a *monomial order* on $\mathcal{M}(x_1, \dots, x_n)$ is a relation $\preccurlyeq$ on $\mathcal{M}(x_1, \dots, x_n)$ satisfying the following three conditions:

**(i)** $\preccurlyeq$ is a total order on $\mathcal{M}(x_1, \dots, x_n)$,
**(ii)** $\preccurlyeq$ is compatible with multiplication, that is, if $u \preccurlyeq u'$, then $uw \preccurlyeq u'w$ for any monomial $w$ in $\mathcal{M}(x_1, \dots, x_n)$,
**(iii)** $\preccurlyeq$ is a well-order on $\mathcal{M}(x_1, \dots, x_n)$, that is, every non-empty subset of $\mathcal{M}(x_1, \dots, x_n)$ has a smallest element with respect to $\preccurlyeq$.

The *leading term*, *leading monomial*, and *leading coefficient* of a polynomial $f$ of $\mathbb{K}[x_1, \dots, x_n]$, with respect to a monomial order $\preccurlyeq$, will be denoted by $\mathrm{lt}_\preccurlyeq(f)$, $\mathrm{lm}_\preccurlyeq(f)$, and $\mathrm{lc}_\preccurlyeq(f)$, respectively. For a set $F$ of polynomials in $\mathbb{K}[x_1, \dots, x_n]$, we will denote by $\mathrm{lm}_{\preccurlyeq}(F)$ the set of leading monomials of the polynomials in $F$. For simplicity, we will use notations $\mathrm{lt}(f)$, $\mathrm{lm}(f)$, $\mathrm{lc}(f)$, and $\mathrm{lm}(F)$ if there is no danger of confusion.

# 2 Exterior Differential Systems

Motivated by problems in analytical mechanics, Euler (1707–1783) and Lagrange (1736–1813) initiated the so-called *variational calculus*, cf. [57], which led to the problem of solving partial differential equations, PDEs for short. In this section, we briefly present the evolution of this theory to serve as a guide to M. Janet's contributions. We follow the history to introduce material on exterior differential systems and various PDE problems. For a deeper discussion of the theory of differential equations and the Pfaff problem, we refer the reader to [22, 92] or [11].

## 2.1 Pfaff's Problem

**2.1.1 Partial Differential Equations for One Unknown Function.** In 1772, Lagrange [56] considered a PDE of the form

$$F(x, y, \varphi, p, q) = 0, \quad \text{with} \quad p = \frac{\partial \varphi}{\partial x} \quad \text{and} \quad q = \frac{\partial \varphi}{\partial y}, \tag{2.1}$$

i.e., a PDE for one unknown function $\varphi$ of two variables $x$ and $y$. Lagrange's method to solve this PDE can be summarized as follows.

**(i)** Express the PDE (2.1) in the form

$$q = F_1(x, y, \varphi, p), \quad \text{with} \quad p = \frac{\partial \varphi}{\partial x} \quad \text{and} \quad q = \frac{\partial \varphi}{\partial y}. \tag{2.2}$$

**(ii)** Ignore for the moment that $p = \frac{\partial \varphi}{\partial x}$ and consider the 1-form

$$\Omega = d\varphi - pdx - qdy = d\varphi - pdx - F_1(x, y, \varphi, p)dy,$$

where $p$ is regarded as some (not yet fixed) function of $x$, $y$, and $\varphi$.

**(iii)** If there exist functions $M$ and $\Phi$ of $x$, $y$, and $\varphi$ satisfying $M\Omega = d\Phi$, then $\Phi(x, y, \varphi) = C$ for some constant $C$. Solving this new equation, we obtain a solution $\varphi = \psi(x, y, C)$ to Eq. (2.2).

**2.1.2 Pfaffian Systems.** In 1814–15, Pfaff (1765–1825) [71] studied a PDE for one unknown function of $n$ variables; this work was then continued by Jacobi (1804–1851) (cf. [46]). Recall that a PDE of any order is equivalent to a system of first-order PDEs. Thus, we may only think of systems of first-order PDEs with $m$ unknown functions

$$F_k\big(x_1,\ldots,x_n,\varphi^1,\ldots,\varphi^m,\frac{\partial\varphi^a}{\partial x_i}\ (1\leqslant a\leqslant m, 1\leqslant i\leqslant n)\big)=0,\quad \text{for}\quad 1\leqslant k\leqslant r.$$

Introducing new variables $p_i^a$, the system lives on the space with coordinates $(x_i, \varphi^a, p_i^a)$ and is given by

$$\begin{cases} F_k(x_i,\varphi^a,p_i^a)=0,\\ d\varphi^a-\displaystyle\sum_{i=1}^n p_i^a dx_i=0,\\ dx_1\wedge\cdots\wedge dx_n\neq 0.\end{cases}$$

Note that the last condition means that the variables $x_1, \ldots, x_n$ are independent. Such a system is called a *Pfaffian system*. One is interested in the questions whether this system admits a solution or not, and if there exists a solution, if it is unique under some conditions. We will refer to these as *Pfaff's problems*.

**2.1.3 Cauchy–Kowalevsky's Theorem.** A naive approach to Pfaff's problems, with applications to mechanics in mind, is the question of the initial conditions. In series of articles published in 1842, A. Cauchy (1789–1857) studied the system of first-order PDEs:

$$\frac{\partial\varphi^a}{\partial t}=f_a(t,x_1,\ldots,x_n)+\sum_{b=1}^m\sum_{i=1}^n f_{a,b}^i(t,x_1,\ldots,x_n)\frac{\partial\varphi^b}{\partial x_i},\quad \text{for}\quad 1\leqslant a\leqslant m,$$

where $f_a$, $f_{a,b}^i$ and $\varphi^1, \ldots, \varphi^m$ are functions of $n+1$ variables $t, x_1, \ldots, x_n$. Kowalevsky (1850–1891) [91] in 1875 considered systems of PDEs of the following form: for some $r_a \in \mathbb{Z}_{>0}$ $(1 \leqslant a \leqslant m)$,

$$\frac{\partial^{r_a}\varphi^a}{\partial t^{r_a}}=\sum_{b=1}^m\sum_{\substack{j=0\\ j+|\alpha|\leqslant r_a}}^{r_a-1} f_{a,b}^{j,\alpha}(t,x_1,\ldots,x_n)\frac{\partial^{j+|\alpha|}\varphi^b}{\partial t^j\partial x^\alpha},$$

where $f_{a,b}^{j,\alpha}$ and $\varphi^1, \ldots, \varphi^m$ are functions of $n+1$ variables $t, x_1, \ldots, x_n$, and where for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ in $(\mathbb{Z}_{\geqslant 0})^n$, we set $|\alpha| = \sum_{i=1}^n \alpha_i$ and $\partial x^\alpha = \partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$. They showed that under the hypothesis of analyticity of the coefficients, such a system admits a unique analytic local solution satisfying a given initial condition. This statement is now called the *Cauchy–Kowalevsky theorem*.

**2.1.4 Completely Integrable Systems.** A first geometric approach to the above problem was undertaken over by Frobenius (1849–1917) [23] and independently by Darboux (1842–1917) [15]. Let $X$ be a differentiable manifold of dimension $n$. We consider the Pfaffian system

$$\omega_i=0\qquad 1\leqslant i\leqslant r,$$

where $\omega_i$ are 1-forms defined on a neighborhood $V$ of a point $x$ in $X$. Suppose that the family

$$\{(\omega_i)_y\}_{1 \leqslant i \leqslant r} \subset T_y^* X$$

is linearly independent for all $y$ in $V$. For $0 \leqslant p \leqslant n$, let us denote by $\Omega_X^p(V)$ the space of differentiable $p$-forms on $V$. A *$p$-dimensional distribution* $\mathcal{D}$ on $X$ is a subbundle of $TX$ with fiber of dimension $p$. A distribution $\mathcal{D}$ is *involutive* if, for any vector fields $\xi$ and $\eta$ taking values in $\mathcal{D}$, the Lie bracket

$$[\xi, , \eta] := \xi\eta - \eta\xi$$

takes values in $\mathcal{D}$ as well. Such a Pfaffian system is said to be *completely integrable*.

G. Frobenius and G. Darboux showed that the ideal $I$ of $\bigoplus_{p=0}^{n} \Omega_X^p(V)$, generated by the 1-forms $\omega_1, \ldots, \omega_r$, is a differential ideal, i.e., $dI \subset I$, if and only if the distribution $\mathcal{D}$ on $V$ defined as the annihilator of $\omega_1, \ldots, \omega_r$ is involutive.

## 2.2 The Cartan–Kähler Theory

Here, we give a brief historically oriented exposition of the so-called Cartan–Kähler theory. In particular, we will present the notion of system in involution. For the original treatment by the founders of the theory, we refer the reader to [14, 52], modern introductions are provided in [6, 62], and a quick survey can be found in [95, Appendix].

**2.2.1 Differential Forms.** Grassmann (1809–1877) [29] introduced in 1844 the first equation-based formulation of the structure of exterior algebra with the anti-commutativity rule

$$x \wedge y = -y \wedge x.$$

Using this setting, Cartan (1869–1951) [11] defined in 1899 the *exterior differential* and *differential $p$-forms*. He showed that these notions are invariant under arbitrary coordinate transformation. Thanks to these differential structures, several results obtained in the nineteenth century were reformulated in a clear manner.

**2.2.2 Exterior Differential Systems.** An *exterior differential system* $\Sigma$ is a finite set of homogeneous differential forms, i.e., $\Sigma \subset \bigcup_p \Omega_X^p$. Cartan [12], in 1901, studied exterior differential systems generated by 1-forms, i.e., Pfaffian systems. Later, Kähler (1906–2000) [52] generalized Cartan's theory to any differential ideal $I$ generated by an exterior differential system. For this reason, the general theory on exterior differential systems is nowadays called the *Cartan–Kähler theory*.

In the rest of this subsection, we discuss briefly the existence theorem for such a system. Since the argument developed here is *local* and we need the Cauchy–

Kowalevsky theorem, we assume that all functions are *analytic* in $x_1, \dots, x_n$ unless otherwise stipulated.

**2.2.3 Integral Elements.** Let $\Sigma$ be an exterior differential system on a real analytic manifold $X$ of dimension $n$ such that the ideal generated by $\Sigma$ is a differential ideal. For $0 \leqslant p \leqslant n$, set $\Sigma^p = \Sigma \cap \Omega_X^p$. We fix $x$ in $X$. For $p > 0$, a pair $(E_p, x)$, with a $p$-dimensional vector subspace $E_p \subset T_x X$, is called an *integral p-element* if $\omega|_{E_p} = 0$ for any $\omega$ in $\Sigma_x^p := \Sigma^p \cap \Omega_{X,x}^p$, where $\Omega_{X,x}^p$ denotes the space of differential $p$-forms defined on a neighborhood of $x$ in $X$. We denote the set of integral elements of dimension $p$ by $I\Sigma_x^p$.

An *integral manifold* $Y$ is a submanifold of $X$ whose tangent space $T_y Y$ at each point $y$ in $Y$ is an integral element. Since the exterior differential system defined by $\Sigma$ is completely integrable, there exists independent $r$-functions $\varphi_1(x), \dots, \varphi_r(x)$, called *integrals of motion* or *first integrals*, defined on a neighborhood $V$ of a point $x \in X$ such that their restrictions on $V \cap Y$ are constants.

The *polar space* $H(E_p)$ of an integral element $E_p$ of $\Sigma$ at the point $x$ is the vector subspace of $T_x X$ generated by the vectors $\xi \in T_x X$ such that $E_p + \mathbb{R}\xi$ is an integral element of $\Sigma$.

**2.2.4 Regular Integral Elements.** Let $E_0$ be the real analytic subvariety of $X$ defined as the zeros of $\Sigma^0$ and let $\mathcal{U}$ be the subset of smooth points. A point in $E_0$ is called *integral point*. A tangent vector $\xi$ in $T_x X$ is called a *linear integral element* if $\omega(\xi) = 0$ for any $\omega \in \Sigma_x^1$ with $x \in \mathcal{U}$. We define inductively the properties called "regular" and "ordinary" as follows:

- **(i)** The zeroth-order *character* is the integer $s_0 = \max_{x\in\mathcal{U}}\{\dim \mathbb{R}\Sigma_x^1\}$. A point $x \in E_0$ is said to be *regular* if $\dim \mathbb{R}\Sigma_x^1 = s_0$, and a linear integral element $\xi \in T_x X$ is called *ordinary* if $x$ is regular.
- **(ii)** Let $E_1 = \mathbb{R}\xi$, where $\xi$ is an ordinary linear integral element. The first-order *character* is the integer $s_1$ satisfying $s_0 + s_1 = \max_{x\in\mathcal{U}}\{\dim H(E_1)\}$. The ordinary integral 1-element $(E_1, x)$ is said to be *regular* if $\dim H(E_1) = s_0 + s_1$. An integral 2-element $(E_2, x)$ is called *ordinary* if it contains at least one regular linear integral element.
- **(iii)** Assume that all these concepts are defined up to $(p-1)$th step and that $s_0 + s_1 + \cdots + s_{p-1} < n - p + 1$.

  The $p$th-order *character* is the integer $s_p$ satisfying

  $$\sum_{i=0}^{p} s_i = \max_{x\in\mathcal{U}} \{\dim H(E_p)\}.$$

  An integral $p$-element $(E_p, x)$ is said to be *regular* if

  $$\sum_{i=0}^{p} s_i = \dim H(E_p).$$

  The integral $p$-element $(E_p, x)$ is called *ordinary* if it contains at least one regular integral element $(E_{p-1}, x)$.

Let $h$ be the smallest positive integer such that $\sum_{i=0}^{h} s_i = n - h$. Then, there does not exist an integral $(h+1)$-element. The integer $h$ is called the *genus* of the system $\Sigma$. For $0 < p \leqslant h$, one has

$$\sum_{i=0}^{p-1} s_i \leqslant n - p.$$

**2.2.5 Theorem** *Let $0 < p \leqslant h$ be an integer.*

**(i)** *The case $\sum_{i=0}^{p-1} s_i = n - p$ : let $(E_p, x)$ be an ordinary integral $p$-element and let $Y_{p-1}$ be an integral manifold of dimension $p-1$ such that $(T_x Y_{p-1}, x)$ is a regular integral $(p-1)$-element contained in $(E_p, x)$. Then, there exists a unique integral manifold $Y_p$ of dimension $p$ containing $Y_{p-1}$ such that $T_x Y_p = E_p$.*

**(ii)** *The case $\sum_{i=0}^{p-1} s_i < n - p$ : let $(E_p, x)$ be an integral $p$-element and let $Y_{p-1}$ be an integral manifold of dimension $p-1$ such that $(T_x Y_{p-1}, x)$ is a regular integral $(p-1)$-element contained in $(E_p, x)$. Then, there is a one-to-one correspondence between the set of real analytic functions $C^\omega(\mathbb{R}^p, \mathbb{R}^{n-p-\sum_{i=0}^{p-1} s_i})$ and the set of $p$-dimensional integral manifolds $Y_p$ containing $Y_{p-1}$ such that $T_x Y_p = E_p$.*

This theorem states that a given chain of ordinary integral elements

$$(E_0, x) \subset (E_1, x) \subset \cdots \subset (E_h, x), \qquad \dim E_p = p \quad (0 \leqslant p \leqslant h),$$

one can inductively find an integral manifold $Y_p$ of dimension $p$ such that $Y_0 = \{x\}$, $Y_{p-1} \subset Y_p$ and $T_x Y_p = E_p$. Notice that to obtain $Y_p$ from $Y_{p-1}$, one applies the Cauchy–Kowalevsky theorem to the PDE system defined by $\Sigma^p$ and the choice of real analytic functions in the above statement provide a datum to define the integral manifold $Y_p$.

**2.2.6 Systems in Involution.** In many applications, the exterior differential systems one considers admit $p$-independent variables $x_1, \ldots, x_p$. In such a case, we are only interested in the $p$-dimensional integral manifolds among which no additional relation between $x_1, \ldots, x_p$ is imposed. In general, an exterior differential system $\Sigma$ for $n - p$ unknown functions and $p$-independent variables $x_1, \ldots, x_p$ is said to be *in involution* if it satisfies the two following conditions:

1. its genus is larger than or equal to $p$,
2. the defining equations of the generic ordinary integral $p$-element introduce no linear relation among $dx_1, \ldots, dx_p$.

**2.2.7 Reduced Characters.** Consider a family $\mathcal{F}$ of integral elements of dimensions $1, 2, \ldots, p-1$ than can be included in an integral $p$-element at a generic integral point $x \in X$. Take a local chart with origin $x$. The *reduced polar system* $H^{\mathrm{red}}(E_i)$ of an integral element at $x$ is the polar system of the restriction of the exterior differential system $\Sigma$ to the submanifold

$$\{x_1 = x_2 = \cdots = x_p = 0\}.$$

The integers $s'_0, \ldots, s'_{p-1}$, called the *reduced characters*, are defined in such a way that $s'_0 + \cdots + s'_i$ is the dimension of the reduced polar system $H^{\mathrm{red}}(E_i)$ at a generic integral element. For convenience, one sets $s'_p = n - p - (s'_0 + \cdots + s'_{p-1})$.

Let $\Sigma$ be an exterior differential system of $n - p$ unknown functions of $p$-independent variables such that the ideal generated by $\Sigma$ is a differential ideal. É. Cartan showed that $\Sigma$ is a *system in involution* iff the most general integral $p$-element in $\mathcal{F}$ depends on $s'_1 + 2s'_2 + \cdots + ps'_p$ independent parameters.

**2.2.8 Recent Developments.** In 1957, Kuranishi (1924 –) [55] considered the problem of the prolongation of a given exterior differential system and treated what É. Cartan called total case. Here, M. Kuranishi as well as É. Cartan worked locally in the analytic category. After an algebraic approach to the integrability was proposed by Guillemin and Sternberg [34], in 1964, Singer and Sternberg, [84], in 1965 studied some classes of infinite-dimensional systems which can be treated even in the $C^\infty$-category. In 1970s, with the aid of jet bundles and the Spencer cohomology, Pommaret (cf. [72]) considered formally integrable involutive differential systems generalizing the work of M. Janet, in the language of sheaf theory. For other geometric aspects not using sheaf theory, see the books by Griffiths (1938-) [31], and Bryant et al. [6].

# 3 Monomial PDE Systems

In this section, we present the method introduced by M. Janet under the name "*calcul inverse de la dérivation*" in his monograph [51]. In [51, Chap. I], M. Janet considered *monomial PDE*, that is, PDE of the form

$$\frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}\varphi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}} = f_{\alpha_1\alpha_2\ldots\alpha_n}(x_1, x_2, \ldots, x_n), \tag{3.1}$$

where $\varphi$ is an unknown function and the $f_{\alpha_1\alpha_2\ldots\alpha_n}$ are analytic functions of several variables. By an algebraic method, he analyzed the solvability of such an equation, namely, the existence and the uniqueness of an analytic solution $\varphi$ of the system. Notice that the analyticity condition guarantees the commutativity of partial differentials operators. This property is crucial for the constructions that M. Janet carried out in the ring of commutative polynomials. Note that the first example of PDE that does not admit any solution was found by Lewy in the 1950s in [58].

## 3.1 Ring of Partial Differential Operators and Multiplicative Variables

**3.1.1 Historical Context.** In the beginning of 1890s, following collaboration with C. Méray (1835–1911), Riquier (1853–1929) initiated his research on finding normal forms of systems of (infinitely many) PDEs for finitely many unknown functions of finitely many independent variables (see [75] and [76] for more details).

In 1894, Tresse [88] showed that such systems can be always reduced to systems of finitely many PDEs. This is the first result on Noeterianity of a module over a ring of differential operators. Based on this result, É. Delassus (1868–19..) formalized and simplified Riquier's theory. In these works, one already finds an algorithmic approach for analyzing ideals of the ring $\mathbb{K}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$.

It was Janet (1888–1983) who already in his thesis [47], published in 1920, had realized that the latter ring is isomorphic to the ring of polynomials with $n$ variables $\mathbb{K}[x_1, \ldots, x_n]$. At that time, several abstract notions on rings were introduced by E. Noether in Germany but by M. Janet in France was not familiar with them. It was only in 1937 that W. Gröbner (1899–1980) proved this isomorphism.

**3.1.2 Proposition [32, Sect. 2.]** *There exists a ring isomorphism*

$$\Phi : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{K}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}],$$

*from the ring of polynomials in $n$ variables $x_1, \ldots, x_n$ with coefficients in an arbitrary field $\mathbb{K}$ to the ring of differential operators with constant coefficients.*

**3.1.3 Derivations and Monomials.** M. Janet considers monomials in the variables $x_1, \ldots, x_n$ and uses implicitly the isomorphism $\Phi$ of Proposition 3.1.2. To a monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, he associates the differential operator

$$D^\alpha := \Phi(x^\alpha) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

In [51, Chap. I], M. Janet considered finite monomial PDE systems. The equations are of the form (3.1) and since the system has a finitely many equations, the set of monomials associated to it is finite. The first result of the monograph is a finiteness result on monomials stating that a sequence of monomials in which none is a multiple of a preceding one is necessarily finite. M. Janet proved this result by induction on the number of variables. We can formulate it as follows.

**3.1.4 Lemma ([51, Sect. 7])** *Let $\mathcal{U}$ be a subset of $\mathcal{M}(x_1, \ldots, x_n)$. If, for any monomials $u$ and $u'$ in $\mathcal{U}$, the monomial $u$ does not divide $u'$, then the set $\mathcal{U}$ is finite.*

This result corresponds to Dickson's Lemma [17], which asserts that every monomial ideal of $\mathbb{K}[x_1, \ldots, x_n]$ is finitely generated.

**3.1.5 Stability of the Multiplication.** M. Janet paid a special attention to families of monomials with the following property. A subset of monomial $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ is called *multiplicatively stable* if for any monomial $u$ in $\mathcal{M}(x_1, \ldots, x_n)$ such that there exists $u'$ in $\mathcal{U}$ that divides $u$, one has that $u$ is in $\mathcal{U}$. In other words, the set $\mathcal{U}$ is closed under multiplication by monomials in $\mathcal{M}(x_1, \ldots, x_n)$.

As a consequence of Lemma 3.1.4, if $\mathcal{U}$ is a multiplicatively stable subset of $\mathcal{M}(x_1, \ldots, x_n)$, then it contains only finitely many elements that are not multiples of any other elements in $\mathcal{U}$. Hence, there exists a finite subset $\mathcal{U}_f$ of $\mathcal{U}$ such that for any $u$ in $\mathcal{U}$, there exists $u_f$ in $\mathcal{U}_f$ such that $u_f$ divides $u$.

**3.1.6 Ascending Chain Condition.** M. Janet observed another consequence of Lemma 3.1.4: the *ascending chain condition* on multiplicatively stable monomial sets, which he formulated as follows. Any ascending sequence of multiplicatively stable subsets of $\mathcal{M}(x_1, \ldots, x_n)$

$$\mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots \subset \mathcal{U}_k \subset \cdots$$

is finite. This corresponds to the Noetherian property on the set of monomials in finitely many variables.

**3.1.7 Inductive Construction.** Let us fix a total order on the variables $x_n > x_{n-1} > \cdots > x_1$. Let $\mathcal{U}$ be a finite subset of $\mathcal{M}(x_1, \ldots, x_n)$. Let us define, for every $0 \leqslant \alpha_n \leqslant \deg_n(\mathcal{U})$,

$$[\alpha_n] = \{u \in \mathcal{U} \mid \deg_n(u) = \alpha_n\}.$$

The family $([0], \ldots, [\deg_n(\mathcal{U})])$ forms a partition of $\mathcal{U}$. We define for every $0 \leqslant \alpha_n \leqslant \deg_n(\mathcal{U})$

$$\overline{[\alpha_n]} = \{u \in \mathcal{M}(x_1, \ldots, x_{n-1}) \mid u x_n^{\alpha_n} \in \mathcal{U}\}.$$

We set for every $0 \leqslant i \leqslant \deg_n(\mathcal{U})$

$$\mathcal{U}'_i = \bigcup_{0 \leqslant \alpha_n \leqslant i} \{u \in \mathcal{M}(x_1, \ldots, x_{n-1}) \mid \text{there exists } u' \in \overline{[\alpha_n]} \text{ such that } u' | u\}.$$

Finally, we set

$$\mathcal{U}_k = \begin{cases} \{u x_n^k \mid u \in \mathcal{U}'_k\}, & \text{if } k < \deg_n(\mathcal{U}), \\ \{u x_n^k \mid u \in \mathcal{U}'_{\deg_n(\mathcal{U})}\}, & \text{if } k \geqslant \deg_n(\mathcal{U}), \end{cases}$$

and $M(\mathcal{U}) = \bigcup_{k \geqslant 0} \mathcal{U}_k$. By this inductive construction, M. Janet obtains the monomial ideal generated by $\mathcal{U}$. Indeed, $M(\mathcal{U})$ coincides with the following set of monomial:

$$\{u \in \mathcal{M}(x_1, \ldots, x_n) \mid \text{there exists } u' \text{ in } \mathcal{U} \text{ such that } u' | u\}.$$

**3.1.8 Example.** Consider the subset $\mathcal{U} = \{x_3 x_2^2, x_3^3 x_1^2\}$ of $\mathcal{M}(x_1, x_2, x_3)$. We have

$$[0] = \emptyset, \quad [1] = \{x_3 x_2^2\}, \quad [2] = \emptyset, \quad [3] = \{x_3^3 x_1^2\}.$$

Hence,

$$\overline{[0]} = \emptyset, \quad \overline{[1]} = \{x_2^2\}, \quad \overline{[2]} = \emptyset, \quad \overline{[3]} = \{x_1^2\}.$$

The set $M(\mathcal{U})$ is defined using of the following subsets:

$$\mathcal{U}'_0 = \emptyset, \quad \mathcal{U}'_1 = \{x_1^{\alpha_1} x_2^{\alpha_2} \mid \alpha_2 \geqslant 2\}, \quad \mathcal{U}'_2 = \mathcal{U}'_1, \quad \mathcal{U}'_3 = \{x_1^{\alpha_1} x_2^{\alpha_2} \mid \alpha_1 \geqslant 2 \text{ ou } \alpha_2 \geqslant 2\}.$$

**3.1.9 Janet's Multiplicative Variables [47, Sect. 7].** Let us fix a total order $x_n > x_{n-1} > \cdots > x_1$ on variables. Let $\mathcal{U}$ be a finite subset of $\mathcal{M}(x_1, \ldots, x_n)$. For all $1 \leqslant i \leqslant n$, we define the following subset of $\mathcal{U}$:

$$[\alpha_i, \ldots, \alpha_n] = \{u \in \mathcal{U} \mid \deg_j(u) = \alpha_j \text{ for all } i \leqslant j \leqslant n\}.$$

That is, $[\alpha_i, \ldots, \alpha_n]$ contains monomials of $\mathcal{U}$ of the form $vx_i^{\alpha_i} \cdots x_n^{\alpha_n}$, with $v$ in $\mathcal{M}(x_1, \ldots, x_{i-1})$. The sets $[\alpha_i, \ldots, \alpha_n]$ with $\alpha_i, \ldots, \alpha_n$ in $\mathbb{N}$ form a partition of $\mathcal{U}$. Moreover, for all $1 \leqslant i \leqslant n-1$, we have $[\alpha_i, \alpha_{i+1}, \ldots, \alpha_n] \subseteq [\alpha_{i+1}, \ldots, \alpha_n]$ and the sets $[\alpha_i, \ldots, \alpha_n]$, where $\alpha_i \in \mathbb{N}$, form a partition of $[\alpha_{i+1}, \ldots, \alpha_n]$.

Given a monomial $u$ in $\mathcal{U}$, the variable $x_n$ is said to be *multiplicative for $u$ in the sense of Janet* if

$$\deg_n(u) = \deg_n(\mathcal{U}).$$

For $i \leqslant n-1$, the variable $x_i$ is said to be *multiplicative for $u$ in the sense of Janet* if

$$u \in [\alpha_{i+1}, \ldots, \alpha_n] \quad \text{and} \quad \deg_i(u) = \deg_i([\alpha_{i+1}, \ldots, \alpha_n]).$$

We will denote by $\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u)$ the set of multiplicative variables of $u$ in the sense of Janet with respect to the set $\mathcal{U}$, also called *$\mathcal{J}$-multiplicative variables*.

Note that, by definition, for any $u$ and $u'$ in $[\alpha_{i+1}, \ldots, \alpha_n]$, we have

$$\{x_{i+1}, \ldots, x_n\} \cap \mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u) = \{x_{i+1}, \ldots, x_n\} \cap \mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u').$$

Accordingly, we will denote this set of multiplicative variables by $\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}([\alpha_{i+1}, \ldots, \alpha_n])$.

**3.1.10 Example.** Consider the subset $\mathcal{U} = \{x_2x_3, x_2^2, x_1\}$ of $\mathcal{M}(x_1, x_2, x_3)$ with the order
$x_3 > x_2 > x_1$. We have $\deg_3(\mathcal{U}) = 1$; hence, the variable $x_3$ is $\mathcal{J}$-multiplicative for $x_3x_2$ and not $\mathcal{J}$-multiplicative for $x_2^2$ and $x_1$.

For $\alpha \in \mathbb{N}$, we have $[\alpha] = \{u \in \mathcal{U} \mid \deg_3(u) = \alpha\}$, hence

$$[0] = \{x_2^2, x_1\}, \qquad [1] = \{x_2x_3\}.$$

We have $\deg_2(x_2^2) = \deg_2([0])$, $\deg_2(x_1) \neq \deg_2([0])$ and $\deg_2(x_2x_3) = \deg_2([1])$, so the variable $x_2$ is $\mathcal{J}$-multiplicative for $x_2^2$ and $x_2x_3$ and not $\mathcal{J}$-multiplicative for $x_1$. Further,

$$[0, 0] = \{x_1\}, \quad [0, 2] = \{x_2^2\}, \quad [1, 1] = \{x_2x_3\},$$