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An Example Based on Harrington's Principle

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Preface

Hilbert proposed his famous list of 23 open problems at the International Congress of Mathematicians in Paris in 1900 [1]. The second problem deals with the question whether the axioms of mathematics are *consistent*, i.e., that no contradiction can be derived. Hilbert later elaborated on the second problem by proposing *Hilbert's program for the foundations of mathematics* [2]. The aim was to find a consistent set of axioms whose consequences comprise all theorems in mathematics.

As is well known, Gödel's first incompleteness theorem shows that for any logical system that can accommodate arithmetic, there are true sentences that cannot be proved. Hence, it is not possible to formalize all of mathematics within a consistent formal system, as any attempt at such a formalism will omit some true mathematical statements. In this light, Gödel's incompleteness theorem is one of the most important theorems in foundations of mathematics and mathematical logic in the twentieth century and has had a huge impact on the development of logic, philosophy, mathematics, computer science, and other fields. In the literature, there are a number of good research books on Gödelian incompleteness (e.g., [3–8]) and a huge number of research articles.

Now, Gödel's true-but-unprovable sentence from the first incompleteness theorem is purely logical in nature, i.e., not mathematically natural or interesting. In this light, an interesting problem is to find mathematically natural and interesting statements that are similarly unprovable. A lot of research has since been done in this direction, most notably by Harvey Friedman. A lot of examples of *concrete incompleteness* with real mathematical content have been found to date. Section 1.1.3 provides an overview of research on incompleteness in higher-order arithmetic.

This book contributes to Harvey Friedman's research program on concrete incompleteness for *higher-order arithmetic*. In a nutshell, I shall introduce the set-theoretic hierarchy Z_n of higher-order arithmetic. Here, Z_2, Z_3 and Z_4 are the corresponding set-theoretical axiomatic systems for second-order arithmetic, third-order arithmetic, and fourth-order arithmetic. I then formulate a concrete mathematical theorem expressible in the language of second-order arithmetic which is neither provable in Z_2 or Z_3, but provable in Z_4. While I do not provide a comprehensive study of incompleteness, Sect. 1.1.3 includes some examples of concrete mathematical theorems about arithmetic which are not provable in PA; examples of concrete mathematical theorems about arithmetic which are not provable in certain sub-systems of second-order arithmetic stronger than PA; and examples of concrete mathematical theorems about analysis provable in third-order arithmetic but not provable in second-order arithmetic.

In this book, I examine the aforementioned Hilbert's program "relativized" to Z_2, which deals with the following question: are all theorems in classic mathematics expressible in second-order arithmetic provable in Z_2? Now, most classic mathematical theorems about real numbers expressible in (the language of) second-order arithmetic are also provable in Z_2. Nonetheless, I shall provide a negative answer to this question in the form of a counterexample which stems from a famous theorem in set theory, namely *the Martin-Harrington Theorem*. The latter expresses between $Det(\Sigma^1_1)$ and the existence of $0^\sharp$, establishing the equivalence between large cardinal and determinacy hypotheses. The Martin-Harrington Theorem is expressible in second-order arithmetic and provable in ZF. In this book, I give a systematic analysis of known proofs of the Martin-Harrington Theorem in higher-order arithmetic.

It is known that Martin's Theorem, i.e., that the existence of $0^\sharp$ implies $Det(\Sigma^1_1)$, is provable in Z_2 (cf. Sect. 3.1). However, in the proof of Harrington's Theorem, i.e., $Det(\Sigma^1_1)$ implies the existence of $0^\sharp$, Harrington makes use of a principle nowadays called *Harrington's Principle* (HP hereafter). All known proofs of Harrington's Theorem are done in the following two steps: first prove that $Det(\Sigma^1_1)$ implies HP and then show that HP implies that $0^\sharp$ exists. Below, I show that the first implication "$Det(\Sigma^1_1)$ implies HP" is provable in Z_2. A natural question is then whether the second implication is provable in Z_2. By way of a negative answer, I show that the statement that the second implication, i.e., HP implies that $0^\sharp$ exists, is not provable in Z_2. This provides the required counterexample for Hilbert's program relativized to Z_2.

Moreover, I show that "HP implies that $0^\sharp$ exists" is also not provable in Z_3, but is provable in Z_4. As part of joint work with Ralf Schindler, I prove in Sect. 2.2 that $Z_2 + \mathsf{HP}$ is equi-consistent with ZFC. In Sect. 2.3, $Z_3 + \mathsf{HP}$ is shown to be equi-consistent with ZFC + there exists a remarkable cardinal. In Sect. 2.4, I show that HP is equivalent to $0^\sharp$ exists in Z_4. Hence, Z_4 is the minimal system from higher-order arithmetic to show that HP implies that $0^\sharp$ exists.

It is unknown whether the Harrington Theorem is provable in Z_2. I show in Chap. 3 that the **boldface** Martin-Harrington Theorem is provable in Z_2. In Chap. 4, we examine the large cardinal strength of the strengthening of HP, called $\mathsf{HP}(\varphi)$, over Z_2 and Z_3. In Sect. 2.3, we force a model of "$Z_3 +$ Harrington's Principle" via class forcing using the reshaping technique assuming the existence of a remarkable cardinal. In Chap. 5, I force a model of "$Z_3 +$ Harrington's Principle" via set forcing without the use of the reshaping technique and assuming there exists a remarkable cardinal with a weakly inaccessible cardinal above it. For the proof of the main Theorem 5.1 in Chap. 5, I introduce the notion of "strong reflecting property" for L-cardinals in Sect. 5.2. In Chap. 6, I develop the full theory of the strong reflecting property for L-cardinals and characterize the strong reflecting property of ω_n for $n \in \omega$.

This book is based on my dissertation [9] and sequent work in [10–12]. In general, Chaps. 2, 4, 5, and 6 are revisions and improvements of dissertation and sequent work in [10–12] to fit into the current theme of concrete incompleteness for higher-order arithmetic.

I felt it was necessary for me to write this book for the following reasons. Firstly, this book contributes to the research program on concrete incompleteness for higher-order arithmetic and gives a systematic analysis of the Martin-Harrington Theorem in higher-order arithmetic. In particular, this book gives a specific example of concrete mathematical theorems which is expressible in second-order arithmetic, but the minimal system in higher-order arithmetic to prove it is Z_4.

Secondly, this book is a significant expansion and improvement over my dissertation. I have strengthened the main results and filled in some technical gaps in my dissertation. The large cardinal strength of "$\mathsf{Z}_2 + \mathsf{HP}$" and "$\mathsf{Z}_3 + \mathsf{HP}$" is not discussed in [9], while I establish the exact large cardinal strength of "$\mathsf{Z}_2 + \mathsf{HP}$" and "$\mathsf{Z}_3 + \mathsf{HP}$" below. Also, the large cardinal hypothesis used in my proof of forcing a model of "$\mathsf{Z}_3 + \mathsf{HP}$" via set forcing in Chap. 5 are much weaker than the hypothesis used in [9]. Furthermore, this book contains some new materials not covered in [9–12]. For these reasons, I felt it was necessary to write this book which contains all my current results on the analysis of the Martin-Harrington Theorem in higher-order arithmetic. I will assume that readers are already familiar with the basics of forcing, large cardinals, effective set theory, determinacy, admissible ordinals, and reverse mathematics. However, I will try my best to make this book self-contained.

This book makes a contribution to the foundations of mathematics and may be relevant for philosophers of mathematics and for three of the four major branches of mathematical logic, namely set theory (large cardinals, descriptive set theory, determinacy), recursion theory (admissible ordinals, higher recursion theory, the analytic hierarchy) and proof theory (reverse mathematics), and therefore certainly relevant for mathematical logicians.

Wuhan, China
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Yong Cheng

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Contents

Chapter 1
Introduction and Preliminaries

Abstract In this chapter, I provide an overview of Incompleteness, Reverse Mathematics, and Incompleteness for higher-order arithmetic, respectively in Sects. 1.1.1, 1.1.2 and 1.1.3. This should provide the reader with a good picture of the background and put the main results in this book into perspective. In Sect. 1.1.4, I review some of the notions and facts from Set Theory used in this book. In Sect. 1.2, I introduce the main research problems and outline the structure of this book.

1.1 Preliminaries

1.1.1 Basics of Incompleteness

In this section, I give a brief introduction to Gödel's incompleteness theorems. In particular, I present different versions of *Gödel's first incompleteness theorem* and *Gödel's second incompleteness theorem*: from the original version to the modern generalized versions.

Gödel's incompleteness theorem is one of the most remarkable and profound discoveries of the 20th century, an important milestone in the history of modern logic, which has had wide and profound influence on the development of logic, philosophy, mathematics, computer science and other fields, substantially shaping mathematical logic and foundations of mathematics after its publication in 1931.

The impact of Gödel's incompleteness theorem is not confined to the community of logic or mathematics. Indeed, Feferman writes the following about the impact of Gödel's incompleteness theorems.

> their relevance to mathematical logic (and its offspring in the theory of computation) is paramount; further, their philosophical relevance is significant, but in just what way is far from settled; and finally, their mathematical relevance outside of logic is very much unsubstantiated but is the object of ongoing, tantalizing efforts. ([1], p. 434).

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