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Mathematical Modelling, Applied Analysis and Computation

ICMMAAC 2018, Jaipur, India, July 6–8

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Preface

This book is based around the conference *International Conference on Mathematical Modelling, Applied Analysis and Computation (ICMMAAC 2018)*, held at JECRC, Jaipur, from 6 to 8 July 2018. The book contains several useful topics of mathematical modelling, applied analysis and computation having different applications in other scientific areas of research and study. It should be useful for graduate students, researcher and educators interested in diverse areas of research in mathematical sciences including modelling, analysis and computation. The general readers interested in tools and techniques from different areas of mathematical sciences having practical applications in real life should also find the book interesting and useful. The book consists of 20 chapters organized as follows:

Chapter "Certain Banach-Space Operators Acting on the Semicircular Elements Induced by Orthogonal Projections" presents certain Banach-space operators acting on (weighted-) semicircular elements induced by mutually orthogonal $|\mathbb{Z}|$-many projections. In particular, it emphasizes in cases where such operators are generated by *-isomorphisms induced by certain shifting processes on the set of all integers. The main results show not only how such Banach-space operators affect the original free-distributional data on (weighted-) semicircular elements, but also how the weighted-semicircular laws are preserved by the operators.

Chapter "Explicit Expressions Related to Degenerate Cauchy Numbers and Their Generating Function" establishes an explicit expression for degenerate Cauchy numbers and finds explicit, meaningful and significant expressions for coefficients in a family of nonlinear differential equations for the generating function of degenerate Cauchy numbers.

Chapter "Statistical Deferred Riesz Summability Mean and Associated Approximation Theorems for Trigonometric Functions" presents an idea of approximation via statistical deferred weighted (Riesz) summability mean for trigonometrical periodic functions defined over a Banach space $C_{2\pi}(\Re)$ and accordingly establishes a new approximation theorem (Korovkin-type). Furthermore, the chapter studies the rate of statistical deferred weighted summability and also establishes another result for the same set of functions by using the modulus of continuity. Finally, it considers a number of special cases and examples to exhibit the relevance of the obtained results and definitions provided in this chapter.

Chapter "On Pointwise Convergence of a Family of Nonlinear Integral Operators" presents some auxiliary theorems concerning existence and pointwise convergence of the certain operators. Then, it presented a Fatou-type convergence theorem for these operators. Finally, the rates of both pointwise and Fatou-type convergences have been established by using the derived results.

Chapter "Existence and Ulam's Type Stability of Integro Differential Equation with Non-instantaneous Impulses and Periodic Boundary Condition on Time Scales" presents existence and stability of integro-differential equation with periodic boundary condition and non-instantaneous impulses on time scales. Banach contraction theorem and nonlinear functional analysis have been used to establish these results. Moreover, to outline the utilization of these outcomes, an example is given.

Chapter "Introduction to Class of Uniformly Fractional Differentiable Functions" presents a new concept of uniformly fractional differentiable functions on an arbitrary interval I of R by using Caputo-type fractional derivative instead of the commonly used first-order derivative. Their interesting properties with few illustrations have been discussed in this chapter.

Chapter "Asymptotically Almost Automorphic Solution for Neutral Functional Integro Evolution Equations on Time Scales" studies the existence, uniqueness with stability consequence of asymptotically almost automorphic (AAA) solution for integro-neutral evolution equation on time scales by applying fixed-point hypothesis. It gives the time scale adaptation of (AAA) functions. Towards the end, a precedent is given for the adequacy of the hypothetical outcomes.

Chapter "An Integral Relation Associated with a General Class of Polynomials and the Aleph Function" reports new finite integral involving two general classes of polynomials with the Aleph function. This integral is supposed to be one of the most universal integrals evaluated until now and act as a key component from which one can obtain as its different special cases, integrals relating a large number of simpler special functions and polynomials. Some useful unique cases of the main outcome have also been discussed in the chapter.

Chapter "On the New Fractional Operator and Application to Nonlinear Bloch System" analyses the nonlinear Bloch system with a new fractional operator without singular kernel proposed by Caputo and Fabrizio. The commensurate and non-commensurate order nonlinear Bloch system is considered. Special solutions using a numerical scheme based on Lagrange interpolations are obtained. It studied the uniqueness and existence of the solutions employing the fixed-point theorem. Novel chaotic attractors with total order less than 3 are obtained.

Chapter "Fractional Order Integration and Certain Integrals of Generalized Multiindex Bessel Function" introduces generalized multiindex Bessel function and some formulas of the Riemann–Liouville fractional integration and differentiation operators. Further, certain integral formulas involving the newly defined generalized multiindex Bessel function have also been derived. It is also proved that such integrals are expressed in terms of the Fox–Wright function. The results presented here are general in nature and easily reducible to new and known results.

Chapter "Fractional Variational Iteration Method for Time Fractional Fourth-Order Diffusion-Wave Equation" applies fractional variational iteration method (FVIM) to solve numerically time-fractional diffusion-wave equation of order four. By using FVIM, a sequence converging rapidly to the exact solution of the fourth-order fractional diffusion-wave equation is obtained. Two test problems are presented to prove the merit of the proposed technique. Plotted graph shows that the numerical solution acquired by employed technique is similar to the exact solution.

Chapter "Analytical Approach to Fractional Navier–Stokes Equations by Iterative Laplace Transform Method" studies iterative Laplace transform scheme to examine fractional Navier–Stokes equations in cylindrical coordinates with initial conditions. The arbitrary ordered derivatives are described in terms of Caputo. By utilizing only the initial conditions, the analytical expressions are derived in the closed form. The results achieved with the aid of the proposed technique are graphically presented.

Chapter "Biological Model of Dengue Spread with Non-Markovian Properties" deals with converting the classical model to fractional model by using the concept of recently established fractional differential operators known as the Caputo-Fabrizio derivative to include into mathematical system the memory and the crossover effects. The new model was subjected to analysis of existence and uniqueness of the system solution to insure the well-posedness of the modified system. Due to the complexity of the new system, a newly introduced numerical scheme was used to solve the system and some numerical simulations were performed to see the effect of the Mittag–Leffler law that brings the crossover effect.

Chapter "Approximate Solution of Higher Order Two Point Boundary Value Problems Using Uniform Haar Wavelet Collocation Method" studies the numerical solution of second- and fourth-order two-point boundary value problems (B.V.P.) based on uniform Haar wavelet. It aims to convert higher order differential equations into a system of differential equations of lower order and then solve it by uniform Haar wavelet, which reduces the time and complexity of the system. The technique introduced in this chapter is easy to apply. The performance of the present method yields more accurate results on increasing the resolution level. To demonstrate the robustness and accuracy of the Haar wavelet collocation method, five problems have been solved and compared with the existing methods present in the literature.

Chapter "Solving Multi-objective Fractional Transportation Problem" deals with optimizing the objective function in the form of one or several ratios subject to some linear constraints. If in multi-objective transportation problem, objective function is in ration of two linear functions under some linear restrictions, then the problem is called multi-objective linear fractional transportation problem. A new method to solve multi-objective linear fractional transportation problem is suggested. Two numerical problems are presented to validate the proposed algorithm.

Chapter "On the Dark and Bright Solitons to the Negative-Order Breaking Soliton Model with (2+1)-Dimensional" studies the complex dynamics of cnoidal waves via the negative-order breaking soliton model with (2+1)-dimensional. This model is arisen in the (2+1)-dimensional interaction of the Riemann wave propagated between y-axis and x-axis. The improved Bernoulli sub-equation function method is used in obtaining some complex and dark solutions with hyperbolic function structure. It presents the interesting contour surfaces along with 2D and 3D graphics of the obtained analytical solutions in this study, plotted by using several computational programmes such as Matlab, Mathematica, and so on.

Chapter "A Reliable Analytical Algorithm for Cubic Isothermal Auto-Catalytic Chemical System" applies an algorithm for the q-homotopy analysis transform method (q-HATM) to solve the Cubic Isothermal Auto-catalytic Chemical System (CIACS). This technique is a combination of the Laplace decomposition method and the homotopy analysis scheme. This method gives the solution in the form of a rapidly convergent series with $\hbar$-curves are employed to determine the intervals of convergent. Averaged residual errors are used to determine the optimal values of $\hbar$. The behaviour of the solutions is shown graphically.

Chapter "Numerical Study of Effects of Adrenal Hormones on Lymphocytes" aims to study a mathematical model to examine the impact of adrenal hormones on the immune system with respect to time evolution and spatial distribution cells in response to hormones concentration. The steady state of the model is studied and found to be uniformly and asymptotically stable subject to the secretion and decay rates of hormones. The numerical experiments using the free diffusion equations further investigates the dynamic behaviour of the "bound" lymphocytes secretion rate of the adrenal hormones induced by psychological stress.

Chapter "Mathematical Modelling of Poor Nutrition in the Human Life Cycle" deals with a mathematical model as a system of nonlinear ordinary differential equations in order to investigate the effects of poor nutrition from conception to adulthood using the poor pregnant woman nutrient status. The steady states are studied and R_0 of poor nutrition in the society are calculated. To keep the society healthy and free of malnutrition, malnourished pregnant females are encouraged to eat foods that contain all the nutrients needed for development. The model is supported with numerical simulation.

Chapter "Characteristics of Homogeneous Heterogeneous Reaction on Flow of Walters' B Liquid Under the Statistical Paradigm" studies the significance of inclined MHD stagnant point flow of Walters B liquid because of stretched surface. Flow phenomenon is studied with Newtonian heating, homogeneous\heterogeneous reactions, Joule heating, and viscous dissipation. The nonlinear PDEs are converted to get nonlinear system of ODEs by invoking suitable transformations and solved by utilizing OHAM. Statistical methodology is used to check the significance and insignificance of the physical parameters via correlation coefficients and probable error. Characteristics of various sundry parameters on velocity, concentration, and temperature fields are studied. Friction and Nusselt numbers are calculated and discussed in detail.

The editors are grateful to the contributors for their cooperation and patience while the manuscripts were being reviewed and processed. The reviewers deserve our sincere gratitude for their wholehearted efforts in evaluating the manuscripts in a timely manner. The editors are also grateful to several colleagues and friends for their kind support during the execution of the task of brining out this volume.

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Certain Banach-Space Operators Acting on the Semicircular Elements Induced by Orthogonal Projections

Ilwoo Cho

Abstract The main purposes of this paper are (i) to construct-and-study (weighted-) semicircular elements induced by mutually orthogonal $|\mathbb{Z}|$-many projections, and the Banach $*$-probability space $\mathbb{L}_Q$ generated by these operators, (ii) to establish $*$-isomorphisms on $\mathbb{L}_Q$ from shifting processes on the set $\mathbb{Z}$ of integers, (iii) to consider the $*$-isomorphisms of (ii) as Banach-space operators acting on $\mathbb{L}_Q$ (by regarding the Banach $*$-algebra $\mathbb{L}_Q$ as a Banach space), and (iv) to compare the free-distributional data affected by the operators of (iii) from the original data.

Keywords Free probability · Weighted-semicircular elements · Semicircular elements · Integer-shifts · The integer-shift group · Integer-shift operators · The integer-shift-operator algebra

1991 Mathematics Subject Classification 46L10 · 46L40 · 46L53 · 46L54 · 47L15 · 47L30 · 47L55

1 Introduction

In this paper, certain Banach-space operators acting on (weighted-)semicircular elements induced by mutually orthogonal $|\mathbb{Z}|$-many projections are constructed-and-considered. In particular, we are interested in the cases where such operators are generated by $*$-isomorphisms induced by certain shifting processes on the set $\mathbb{Z}$ of all integers. The main results show not only how such Banach-space operators affect the original free-distributional data on (weighted-)semicircular elements, but also how our weighted-semicircular laws are preserved by the operators.

To study our topics, we (i) construct *weighted-semicircular*, and *semicircular elements* in a certain *Banach $*$-probability space* $\mathfrak{L}_Q(\mathbb{Z})$ induced by a fixed C^*-

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probability space (A, ψ) containing $|\mathbb{Z}|$-many mutually orthogonal projections, (ii) study *free distributions* of certain free reduced words in the Banach $*$-probabilistic sub-structure $\mathbb{L}_Q$ of $\mathfrak{L}_Q(\mathbb{Z})$, generated by the (weighted-)semicircular elements of (i), providing "non-zero" free distributions in $\mathfrak{L}_Q(\mathbb{Z})$, (iii) define suitable shift processes on $\mathbb{Z}$, and (iv) establish certain $*$-homomorphisms on $\mathbb{L}_Q$ induced by the shift processes of (iii), and then (v) compare the free-distributional data of the $*$-homomorphic images of the $*$-homomorphisms in (iv), and the original free-distributional data of (ii).

1.1 Motivations

There are many different approaches to construct semicircular elements (e.g., [1, 4, 6, 12, 15, 19–21]) in topological $*$-probability spaces (e.g., C^*-*probability spaces*, or W^*-*probability spaces*, or *Banach* $*$-*probability spaces*, etc.). The construction of semicircular elements in this paper is highly motivated by those of *weighted-semicircular elements* in certain Banach $*$-probability spaces of [5, 7, 8] induced by p-adic analysis on the *p-adic number fields* $\mathbb{Q}_p$, for *primes* p. So, our construction is different from those of earlier works (also, see [6]).

In [5, 8], we studied weighted-semicircular elements induced by measurable functions on p-adic number fields $\mathbb{Q}_p$ (e.g., [8]), and those from a *free product* Banach $*$-algebra induced from weighted-semicircular elements of [8] (e.g., [5]). The author and Jorgensen applied number-theoretic results (e.g., [10, 18]), and free-probabilistic techniques (e.g., [2–4, 13, 14, 16]) to consider free-probabilistic models of [8], and they realized that there are well-defined *weighted-semicircular elements*. Interestingly, these operators automatically generate corresponding semicircular elements. In [5], the author extended the constructions and the main results of [8] under *free product* over *primes*. The detailed properties of "p-adic" weighted-semicircular elements, and those of corresponding semicircular elements were studied there.

Motivated by [5, 8], the author considered the similar constructions of (weighted-) semicircular elements from arbitrary C^*-probability spaces containing $|\mathbb{Z}|$-many mutually orthogonal projections in [6, 7], by mimicking the constructions of [5, 8]. The main results of [6] show that whenever one can have mutually orthogonal $|\mathbb{Z}|$-many projections in a C^*-probability space, the corresponding weighted-semicircular elements whose weights are characterized by the free-distributional data of the projections; moreover, under suitable (additional) conditions, semicircular elements are well-defined (see short Sects. 3, 4 and 5, below).

In this paper, we are interested in certain Banach-space adjointable operators (in the sense of [9]) acting on weighted-semicircular elements of [6, 7]. Especially, they are induced by certain shift processes on the set $\mathbb{Z}$ of integers, and corresponding well-defined $*$-homomorphisms on the (weighted-)semicircular elements.

1.2 Overview

In Sect. 2, we briefly mention about backgrounds of our proceeding works. In short Sects. 3, 4 and 5, weighted-semicircular elements, and semicircular elements are constructed from mutually orthogonal $|\mathbb{Z}|$-many projections (e.g., [6, 7]).

In Sect. 6, we construct a suitable free-probabilistic, operator-algebraic structure $\mathbb{L}_Q$ generated by our (weighted-)semicircular elements under free product.

In Sect. 7, we define-and-study Banach-space adjointable operators acting on $\mathbb{L}_Q$. In particular, certain shifting processes on $\mathbb{Z}$ are defined in Sect. 7.1, and the corresponding $*$-isomorphisms are determined on $\mathbb{L}_Q$ in Sect. 7.2. We realize that the collection of such $*$-isomorphisms forms a subgroup $\mathfrak{B}$ of the automorphism group $Aut(\mathbb{L}_Q)$ of $\mathbb{L}_Q$. The structure theorem of this group $\mathfrak{B}$ is provided in Sect. 7.2: $\mathfrak{B}$ is group-isomorphic to the infinite cyclic abelian group $(\mathbb{Z}, +)$. We then study how the group $\mathfrak{B}$ generate our Banach-space adjointable operators (in the sense of [9]) on $\mathbb{L}_Q$, and how they affects the free-probabilistic information on $\mathbb{L}_Q$ in Sect. 7.3.

In Sect. 8, we re-characterize the free-distributional data of Sect. 7.3 with help of the group-isomorphic relation of Sect. 7.2. Also, group dynamical systems of $\mathfrak{B}$ are studied.

2 Preliminaries

Readers can review fundamental analytic-and-combinatorial free probability theory from [17, 19] (and the cited papers therein). *Free probability* is understood as the noncommutative operator-algebraic version of classical *measure theory* and *statistics*. The classical *independence* is replaced by so-called the *freeness*, by replacing measures on sets to linear functionals on algebras. It has various applications not only in pure mathematics (e.g., [2–4, 12, 14, 15]), but also in related fields (e.g., [5–8, 13, 16, 20, 21]).

In particular, we use combinatorial free-probabilistic approach of *Speicher* (e.g., [17]). *Free moments* and *free cumulants* of operators will be computed without introducing detailed concepts. Also, *free product* (in the sense of [17, 19]) will be used without precise introduction.

3 Fundamental Settings

In this section, we establish backgrounds of our proceeding works. Let (B, φ) be a topological $*$-probability space (a C^*-probability space, or a W^*-probability space, or a Banach $*$-probability space, etc), where B is a topological $*$-algebra (a C^*-algebra, resp., a W^*-algebra, resp., a Banach $*$-algebra, etc), and φ is a (bounded or unbounded) linear functional on B.

An operator a of B is said to be a *free random variable*, whenever it is regarded as an element of (B, φ). As usual in *operator theory*, an operator a is said to be *self-adjoint*, if $a^* = a$ in B, where a^* is the *adjoint of* a (e.g., [11]).

Definition 3.1 A self-adjoint free random variable a is said to be weighted-semicircular in (B, φ) with its weight $t_0 \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ (or, in short, t_0-semicircular), if a satisfies the free cumulant computations,

$$k_n(a, \ldots, a) = \begin{cases} k_2(a, a) = t_0 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $k_n(\ldots)$ is the free cumulant on B in terms of φ under the Möbius inversion of [17].

If $t_0 = 1$ in (3.1), the 1-semicircular element a is said to be semicircular in (B, φ), i.e., a is semicircular in (B, φ), if a satisfies

$$k_n(a, \ldots, a) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

for all $n \in \mathbb{N}$.

By the Möbius inversion of [17], one can characterize the weighted-semicircularity (3.1) as follows: a self-adjoint operator a is t_0-semicircular in (B, φ), if and only if

$$\varphi(a^n) = \omega_n \left(t_0^{\frac{n}{2}} c_{\frac{n}{2}} \right), \tag{3.3}$$

where

$$\omega_n \overset{def}{=} \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$, and

$$c_k = \frac{1}{k+1} \binom{2k}{k} = \frac{(2k)!}{k!(k+1)!}$$

are the k-th Catalan numbers for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Similarly, a self-adjoint free random variable a is semicircular in (B, φ), if and only if a is 1-semicircular in (B, φ), if and only if

$$\varphi(a^n) = \omega_n c_{\frac{n}{2}}, \tag{3.4}$$

by (3.3), for all $n \in \mathbb{N}$, where ω_n are in the sense of (3.3).

So, we will use the t_0-semicircularity (3.1) (or the semicircularity (3.2)) and its characterization (3.3) (resp., (3.4)) alternatively from below.

If a is a self-adjoint free random variable in (B, φ), then

$$\text{the } \textit{free moments } \left(\varphi(a^n)\right)_{n=1}^{\infty},$$

and

$$\text{the}\,\textit{free cumulants}\ \ (k_n(a, \ldots, a))_{n=1}^{\infty}$$

provide equivalent free-distributional data of a in (B, φ) (e.g., [17]). Indeed, the *Möbius inversion* makes us have

$$\varphi(a^n) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} k_{|V|}(a, \ldots, a) \right),$$

and

$$k_n(a, \ldots, a) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \theta} \varphi(a^{|V|}) \right) \mu_\pi,$$

where $NC(n)$ is the *lattice* consisting of all *noncrossing partitions* over $\{1, \ldots, n\}$, and "$V \in \pi$" means "V is a *block of* π," and where

$$\mu_\pi = \mu(\pi, 1_n)$$

the Möbius functional value at $(\pi, 1_n)$, where 1_n is the maximal partition of $NC(n)$ consisting of only one block, for all $n \in \mathbb{N}$.

In the rest of this section, we fix a C^*-probability space (A, ψ), and assume that there are $|\mathbb{Z}|$-many projections $\{q_j\}_{j \in \mathbb{Z}}$ in the C^*-algebra A, i.e., the operators q_j satisfy

$$q_j^* = q_j = q_j^2 \text{ in } A,$$

for all $j \in \mathbb{Z}$. Assume further that these projections $\{q_j\}_{j \in \mathbb{Z}}$ are *mutually orthogonal* from each other in A, in the sense that:

$$q_i q_j = \delta_{i,j} q_j \text{ in } A, \ \text{ for all } i, j \in \mathbb{Z}, \tag{3.5}$$

where δ is the *Kronecker delta*.

Now, we fix the family $\{q_j\}_{j \in \mathbb{Z}}$ of mutually orthogonal projections (3.5) of A, and we denote it by $\mathbf{Q}$, i.e.,

$$\mathbf{Q} = \{q_j : j \in \mathbb{Z}\} \text{ in } A, \tag{3.6}$$

satisfying (3.5).

Remark 3.1 One can have such a C^*-algebraic structure A containing a family $\mathbf{Q}$ in the sense of (3.6), naturally, or artificially. Clearly, in the settings of [5, 8], one can naturally take such structures.

Suppose there is a C^*-algebra A_0 containing a family $\mathbf{Q}_N = \{q_1, \ldots, q_N\}$ of mutually orthogonal N-many projections $q_1, \ldots, q_N$, for $N \in \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Then, under suitable direct product, or tensor product, or free product of copies of A_0 with product topology, one can construct a C^*-algebra A containing a family $\mathbf{Q}$ with

$|\mathbb{Z}|$-many mutually orthogonal projections, where $\mathbf{Q}_0$ is contained in $\mathbf{Q}$, and every projection of $\mathbf{Q}$ is unitarily equivalent to a projection of $\mathbf{Q}_0$ in A.

And let Q be the C^*-subalgebra of A generated by the family $\mathbf{Q}$ of (3.6),

$$Q \overset{def}{=} C^* (\mathbf{Q}) \subseteq A. \tag{3.7}$$

Then it is easy to check that:

Proposition 3.1 *Let Q be a C^*-subalgebra (3.7) of a C^*-algebra A, generated by* $\mathbf{Q}$ *of (3.6). Then*

$$Q \overset{*\text{-}iso}{=} \underset{j \in \mathbb{Z}}{\oplus} \left(\mathbb{C} \cdot q_j\right) \overset{*\text{-}iso}{=} \mathbb{C}^{\oplus |\mathbb{Z}|}, \tag{3.8}$$

in A.

Proof The proof of (3.8) is straightforward by the mutual-orthogonality (3.5) of the generator set $\mathbf{Q}$ of Q in A. ■

Define now linear functionals ψ_j on the C^*-algebra Q by

$$\psi_j (q_i) = \delta_{ij} \psi (q_j), \text{ for all } i \in \mathbb{Z}, \tag{3.9}$$

for all $j \in \mathbb{Z}$, where ψ is the linear functional of the fixed C^*-probability space (A, ψ). The linear functionals $\{\psi_j\}_{j \in \mathbb{Z}}$ of (3.9) are well-defined on Q by the structure theorem (3.8).

Assumption Let (A, ψ) be a fixed C^*-probability space, and let Q be the C^*-subalgebra (3.7) of A. In the rest of this paper, we further assume that

$$\psi (q_j) \in \mathbb{C}^{\times}, \text{ for all } j \in \mathbb{Z}.$$

□

By (3.7) and (3.8), if $T \in Q$, then

$$T = \sum_{j \in \mathbb{Z}} t_j q_j \qquad (\text{with } t_j \in \mathbb{C}),$$

and hence,

$$\psi_j(T) = t_j \psi (q_j),$$

by (3.9), for all $j \in \mathbb{Z}$.

Definition 3.2 The C^*-probability spaces $\left(Q, psi_j\right)$ are called the j-th C^*-probability spaces of Q in a given C^*-probability space (A, ψ), where Q is in the sense of (3.7), and ψ_j are in the sense of (3.9), for all $j \in \mathbb{Z}$.

Now, let's define bounded linear transformations c and a acting on the C^*-algebra Q, by linear morphisms satisfying

$$c\left(q_j\right) = q_{j+1}, \text{ and } a\left(q_j\right) = q_{j-1}, \tag{3.10}$$

for all $j \in \mathbb{Z}$. Then c and a are well-defined bounded linear operators "on Q." One can understand they are *Banach-space operators* in the *operator space* $B(Q)$ consisting of all bounded linear transformations acting on Q, by regarding Q as a Banach space equipped with its C^*-norm (e.g., [9]).

Definition 3.3 We call these Banach-space operators c and a of (3.10), the *creation*, respectively, the *annihilation* on Q.

The creation c and the annihilation a on Q are indeed well-defined because of the structure theorem (3.8) of Q. Define now a new Banach-space operator l on Q by

$$l = c + a \in B(Q). \tag{3.11}$$

Definition 3.4 We call the Banach-space operator $l \in B(Q)$ of (3.11), the *radial operator on* Q.

By the definition (3.11), one has

$$l\left(\sum_{j\in\mathbb{Z}} t_j q_j\right) = \sum_{j\in\mathbb{Z}} t_j\left(q_{j+1} + q_{j-1}\right), \text{ on } Q.$$

Now, define a closed subspace $\mathfrak{L}$ of $B(Q)$ by

$$\mathfrak{L} \overset{def}{=} \overline{\mathbb{C}[\{l\}]}^{\|.\|}, \tag{3.12}$$

generated by the radial operator l of (3.11), where the operator norm $\|.\|$ on the operator space $B(Q)$ is defined to be

$$\|T\| = \sup\{\|Tq\|_Q : \|q\|_Q = 1\},$$

for all $T \in B(Q)$, where $\|.\|_Q$ is the C^*-*norm on* Q (inherited from the C^*-norm on A), and where $\overline{X}^{\|.\|}$ mean the *operator-norm closures* of subsets X of the *operator space* $B(Q)$ (e.g., [9]). It is not difficult to check that, by the definition (3.12), this subspace $\mathfrak{L}$ forms an algebra in the vector space $B(Q)$, i.e., it forms a Banach algebra.

On this Banach algebra $\mathfrak{L}$ of (3.12), define a unary operation $(*)$ by

$$\left(\sum_{n=0}^{\infty} t_n l^n\right)^* = \sum_{n=0}^{\infty} \overline{t_n} l^n \text{ in } \mathfrak{L}, \tag{3.13}$$

where $\overline{z}$ mean the *conjugates of* $z \in \mathbb{C}$.

Then this operation (3.13) becomes a well-defined *adjoint on* the Banach algebra $\mathfrak{L}$ of (3.12) (e.g., [11]), and hence, every element of $\mathfrak{L}$ is *adjointable* in $B(Q)$ (e.g., [9]). So, the algebra $\mathfrak{L}$ forms a *Banach* $*$*-algebra* in $B(Q)$ with the adjoint (3.13).

Definition 3.5 We call the Banach $*$-algebra $\mathfrak{L}$ of (3.12), the *radial* (*Banach* $*$*-)algebra on* Q (or, in the operator space $B(Q)$).

Now, let $\mathfrak{L}$ be the radial algebra on Q. Define the *tensor product Banach* $*$*-algebra* $\mathfrak{L}_Q$,

$$\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q, \tag{3.14}$$

where $\otimes_{\mathbb{C}}$ is the tensor product of Banach $*$-algebras.

Since $\mathfrak{L}$ is a Banach $*$-algebra, and Q is a C^*-algebra, the tensor product $\mathfrak{L}_Q$ of (3.14) is a well-defined Banach $*$-algebra under product topology.

Definition 3.6 We call the tensor product Banach $*$-algebra $\mathfrak{L}_Q$ of (3.14), the radial projection (Banach $*$-)algebra on Q.

4 Weighted-Semicircular Elements Induced by Q

Throughout this section, let's fix the settings of Sect. 3. We here construct weighted-semicircular elements induced by the family $\mathbf{Q}$ of mutually orthogonal projections generating the radial projection algebra $\mathfrak{L}_Q$ of (3.14). Let (Q, ψ_j) be the j-th C^*-probability spaces of Q in (A, ψ), where ψ_j are in the sense of (3.9), for all $j \in \mathbb{Z}$.

Remark that, if u_j are the generating operators of $\mathfrak{L}_Q$,

$$u_j \overset{def}{=} l \otimes q_j \in \mathfrak{L}_Q, \text{ for all } j \in \mathbb{Z}, \tag{4.1}$$

then

$$u_j^n = \left(l \otimes q_j\right)^n = l^n \otimes q_j, \text{ for all } n \in \mathbb{N},$$

since $q_j^n = q_j$, for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$.

Then one can construct a linear functional φ_j on $\mathfrak{L}_Q$ by a linear morphism satisfying that

$$\varphi_j\left((l \otimes q_i)^n\right) \overset{def}{=} \psi_j\left(l^n(q_i)\right), \tag{4.2}$$

for all $n \in \mathbb{N}$, for all $i, j \in \mathbb{Z}$.

These linear functionals φ_j of (4.2) are well-defined by (3.8), (3.12) and (3.14), for all $j \in \mathbb{Z}$.

Definition 4.1 We call the Banach $*$-probability spaces

$$\left(\mathfrak{L}_Q, \varphi_j\right), \textit{ for all } j \in \mathbb{Z}, \tag{4.3}$$

the j-th (*Banach-*$*$*-)probability spaces on* Q.

Observe that, if c and a are the creation, respectively, the annihilation on Q of (3.10), then

$$ca = 1_Q = ac, \text{ the identity operator on } Q. \tag{4.4}$$

Indeed, for any generators $q_j \in \mathbf{Q}$ of Q,

$$ca\left(q_j\right) = c\left(a\left(q_j\right)\right) = c\left(q_{j-1}\right) = q_{j-1+1} = q_j,$$

and

$$ac\left(q_j\right) = a\left(c\left(q_j\right)\right) = a\left(q_{j+1}\right) = q_{j+1-1} = q_j,$$

for all $j \in \mathbb{Z}$. More generally, one has

$$\begin{array}{l} c^n a^n = 1_Q = a^n c^n, \text{ for all } n \in \mathbb{N}, \quad \text{and} \\ c^{n_1} a^{n_2} = a^{n_2} c^{n_1}, \text{ for all } n_1, n_2 \in \mathbb{N}, \end{array} \tag{4.4$'$}$$

by (4.4).

Thus, one obtains that

$$l^n = (c+a)^n = \sum_{k=0}^{n} \binom{n}{k} c^k a^{n-k}, \tag{4.5}$$

for all $n \in \mathbb{N}$, by (4.4)$'$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \forall k \leq n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Note that, for any $n \in \mathbb{N}$,

$$l^{2n-1} = \sum_{k=0}^{2n-1} \binom{2n-1}{k} c^k a^{n-k}, \tag{4.6}$$

by (4.5). So, the formula (4.6) does not contain 1_Q-terms by (4.4) and (4.4)$'$.

Note also that, for any $n \in \mathbb{N}$, one has

$$l^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} c^k a^{n-k} = \binom{2n}{n} c^n a^n + [\text{Rest terms}], \tag{4.7}$$

by (4.5). So, l^{2n} contains $\binom{2n}{n}$-many 1_Q-terms by (4.4)′ and (4.7).

Proposition 4.1 *Let l be the radial operator (3.1) on Q. Then*

(4.8) l^{2n-1} does not contain $1_Q -$ terms in $\mathfrak{L}$,
(4.9) l^{2n} contains $\binom{2n}{n} \cdot 1_Q$ in $\mathfrak{L}$.

Proof The statements (4.8) and (4.9) are proven by (4.6), respectively, by (4.7). ■

Remark that, since

$$u_j^n = \left(l \otimes q_j\right)^n = l^n \otimes q_j,$$

one has

$$\varphi_j\left(u_j^{2n-1}\right) = \psi_j\left(l^{2n-1}\left(q_j\right)\right) = 0, \tag{4.10}$$

for all $n \in \mathbb{N}$, by (3.9) and (4.8).

Similarly, we have

$$\varphi_j\left(u_j^{2n}\right) = \psi_j\left(l^{2n}\left(q_j\right)\right) = \psi_j\left(\binom{2n}{n} q_j + [\text{Rest terms}]\right)$$

by (4.7)

$$= \binom{2n}{n} \psi_j\left(q_j\right) = \binom{2n}{n} \psi\left(q_j\right),$$

by (3.9) and (4.9). I.e.,

$$\varphi_j\left(u_j^{2n}\right) = \binom{2n}{n} \psi\left(q_j\right), \tag{4.11}$$

for all $n \in \mathbb{N}$.

Thus, one obtains the following free-distributional data on the j-th probability space $\left(\mathfrak{L}_Q, \varphi_j\right)$, for $j \in \mathbb{Z}$.

Theorem 4.2 *Fix $j \in \mathbb{Z}$, and let $u_k = l \otimes q_k$ be the k-th generating operators of the j-th probability space $(\mathfrak{L}_Q, \varphi_j)$, for all $k \in \mathbb{Z}$, for $j \in \mathbb{Z}$. Then*

$$\varphi_j\left(u_k^n\right) = \delta_{j,k}\omega_n\left(\left(\frac{n}{2}+1\right)\psi\left(q_j\right)\right)c_{\frac{n}{2}}, \tag{4.12}$$

where ω_n are in the sense of (3.3) for all $n \in \mathbb{N}$, and c_k are the k-th Catalan numbers for all $k \in \mathbb{N}$.

Proof First, take the j-th generating operator u_j in the j-th probability space $\left(\mathfrak{L}_Q, \varphi_j\right)$, for $j \in \mathbb{Z}$. By (4.10) and (4.11), one can get that:

$$\varphi_j\left(u_j^{2n-1}\right) = 0,$$

and

$$\begin{aligned}\varphi_j\left(u_j^{2n}\right) &= \binom{2n}{n}\psi\left(q_j\right) = \binom{n+1}{n+1}\binom{2n}{n}\psi\left(q_j\right) \\ &= \left((n+1)\psi\left(q_j\right)\right)\left(\frac{1}{n+1}\binom{2n}{n}\right) \\ &= \left((n+1)\psi\left(q_j\right)\right)c_n,\end{aligned}$$

for all $n \in \mathbb{N}$. So,

$$\varphi_j\left(u_j^n\right) = \omega_n\left((n+1)\psi(q_j)\right)c_n, \text{ for all } n \in \mathbb{N}.$$

Assume now that $k \neq j$ in $\mathbb{Z}$. Then, by the definition (4.2) of φ_j (and by the definition (3.9) of ψ_j),

$$\varphi_j\left(u_k^n\right) = 0, \text{ for all } n \in \mathbb{N}.$$

Therefore, the formula (4.12) holds. ■

Motivated by (4.12), we define a linear morphism,

$$E_{j,Q} : \mathfrak{L}_Q \to \mathfrak{L}_Q$$

by a surjective linear transformation satisfying

$$E_{j,Q}\left(u_i^n\right) \overset{def}{=} \begin{cases} \frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)}u_j^n & \text{if } i = j \\ 0_{\mathfrak{L}_Q}, \text{ the zero operator of } \mathfrak{L}_Q & \text{otherwise,}\end{cases} \tag{4.13}$$

for all $n \in \mathbb{N}$, $i, j \in \mathbb{Z}$, where $[\frac{n}{2}]$ mean the *minimal integers* greater than or equal to $\frac{n}{2}$, for example,

$$\left[\frac{3}{2}\right] = 2 = \left[\frac{4}{2}\right].$$

The linear transformations $E_{j,Q}$ of (4.13) are well-defined bounded linear transformations on $\mathfrak{L}_Q$, because of the cyclicity (3.12) of the tensor factor $\mathfrak{L}$ of $\mathfrak{L}_Q$, and the structure theorem (3.8) of the other tensor factor Q of $\mathfrak{L}_Q$, for all $j \in \mathbb{Z}$.

Define now new linear functionals τ_j on $\mathfrak{L}_Q$ by

$$\tau_j \overset{def}{=} \varphi_j \circ E_{j,Q} \text{ on } \mathfrak{L}_Q, \text{ for all } j \in \mathbb{Z}, \tag{4.14}$$

where φ_j are in the sense of (4.2), and $E_{j,Q}$ are in the sense of (4.13).

Definition 4.2 The well-defined Banach $*$-probability spaces

$$\mathfrak{L}_Q(j) \overset{denote}{=} \left(\mathfrak{L}_Q, \tau_j\right) \tag{4.15}$$

are called the j-th filtered (Banach-$*$-)probability spaces of $\mathfrak{L}_Q$, where τ_j are the linear functionals (4.14) on $\mathfrak{L}_Q$, for all $j \in \mathbb{Z}$.

On the j-th filtered probability space $\mathfrak{L}_Q(j)$ of (4.15), One can get that

$$\begin{aligned}\tau_j\left(u_j^n\right) &= \varphi_j\left(E_{j,Q}\left(u_j^n\right)\right)\\ &= \varphi_j\left(\frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)}\left(u_j^n\right)\right) = \frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)}\varphi_j\left(u_j^n\right)\\ &= \frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)}\omega_n\left(\left(\frac{n}{2}+1\right)\psi\left(q_j\right)\right)c_{\frac{n}{2}},\end{aligned}$$

by (4.12), i.e.,

$$\tau_j\left(u_j^n\right) = \omega_n \psi(q_j)^n c_{\frac{n}{2}}, \tag{4.16}$$

for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$, where ω_n are in the sense of (3.3).

Lemma 4.3 *Let $\mathfrak{L}_Q(j) = (\mathfrak{L}_Q, \tau_j)$ be the j-th filtered probability space of $\mathfrak{L}_Q$, for an arbitrarily fixed $j \in \mathbb{Z}$. Then*

$$\tau_j\left(u_i^n\right) = \delta_{j,i}\left(\omega_n \psi(q_j)^n c_{\frac{n}{2}}\right), \tag{4.17}$$

where ω_n are in the sense of (3.3), for all $n \in \mathbb{N}$, for all $i \in \mathbb{Z}$.

Proof If $i = j$ in $\mathbb{Z}$, then the free-momental data (4.17) holds true by (4.16), for all $n \in \mathbb{N}$.

If $i \neq j$ in $\mathbb{Z}$, then, by the very definition (4.13) of the j-th filterization $E_{j,Q}$, and also by the definition (4.2) of φ_j,

$$\tau_j\left(u_i^n\right) = 0, \text{ for all } n \in \mathbb{N}.$$

Therefore, the free-distributional data (4.17) holds true, for all $i \in \mathbb{Z}$. ■

The following theorem is proven by the above free-distributional data (4.17) in terms of the weighted-semicircularity characterization (3.3) of the weighted-semicircularity (3.1).

Theorem 4.4 *Let $\mathfrak{L}_Q(j)$ be the j-th filtered probability space $\left(\mathfrak{L}_Q, \tau_j\right)$ of $\mathfrak{L}_Q$, for $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be the "j-th" generating operator of $\mathfrak{L}_Q$. Then u_j is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$.*

Proof First of all, the operator u_j is self-adjoint in $\mathfrak{L}_Q$ (for all $j \in \mathbb{Z}$). Indeed,

$$u_j^* = \left(l \otimes q_j\right)^* = l \otimes q_j = u_j$$

(for all $j \in \mathbb{Z}$) by (3.13).

Let's fix $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be the j-th generating operator of the j-th filtered probability space $\mathfrak{L}_Q(j)$. Then, by (4.17), we have that

$$\tau_j\left(u_j^n\right) = \omega_n \left(\psi\left(q_j\right)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, and where c_k are the k-th Catalan numbers, for all $k \in \mathbb{N}_0$.

Therefore, by the characterization (3.3) of the weighted-semicircularity (3.1), this self-adjoint element u_j is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$. ■

The above theorem shows that, for any $j \in \mathbb{Z}$, the j-th generating operator u_j is $\psi(q_j)^2$-semicircular in the j-th filtered probability space $\mathfrak{L}_Q(j)$ of Q, by (4.17). Meanwhile, also by (4.17), one can verify the following result, too.

Theorem 4.5 *Let $u_i = l \otimes u_i$ be the i-th generating operators of the j-th filtered probability space $\mathfrak{L}_Q(j)$, for all $j \neq i \in \mathbb{Z}$. Then u_i have the zero free distribution in $\mathfrak{L}_Q(j)$.*

Proof Let $\mathfrak{L}_Q(j)$ be the j-th filtered probability space for a fixed $j \in \mathbb{Z}$, and assume $i \neq j$ in $\mathbb{Z}$. Consider the i-th generating operators u_i of $\mathfrak{L}_Q(j)$. It is shown already that u_i are self-adjoint in $\mathfrak{L}_Q$, and hence, the free distributions of u_i are completely characterized by the free-momental sequences

$$\left(\tau_j(u_i^n)\right)_{n=1}^{\infty} = (0, 0, 0, \ldots),$$

the zero sequence, by (4.17). It guarantees that the free distributions of $u_i \in \mathfrak{L}_Q(j)$ are the zero free distribution, for all $j \neq i \in \mathbb{Z}$. ■

The above two theorems characterize the free-probabilistic information of the generators $\{u_i\}_{i \in \mathbb{Z}}$ of our j-th filtered probability space $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$. From below, we focus on "non-zero" free-distributional data on $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$.

By the Möbius inversion of [17], if u_i are the i-th generating operators of the j-th filtered probability space $\mathfrak{L}_Q(j)$, then