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Nonlinear Dispersive Partial Differential Equations and Inverse Scattering

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FIELDS The Fields Institute for Research in Mathematical Sciences

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Preface

This volume contains lectures and invited papers from the Focus Program on "Nonlinear Dispersive Partial Differential Equations and Inverse Scattering" held at the Fields Institute from July 31 to August 18, 2017.

Our conference coincided with the fiftieth anniversary of the discovery by Gardner, Greene, Kruskal, and Miura[1] that the Korteweg-de Vries (KdV) equation could be integrated by exploiting a remarkable connection between KdV and the spectral theory of Schrödinger's equation in one space dimension. This led to the discovery of a number of completely integrable models of dispersive wave propagation including the one-dimensional cubic nonlinear Schrödinger (NLS) equation, the derivative NLS equation and, in two dimensions, the Davey-Stewartson, Kadomtsev-Petviashvili, and the Novikov-Veselov equations. These models have been extensively studied and, in some cases, the inverse scattering theory has been put on rigorous footing and used as a powerful analytical tool to study global well-posedness and elucidate long-time asymptotic behavior of the solutions, including dispersion, soliton solutions, and semiclassical limits.

Pioneering works in this literature are the papers of Deift and Zhou which establish the "nonlinear steepest descent" method for solving the Riemann-Hilbert problems at the heart of inverse scattering. More recently, rigorous treatments of inverse scattering have led to advances in the understanding of dispersive partial differential equations (PDEs), in particular addressing questions concerning asymptotic stability of solitons and the soliton resolution conjecture. Motivated by completely integrable models as well as by considerations stemming from the physical origin of the equations, the existence and stability properties of special solutions such as traveling waves and solitary waves have been thoroughly investigated by pure variational and PDE techniques. A common theme in many PDE results is that equations which are very far from the integrable cases nonetheless exhibit similar qualitative and asymptotic behavior.

[1]Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura, *Phys. Rev. Lett.* **19**, 1095 (1967).

This conference brought together researchers in completely integrable systems and PDE with the goal of advancing the understanding of qualitative and long-time behavior in dispersive nonlinear equations.

Percy Deift's Coxeter Lectures, "Fifty Years of KdV: An integrable system," provided an introduction and overview of the completely integrable method and its applications in dynamical systems, probability, statistical mechanics, and many other areas of applied mathematics. The first week of the focus program consisted of expository lectures by Walter Craig, Patrick Gérard, Peter D. Miller, Peter A. Perry, and Jean-Claude Saut. Walter Craig presented a series of lectures on Hamiltonian PDEs. The notion of phase space, flows for PDEs and conserved integrals of motion were introduced as well as normal forms transformations and the concept of Nekhoroshev stability. Patrick Gérard's lectures on "Wave Turbulence and Complete Integrability" discussed a completely integrable model, the cubic Szegő equation, for which the growth of Sobolev norms, thought to characterize turbulent behavior, can be studied explicitly. Jean-Claude Saut provided a comprehensive survey of results for the Benjamin-Ono and intermediate long-wave equations, both completely integrable models for one-dimensional wave propagation, by inverse scattering and PDE methods. Peter A. Perry's lectures gave a rigorous treatment of the inverse scattering method for the cubic defocussing nonlinear Schrödinger equation in one dimension (based on the foundational work of Deift and Zhou) and for the defocussing Davey-Stewartson equation in two space dimensions (based on Perry's work and more recent work of Nachman, Regev, and Tataru). Peter D. Miller described some theory of Riemann-Hilbert problems, culminating in a description of the Deift-Zhou steepest descent method. The paper of Dieng, McLaughlin, and Miller in this volume provides a detailed exposition of a useful generalization, namely the $\overline{\partial}$-steepest descent method for Riemann-Hilbert problems, pioneered by Dieng and McLaughlin. This method has become an effective tool for attacking soliton resolution for completely integrable, dispersive PDEs.

The mini-school was followed by two workshops on various aspects of dispersive PDEs. The research papers collected here include new results on the focusing NLS equation, the massive Thirring model, and the BBM equation as dispersive PDE in one space dimension, as well as the KP-II equation, the Zakharov-Kuznetsov equation, and the Gross-Pitaevskii equation as dispersive PDE in two space dimensions.

The editors of this volume would like to thank the Fields Institute for Research in the Mathematical Sciences and its Director, Dr. Ian Hambleton, for their generous support. We are grateful to Esther Berzunza and Dr. Huaxiong Huang for their assistance with the organization of the conference as well as to Tyler Wilson and the Springer team for their assistance with the publication of this special volume. We are also grateful to the participants of the conference and to the authors for their contributions to this volume as well as to the referees for their invaluable help during the review process.

Finally, we dedicate this volume to our friend and colleague Walter Craig, who sadly passed away on January 18, 2019. Walter was a world renowned scholar for his work on nonlinear partial differential equations, infinite dimensional Hamiltonian systems, and their applications, in particular, to fluid dynamics. A constant source of inspiration, Walter was a generous mentor and a wonderful collaborator. He will be greatly missed by all who had the privilege of knowing him as a mathematician, colleague, and dear friend.

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Part I
Lectures

Three Lectures on "Fifty Years of KdV: An Integrable System"

Percy A. Deift

The goal in the first two Coxeter lectures was to give an answer to the question

"What is an integrable system?"

and to describe some of the tools that are available to identify and integrate such systems. The goal of the third lecture was to describe the role of integrable systems in certain numerical computations, particularly the computation of the eigenvalues of a random matrix. This paper closely follows these three Coxeter lectures, and is written in an informal style with an abbreviated list of references. Detailed and more extensive references are readily available on the web. The list of authors mentioned is not meant in any way to be a detailed historical account of the development of the field and I ask the reader for his'r indulgence on this score.

The notion of an integrable system originates in the attempts in the seventeenth and eighteenth centuries to integrate certain specific dynamical systems in some explicit way. Implicit in the notion is that the integration reveals the long-time behavior of the system at hand. The seminal event in these developments was Newton's solution of the two-body problem, which verified Kepler's laws, and by the end of the nineteenth century many dynamical systems of great interest had been integrated, including classical spinning tops, geodesic flow on an ellipsoid, the Neumann problem for constrained harmonic oscillators, and perhaps most spectacularly, Kowalewski's spinning top. In the nineteenth century, the general and very useful notion of *Liouville integrability* for Hamiltonian systems, was introduced: If a Hamiltonian system with Hamiltonian H and n degrees of freedom

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has n independent, Poisson commuting integrals, $I_1, \ldots, I_n$, then the flow $t \mapsto z(t)$ generated by H can be integrated explicitly by quadrature, or symbolically,

$$\begin{cases} I_k\,(z(t)) \quad = \text{const}\,,\ 1 \le k \le n,\ \text{rank}\ (dI_1, \ldots, dI_n) = n,\ \left\{I_k, I_j\right\} = 0, \\ \qquad\qquad 1 \le j,\ k \le n \quad \Rightarrow\ \text{explicit integration.} \end{cases} \tag{1}$$

Around the same time the Hamilton-Jacobi equation was introduced, which proved to be equally useful in integrating systems.

The modern theory of integrable systems began in 1967 with the discovery by Gardner et al. [19] of a method to solve the Korteweg de Vries (KdV) equation

$$q_t + 6qq_x - q_{xxx} = 0 \tag{2}$$
$$q(x,t)_{t=0} = q_0(x) \to 0 \quad \text{as} \quad |x| \to \infty\,.$$

The method was very remarkable and highly original and expressed the solution of KdV in terms of the spectral and scattering theory of the Schrödinger operator $L(t) = -\partial_x^2 + q(x,t)$, acting in $L^2(-\infty < x < \infty)$ for each t. In 1968 Peter Lax [26] reformulated [19] in the following way. For $L(t) = -\partial_x^2 + q(x,t)$ and $B(t) \equiv 4\partial_x^3 - 6q\,\partial_x - 3q_x$.

$$\begin{aligned} \text{KdV}\ &\equiv \partial_t\, L = [B, L] = BL - LB \\ &\equiv\ \text{isospectral deformation of } L(t) \end{aligned} \tag{3}$$

$$\Rightarrow\ \text{spec}\ (L(t)) = \text{spec}\ (L(0)) \Rightarrow\ \text{integrals of the motion for KdV.}$$

L, B are called *Lax pairs*: By the 1970s, Lax pairs for the Nonlinear Schrödinger Equation (NLS), the Sine-Gordon equation, the Toda lattice, ..., had been found, and these systems had been integrated as in the case of KdV in terms of the spectral and scattering theory of their associated "L" operators.

Over the years there have been many ideas and much discussion of what it means for a system to be integrable, i.e. explicitly solvable. Is a Hamiltonian system with n degrees of freedom integrable if and only if the system is Liouville integrable, i.e. the system has n independent, commuting integrals? Certainly as explained above, Liouville integrability implies explicit solvability. But is the converse true? If we can solve the system in some explicit fashion, is it necessarily Liouville integrable? We will say more about this matter further on. Is a system integrable if and only if it has a Lax pair representation as in (3)? There is, however, a problem with the Lax-pair approach from the get-go. For example, if we are investigating a flow on $n \times n$ matrices, then a Lax-pair would guarantee at most n integrals, viz., the eigenvalues, whereas an $n \times n$ system has $O(n^2)$ degrees of freedom—too little, a priori, for Liouville integrability. The situation is in fact even more complicated. Indeed, suppose we are investigating a flow on real skew-symmetric $n \times n$ matrices A—i.e. a flow for a generalized top. Such matrices constitute the dual Lie algebra of the orthogonal group O_n, and so carry a natural Lie-Poisson structure. The symplectic leaves of this structure are the co-adjoint orbits of O_n

$$\mathcal{A} = \mathcal{A}_A = \left\{ O\ A\ O^T : O\ \in\ O_n \right\} \tag{4}$$

Thus **any** Hamiltonian flow $t \to A(t)$ on $\mathcal{A}$, $A(t = 0) = A$, must have the form

$$A(t) = O(t)\ A\ O(t)^T \tag{5}$$

for some $O(t) \in \mathcal{A}$ and hence has Lax-pair form

$$\frac{dA}{dt} = \dot{O}\ A\ O^T + O\ A\ \dot{O}^T = [B, A] \tag{6}$$

where

$$B = \dot{O}\ O^T = -B^T \tag{7}$$

The Lax-pair form guarantees that the eigenvalues $\{\lambda_i\}$ of A are constants of the motion. But we see from (4) that the co-adjoint orbit through A is simply specified by the eigenvalues of A. In other words the eigenvalues of A are just parameters for the symplectic leaves under considerations: They are of no help in integrating the system: Indeed $d\lambda_i|_{\mathcal{A}_A} = 0$ for all i. So for a Lax-pair formulation to be useful, we need

$$\text{Lax pair} + \text{ “something”} \tag{8}$$

So, what is the "something"? A Lax-pair is a proclamation, a marker, as it were, on a treasure map that says "Look here!" The real challenge in each case is to turn the Lax-pair, if possible, into an effective tool to solve the equation. In other words, the real task is to find the "something" to dig up the treasure! Perhaps the best description of Lax-pairs is a restatement of Yogi Berra's famous dictum "If you come to a fork in the road, take it". So if you come upon a Lax-pair, take it!

Over the years, with ideas going back and forth, Liouville integrability, Lax-pairs, "algebraic integrability", "monodromy", the discussion of what is an integrable system has been at times, shall we say, quite lively. There is, for example, the story of Henry McKean and Herman Flashka discussing integrability, when one of them, and I'm not sure which one, said to the other: "So you want to know what is an integrable system? I'll tell you! You didn't think I could solve it. But I can!"

In this "wild west" spirit, many developments were taking place in integrable systems. What was not at all clear at the time, however, was that these developments would provide tools to analyze mathematical and physical problems in areas **far removed from their original dynamical origin**. These tools constitute what may now be viewed as an **integrable method (IM)**.

There is a picture that I like that illustrates, very schematically, the intersection of IM with different areas of mathematics. Imagine some high dimensional space, the "space of problems". The space contains a large number of "parallel" planes, stacked one on top of the other and separated. The planes are labeled as follows: dynamical systems, probability theory and statistical mechanics, geometry, combinatorics, statistical mechanics, classical analysis, numerical analysis, representation theory, algebraic geometry, transportation theory, ... In addition, there is another plane in the space labeled "the integrable method (IM)": Any problem lying on IM can be solved/integrated by tools taken from the integrable method. Now the fact of the matter is that the IM-plane intersects all of the parallel planes described above: Problems lying on the intersection of any one of these planes with the IM-plane are thus solvable by the integrable method (Fig. 1).

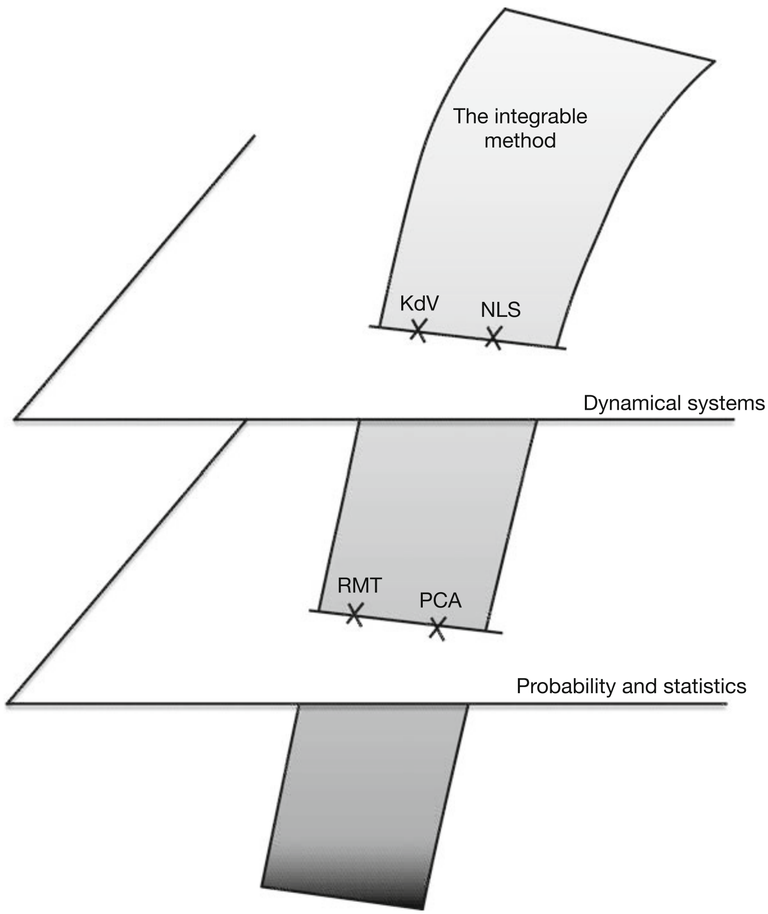

Fig. 1 Intersections of the integrable method

For each parallel plane we have, for example, the following intersection points:

- dynamical systems: Korteweg-de Vries (KdV), Nonlinear Schrödinger (NLS), Toda, Sine-Gordon, ...
- probability theory and statistics: Random matrix theory (RMT), Integrable probability theory, Principal component analysis (PCA), ...
- geometry: spaces of constant negative curvature R, general relativity in $1 + 1$ dimensions, ...
- combinatorics: Ulam's increasing subsequence problem, tiling problems, (Aztec diamond, hexagon tiling, ...), random particle systems (TASEP, ...), ...
- statistical mechanics: Ising model, XXZ spin chain, six vertex model, ...
- classical analysis: Riemann-Hilbert problems, orthogonal polynomials, (modern) special function theory (Painlevé equations), ...
- numerical analysis: QR, Toda eigenvalue algorithm, Singular value decomposition, ...
- representation theory: representation theory of large groups (S_∞, U_∞ ...), symmetric function theory, ...
- algebraic geometry: Schottky problem, infinite genus Riemann surfaces, ...
- transportation theory: Bus problem in Cuernavaca, Mexico, airline boarding, ...

The list of such intersections is long and constantly growing.

The singular significance of KdV is just that the **first intersection** that was observed and understood as such, was the junction of IM with dynamical systems, and that was at the point of KdV.

How do we come to such a picture? First we will give a precise definition of what we mean by an integrable system. Consider a simple harmonic oscillator:

$$\dot{x} = y \qquad , \qquad \dot{y} = -\omega^2 x \tag{9}$$
$$x(t)|_{t=0} = x_0 \qquad , \qquad y(t)|_{t=0} = y_0$$

The solution of (9) has the following form:

$$\begin{cases} x(t; x_0, y_0) & = \frac{1}{\omega}\sqrt{w^2 x_0^2 + y_0^2}\ \sin\left(wt + \sin^{-1}\left(\frac{\omega x_0}{\sqrt{\omega^2 x_0^2 + y_0^2}}\right)\right) \\ y(t; x_0, y_0) & = \sqrt{\omega^2 x_0^2 + y_0^2}\ \cos\left(wt + \sin^{-1}\left(\frac{\omega x_0}{\sqrt{\omega^2 x_0^2 + y_0^2}}\right)\right) \end{cases} \tag{10}$$

Note the following features of (10): Let $\varphi : \mathbb{R}^2 \to \mathbb{R}_+ \times (\mathbb{R}/2\pi\mathbb{Z})$

$$(\alpha, \beta) \longmapsto A = \frac{1}{\omega}\sqrt{\omega^2 \alpha^2 + \beta^2},\ \theta = \sin^{-1}\left(\frac{\omega \alpha}{\sqrt{\omega^2 \alpha^2 + \beta^2}}\right).$$

Then

$$\varphi^{-1} : \mathbb{R}_+ \times (\mathbb{R}/2\pi\mathbb{Z}) \to \mathbb{R}^2$$

has the form

$$\varphi^{-1}(A, \theta) = (A \sin\theta,\ \omega A \cos\theta)$$

Thus (10) implies

$$\begin{cases} \eta(t; \eta_0) = \varphi^{-1}\left(\varphi(\eta_0) + \boldsymbol{\omega}\, t\right) \\ \text{where} \quad \eta(t) = (x(t), y(t))\,, \eta_0 = (x_0, y_0), \quad \boldsymbol{\omega} = (0, \omega) \end{cases} \tag{11}$$

In other words:

There exists a bijective change of variables $\eta \longmapsto \varphi(\eta)$ such that (12a)

$\eta(t, \eta_0)$ evolves according to (9) $\Rightarrow$ (12b)

$$\varphi(\eta(t); \eta_0) = \varphi(\eta_0) + t\, \boldsymbol{\omega}$$

i.e., in the variables $(A, \theta) = \varphi(\alpha, \beta)$, solutions of (9) move linearly.

$$\left\{ \begin{array}{l} \eta(t, \eta_0) \text{ is recovered from formula (11) via a map} \\ \quad \varphi^{-1}(A, \theta) = (A \sin\theta, \omega A \cos\theta) \\ \\ \text{in which the behavior of } \sin\theta, \cos\theta \text{ is very well understood.} \\ \text{The same is true for } \varphi\text{. What we learn, in particular, based on this} \\ \text{knowledge of } \varphi \text{ and } \varphi^{-1}, \quad \text{is that} \\ \quad \eta(t; \eta_0) \quad \text{evolves periodically in time with period } 2\pi/\omega \end{array} \right. \tag{12c}$$

We are led to the following:

We say that a dynamical system $t \mapsto \eta(t)$ is **integrable** if

$$\left\{ \begin{array}{l} \text{There exists a bijective map } \varphi : \eta \mapsto \varphi(\eta) \equiv \zeta \\ \quad \text{such that } \varphi \text{ linearizes the system} \\ \qquad \varphi\left(\eta(t)\right) = \varphi\left(\eta(t=0)\right) + \boldsymbol{\omega}\, t \\ \text{and so} \\ \quad \eta\left(t; \eta\left(t=0\right)\right) = \varphi^{-1}\left(\varphi\left(\eta(t=0)\right) + \boldsymbol{\omega}\, t\right) \end{array} \right. \tag{13a}$$

AND

$$\left\{ \begin{array}{l} \text{The behavior of } \varphi, \varphi^{-1} \text{ are well enough understood so that} \\ \text{the behavior of } \eta\left(t; \eta\left(t=0\right)\right) \text{ as } t \to \infty \text{ is clearly revealed.} \end{array} \right. \tag{13b}$$

More generally, we say a system η which depends on some parameters $\eta = \eta\,(a, b, \dots)$ is **integrable** if

$$\left\{\begin{array}{l}\text{There exists a bijective change of variables } \eta \to \zeta = \varphi(\eta) \text{ such} \\ \text{that the dependence of } \zeta \text{ on } a, b, \dots . \\ \qquad\qquad \zeta\,(a, b, \dots) = \varphi\,(\eta\,(a, b, \dots)) \\ \text{is simple/well-understood}\end{array}\right. \tag{14a}$$

and

$$\left\{\begin{array}{l}\text{The behavior of the function theory} \\ \qquad \eta \mapsto \zeta \equiv \varphi(\eta) \quad , \quad \zeta \longmapsto \eta = \varphi^{-1}(\zeta) \\ \text{is well-enough understood so that the behavior of} \\ \qquad \eta\,(a, b \dots) = \varphi^{-1}\,(\zeta\,(a, b \dots,))\end{array}\right. \tag{14b}$$

is revealed in an explicit form as $a, b, \dots$ vary, becoming, in particular, large or small.

Notice that in this definition of an integrable system, various sufficient conditions for integrability such as commuting integrals, Lax-pairs, ..., are conspicuously absent. A system is integrable, if you can solve it, but subject to certain strict guidelines. This is a return to McKean and Flaschka, an institutionalization, as it were, of the "Wild West".

According to this definition, progress in the theory of integrable systems is made

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by discovering how to linearize a new system

$$\eta \to \zeta = \varphi(\eta)$$

using a **known** function theory φ. For example: Newton's problem of two gravitating bodies, is solved in terms of trigonometric functions/ellipses/parabolas—mathematical objects already well-known to the Greeks. In the nineteenth century, Jacobi solved geodesic flow on an ellipsoid using newly minted hyperelliptic function theory, and so on, ...

OR

by discovering/inventing a new function theory which linearizes the given problem at hand. For example: To facilitate numerical calculations in spherical geometry, Napier, in the early 1700s, realized that what he needed to do was to linearize multiplication

$$\eta\,\tilde{\eta} \longrightarrow \varphi\,(\eta\,\tilde{\eta}) = \varphi(\eta) + \varphi(\tilde{\eta})$$

which introduced a new function theory—the logarithm. Historically, **no** integrable system has had greater impact on mathematics and science, than multiplication!

There is a similar story for all the classical special functions, Bessel, Airy, ..., each of which was introduced to address a particular problem.

The following aspect of the above evolving integrability process is crucial and gets to the heart of the Integrable Method (IM): Once a new function theory has been discovered and developed, it enters the **toolkit** of IM, finding application in problems far removed from the original discovery.

Certain philosophical points are in order here.

(1) There is **no difference** in spirit, philosophically, between our definition of an integrable system and what we do in ordinary life. We try to address problems by rephrasing them (read "change of variables") so we can recognize them as something we know. After all, what else is a "precedent" in a law case? We introduce new words—a new "function theory"—to capture new developments and so extend and deepen our understanding. Recall that Adam's first cognitive act in Genesis was to give the animals names. The only difference between this progression in ordinary life versus mathematics, is one of degree and precision.

(2) This definition presents "integrability" **not as a predetermined** property of a system frozen in time. Rather, in this view the status of a system as integrable depends on the technology/function theory available **at the time**. If an appropriate new function theory is developed, the status of the system may change to integrable.

How does one determine if a system is integrable and how do you integrate it? Let me say at the outset, and categorically, that I believe there is no systematic answer to this question. Showing a system is integrable is **always** a matter of luck and intuition.

We do, however, have a **toolkit** which one can bring to a problem at hand.

At this point in time, the toolkit contains, amongst others, the following components:

(a) a broad and powerful **set of functions/transforms/constructions**

$$\eta \to \zeta = \varphi(\eta)$$

that can be used to convert a broad class of problems of interest in mathematics/physics, into "known" problems: In the simplest cases $\eta \to \varphi(\eta)$ linearizes the problem.

(b) **powerful techniques** to analyze φ, φ^{-1} such that the asymptotic behavior of the original η-system can be inferred explicitly from the known asymptotic behavior of the ζ-system, as relevant parameters, e.g. time, become large.

(c) a **particular, versatile** class of functions, the Painlevé functions, which play the same role in modern (nonlinear) theoretical physics that classical special functions played in (linear) nineteenth century physics. Painlevé functions form the **core of modern special function**, and their behavior is known with the same precision as in the case of the classical special functions. We note that the Painlevé equations **are themselves integrable** in the sense of Definition (14a).

(d) a class of **"integrable" stochastic models**—random matrix theory (RMT). Instead of modeling a stochastic system by the roll of a die, say, we now have the possibility to model a whole new class of systems by the **eigenvalues of a random matrix**. Thus RMT plays the role of a **stochastic special function theory**. RMT is "integrable" in the sense that key statistics such as the gap probability, or edge statistics, for example, are given by functions, e.g. Painlevé functions, that describe (deterministic) integrable problems as above. We will say more about this later.

We will now show how all this works in concrete situations. Note, however, by no means all known integrable systems can be solved using tools from the IM-toolkit. For example the beautiful system that Patrick Gérard et al. have been investigating recently (see e.g. [21]), seems to be something completely different. We will consider various examples. The first example is taken from dynamics, viz., the NLS equation.

To show that NLS is integrable, we **first** extract a particular tool from the toolkit—the Riemann-Hilbert Problem (RHP): Let $\Sigma \subset \mathbb{C}$ be an oriented contour and let $v : \Sigma \to G\ell\,(n, \mathbb{C})$ be a map (the "jump matrix") from Σ to the invertible $n \times n$ matrices, $v,\ v^{-1} \in L^\infty(\Sigma)$. By convention, at a point $z \in \Sigma$, the $(+)$ side (respectively $(-)$ side) lies to the left (respectively right) as one traverse Σ in the direction of the orientation, as indicated in Fig. 2. Then the (normalized) RHP (Σ, v) consists in finding an $n \times n$ matrix-valued function $m = m(z)$ such that

- $m(z)$ is analytic in $\mathbb{C}/\Sigma$
- $m_+(z) = m_-(z)\, v(z) \quad , \quad z \in \Sigma$

  where $\quad m_\pm(z) = \lim_{z' \to z_\pm} m(z')$
- $m(z) \to I_n \quad$ as $\quad z \to \infty$

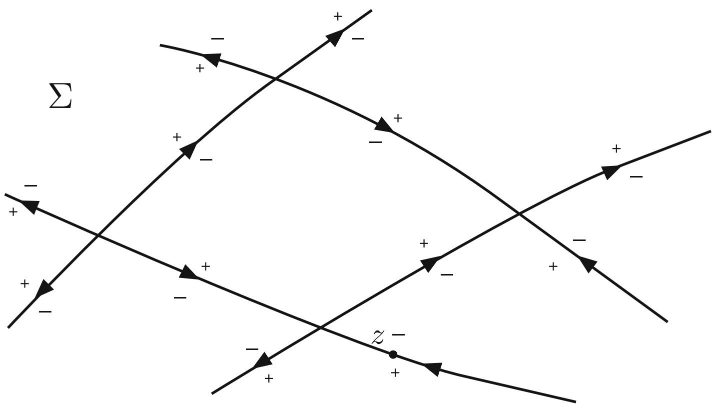

Fig. 2 Oriented contour Σ

Here "$z' \to z_\pm$" denotes the limit as $z' \in \mathbb{C}/\Sigma$ approaches $z \in \Sigma$ from the $(\pm)$-side, respectively. The particular contour Σ and the jump matrix v are tailored to the problem at hand.

There are many technicalities involved here: Does such an $m(z)$ exist? In what sense do the limits $m_\pm$ exist? And so on … . Here we leave such issues aside. RHP's play an analogous role in modern physics that integral representations play for classical special functions, such as the Airy function $Ai(z)$, Bessel function $J_n(z)$, etc. For example, $Ai(z) = \frac{1}{2\pi i} \int_{\mathscr{C}} \exp\left(\frac{t^3}{3} - z\,t\right) dt$ for some appropriate contour $\mathscr{C} \subset \mathbb{C}$, which makes it possible to analyze the behavior of $Ai(z)$ as $z \to \infty$, using the classical steepest descent method.

Now consider the defocusing NLS equation on $\mathbb{R}$

$$\begin{cases} i\,u_t + u_{xx} - 2|u|^2\,u = 0 \\ u(x,t)\Big|_{t=0} = u_0(x) \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{15}$$

In 1972, Zakharov and Shabat [35] showed that NLS has a Lax-pair formulation, as follows: Let

$$L(t) = (i\,\sigma)^{-1}\,(\partial_x - Q(t))$$

where

$$\sigma = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(t) = \begin{pmatrix} 0 & u(x,t) \\ -\overline{u(x,t)} & 0 \end{pmatrix}.$$

For each t, $L(t)$ is a self-adjoint operator acting on vector valued function in $\left(L^2(\mathbb{R})\right)^2$. Then for some explicit $B(t)$, constructed from $u(x,t)$ and $u_x(x,t)$,

$$u(x,t) \quad \text{solves NLS} \quad \Longleftrightarrow \quad \frac{d\,L(t)}{dt} = [B(t), L(t)]\,. \tag{16}$$

This the **second** tool we extract from our toolkit. So the Lax operator $L(t)$ marks a point, as it were, on our treasure map. How can one use $L(t)$ to solve the system?

One proceeds as follows: This crucial step was first taken by Shabat [30] in the mid-1970s in the case of KdV and developed into a general scheme for ordinary differential operators by Beals and Coifman [4] in the early 1980s.

The **map** $\boldsymbol{\varphi}$ in (13a) above for NLS is the scattering map constructed as follows: Suppose $u = u(x)$ is given, $u(x) \to 0$ sufficiently rapidly as $|x| \to \infty$. For *fixed* $z \in \mathbb{C}/\mathbb{R}$, there exists a *unique* 2×2 solution of the scattering problem

$$L\psi = z\psi \tag{17}$$

where

$$m = m(x, z;\ u) \equiv \psi(x, z)\, e^{-iz,x\sigma} \tag{18}$$

is bounded on $\mathbb{R}$ and

$$m(x, z; u) \to I \quad \text{as } x \to -\infty.$$

For **fixed** $\boldsymbol{x} \in \mathbb{R}$, such so-called *Beals-Coifman solutions* also have the following properties:

$$\begin{gathered} m(x, z; u) \ \text{ is analytic in } z \text{ for } \ z \in \mathbb{C}/\mathbb{R} \\ \text{and continuous in } \overline{\mathbb{C}}_+ \text{ and in } \overline{\mathbb{C}}_- \end{gathered} \tag{19}$$

$$m(x, z; u) \to I \quad \text{as} \quad z \to \infty \quad \text{in} \quad \mathbb{C}/\mathbb{R}. \tag{20}$$

Now both $\psi_\pm(x, z; u) = \lim_{z' \to z_\pm} \ \psi(x, z; u)\ ,\ z \in \mathbb{R}$ clearly solve $L\,\psi_\pm = z\,\psi_\pm$ which implies that there exists $v = v(z) = v(z; u)$ independent of x, such that for all $x \in \mathbb{R}$

$$\psi_+\,(x, z) = \psi_-\,(x, z)\, v(z) \quad , \quad z \in \mathbb{R} \tag{21}$$

or in terms of

$$m_\pm = \psi_\pm(x, z)\, e^{-ixz\sigma} \tag{22}$$

we have

$$m_+(x, z) = m_-(x, z)\, v_x(z) \quad , \quad z \in \mathbb{R} \tag{23}$$

where

$$v_x(z) = e^{ixz\,\sigma}\ v(z)\, e^{-ixz\,\sigma} \tag{24}$$

Said differently, for each $x \in \mathbb{R}$, $m(x, z)$ solves the normalized RHP (Σ, v_x) where $\Sigma = \mathbb{R}$, oriented from $-\infty$ to $+\infty$, and v is as above. In this way, a RHP enters naturally into the picture introduced by the Lax operator L.

It turns out that v has a special form

$$\begin{cases} v(z) & = \begin{pmatrix} 1 - |r(z)|^2 & r(z) \\ -\overline{r(z)} & 1 \end{pmatrix} \\ v_x(z) & = \begin{pmatrix} 1 - |r|^2 & r\,e^{ixz} \\ -\bar{r}e^{-ixz} & 1 \end{pmatrix} \end{cases} \tag{25}$$

where $r(z)$, the **reflection coefficient**, satisfies $\|r\|_\infty < 1$. We define the map φ for NLS as follows:

$$u \mapsto \varphi(u) \equiv r \tag{26}$$

Suppose r is given and x fixed. To construct $\varphi^{-1}(r)$ we must solve the RHP $(\mathbb{R}, v_x)$ with v_x as in (25). If $m = m(x, z)$ is the solution of the RHP, then expanding at $z = \infty$, we have

$$m(x, z) = I + \frac{(m_1(x))}{z} + O\left(\frac{1}{z^2}\right) \quad , \qquad z \to \infty.$$

A simple calculation then shows that

$$u(x) = \varphi^{-1}(r) = -i(m_1(x))_{12}. \tag{27}$$

Thus

$$\varphi \leftrightarrow \quad \text{scattering map} \quad ; \quad \varphi^{-1} \leftrightarrow \text{ RHP}.$$

Now the key fact of the matter is that

$$\varphi \;\text{ linearizes NLS.} \tag{28}$$

Indeed if $u(t) = u(x, t)$ solves NLS with $u(x, t)\big|_{t=0} = u_0(x)$, then

$$r(t) = \varphi(u(t)) = r(z; u_0)\, e^{-itz^2} = \varphi(u_0)(z)\, e^{-itz^2} \tag{29}$$

$$\text{or} \qquad \log r(t) = \log r(z, u_0) - i\,t\,z^2$$

which is linear motion!

This leads to the celebrated solution procedure

$$u(t) = \varphi^{-1}\left(\varphi(u_0)(\cdot)\, e^{-i\,t\,(\cdot)^2}\right). \tag{30}$$

Thus condition (13a) for the integrability of NLS is established.

But condition (13b) is also satisfied. Indeed the analysis of the scattering map $u \to r = \varphi(u)$ is classical and well-understood. The inverse scattering map is **also** well-understood because of the nonlinear steepest descent method for RHP's introduced by Deift and Zhou in 1993 [13].[1] This is the **third tool** we extract from our toolkit. One finds, for example, that as $t \to \infty$

[1]This paper also contains some history of earlier approaches to analyze the behavior of solutions of integrable systems asymptotically.

$$u(x,t) = \frac{1}{\sqrt{t}}\, \alpha(z_0) \quad e^{i\,x^2/4t - i\nu(z_0)\,\log 2t} \qquad (31)$$

$$+\, O\left(\frac{\ell n\ t}{t}\right)$$

where

$$z_0 = x/2t \quad , \quad \nu(z) = -\frac{1}{2\pi}\,\log\left(1 - |r(z)|^2\right) \qquad (32)$$

and

$\alpha(z)$ is an explicit function of r.

We see, in particular, that the long-time behavior of $u(x,t)$ is given with the same accuracy and detail as the solution of the linear Schroödinger equation $i\, u_t^0 + u_{xx}^0 = 0$ which can be obtained by applying the classical steepest descent method to the Fourier representation of $u^0(x,t)$

$$u^0(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{u^0}(z)\; e^{i(xz - tz^2)}\, dz$$

where $\widehat{u^0}$ is the Fourier transform of $u^0(x, t = 0)$. As $t \to \infty$, one finds

$$u^0(x,t) = \frac{\widehat{u^0}(z_0)}{\sqrt{2it}}\; e^{ix^2/4t} + o\left(\frac{1}{\sqrt{t}}\right). \qquad (33)$$

We see that NLS is an integrable system in the sense advertised in (13a,b). It is interesting and important to note that NLS is also integrable in the sense of Liouville. In 1974, Zakharov and Manakov in a celebrated paper [33], not only showed that NLS has a complete set of commuting integrals (read "actions"), but also computed the "angle" variables canonically conjugate to these integrals, thereby displaying NLS explicitly in so-called "action-angle" form. This effectively integrates the system by "quadrature" (see page 2). The first construction of action-angle variables for an integrable PDE is due to Zakharov and Faddeev in their landmark paper [32] on the Korteweg-de Vries equation in 1971.

We note that the asymptotic formula (31) and (32) for NLS was first obtained by Zakharov and Manakov in 1976 [34] using inverse scattering techniques, also taken from the IM toolbox, but without the rigor of the nonlinear steepest descent method.

The next example, taken from Statistical Mechanics, utilizes another tool from the toolkit, viz. the theory of integrable operators, IO's.

IO's were first singled out as a distinguished class of operators by Sakhnovich in the 1960s, 1970s and the theory of such operators was then fully developed by Its et al. [23] in the 1990s. Let Σ be an oriented contour in $\mathbb{C}$. We say an operator K acting on measurable functions h on Σ is *integrable* if it has a kernel of the form

$$K(x, y) = \frac{\sum_{i=1}^{n} f_i(x)\, g_i(y)}{x - y} \quad , \quad n < \infty, \quad x,\ y \in \Sigma, \tag{34}$$

where

$$f_i,\ g_i \quad \in L^{\infty}(\Sigma), \quad \text{and} \tag{35}$$

$$Kh(x) = \int_{\Sigma} K(x, y)\, h(y)\, dy.$$

If Σ is a "good" contour (i.e. Σ is a **Carleson curve**), K is bounded in $L^p(\Sigma)$ for $1 < p < \infty$.

Integral operators have many remarkable properties. In particular the integrable operators form an algebra and $(I + K)^{-1}$, if it exists, is also integrable if K is integrable. But most remarkably, $(I + K)^{-1}$ can be computed in terms of a naturally associated RHP on Σ. It works like this. If $K(x, y) = \sum_{i=1}^{n} f_i(x)\ g_i(y)/x - y$, then

$$(I + K)^{-1} = I + R \tag{36}$$

$$\text{where} \quad R(x, y) = \sum_{i=1}^{n} F_i(x)\ G_i(y) \Big/ x - y$$

for suitable $F_i,\ G_i$. Now assume for simplicity that $\sum_{i=1}^{n} f_i(x)\, g_i(x) = 0$ and let

$$v(z) = I - 2\pi\ f(z)\, g(z)^T \quad , \quad z \in \Sigma, \tag{37}$$

where $\quad f = (f_i, \ldots, f_n)^T\ , \quad g = (g_i, \ldots, g_n)^T$
and suppose $m(z)$ solves the normalized RHP (Σ, v). Then

$$F(z) = m_+(z)\, f(z) = m_-(z)\, f(z) \tag{38}$$

and

$$G(z) = \left(m_+^{-1}\right)^T\ g(z) = \left(m_-^{-1}\right)^T\ g(z) \tag{39}$$

Here is an example how integrable operators arise. Consider the spin–$\frac{1}{2}$ XY model in a magnetic field with Hamiltonian

$$H = -\frac{1}{2} \sum_{\ell \in \mathbb{Z}} \left(\sigma_\ell^x \, \sigma_{\ell+1}^x + \sigma_\ell^z \right) \tag{40}$$

where σ_ℓ^x, σ_ℓ^z are the standard Pauli matrices at the ℓth site of a 1-d lattice.

As shown by McCoy et al. [28] in 1983, the auto-correlation function $X(t)$

$$X(t) = \langle \sigma_0^x(t) \, \sigma_0^x \rangle_T = \frac{tr \left(e^{-\beta H} \left(e^{-iHt} \, \sigma_0^x \, e^{i\,Ht} \right) \sigma_0^x \right)}{tr \; e^{-\beta H}}$$

where $\beta = \frac{1}{T}$, can be expressed as follows:

$$X(t) = e^{-t^2/2} \, \det(1 - K_t)$$

Here K_t is the operator on $L^2(-1, 1)$ with kernel

$$K_t(z, z') = \varphi(z) \, \frac{\sin it(z - z')}{\pi \, (z - z')} \quad , \quad -1 \le z, \; z' \le 1, \tag{41}$$

and

$$\varphi(z) = \tanh\left(\beta \, \sqrt{1 - z^2}\right) \quad , \quad -1 < z < 1. \tag{42}$$

Observe that

$$K_t(z, z') = \frac{\sum_{i=1}^2 f_i(z) \, g_i(z')}{z - z'} \tag{43}$$

where

$$f = (f_1, f_2)^T = \left(\frac{-e^{tz} \, \varphi(z)}{2\pi \, i} \, , \, \frac{-e^{-tz} \, \varphi(z)}{2\pi \, i} \right)^T$$

$$g = (g_1, g_2)^T = \left(e^{-tz}, \; -e^{tz} \right)^T$$

so that K_t is an integrable operator. We have

$$v = v_t = I - 2\pi i \; f \, g^T = \begin{pmatrix} 1 + \varphi(z) & -\varphi(z) \, e^{2zt} \\ \varphi(z) \, e^{-2zt} & 1 - \varphi(z) \end{pmatrix} \, , \quad z \in (-1, 1) \, . \tag{44}$$

As

$$\frac{d}{dt}\ \log\det(1-K_t) = \frac{d}{dt}\ tr\log(1-K_t)$$
$$= -tr\left(\frac{1}{1-K_t}\ \dot{K}_t\right)$$

we see that $\frac{d}{dt}$ $\log\det(1-K_t)$, and ultimately $X(t)$, can be expressed via (36), (38), and (39) in terms of the solution m_t of the RHP $\left(\sum=(-1,1), v_t\right)$

Applying the nonlinear steepest descent method to this RHP as $t\to\infty$, one finds (Deift and Zhou [14]) that

$$X(t) = \exp\left(\frac{t}{\pi}\int_{-1}^{1}\ \log|\tanh\beta s|\,ds + o(t)\right) \tag{45}$$

This shows that H in (40) is integrable in the sense that key statistics for H such as the autocorrelation function $X(t)$ for the spin σ_0^x is integrable in the sense of (14a,b)

$$X(t) \overset{\varphi}{\mapsto}\ K_t \in\ \text{integrable operators}$$

and φ^{-1} is computed with any desired precision using RH-steepest descent methods to obtain (45). Note that the appearance of the terms $\varphi(z)\,e^{\pm 2zt}$ in the jump matrix v_t for $K_t=\varphi(X(t))$, makes explicit the linearizing property of the map φ.

Another famous integrable operator appears in the bulk scaling limit for the gap probability for invariant Hermitian ensembles in random matrix theory. More precisely, consider the ensemble of $N\times N$ Hermitian matrix $\{M\}$ with invariant distribution

$$P_N(M)\,dM = \frac{e^{-N\,tr\,V(M)}\,dM}{\int e^{-N\,tr\,V(M)}\,dM},$$

where $V(x)\to+\infty$ as $|x|\to\infty$ and dM is Lebesgue measure on the algebraically independent entries of M.

Set $P_N\left([\alpha,\ \beta]\right) =$ gap probability $\equiv$ Prob $\{M$ has no eigenvalues in $[\alpha,\ \beta]\}$, $\alpha<\beta$.

We are interested in the scaling limit of $P_N\left([\alpha,\ \beta]\right)$ i.e.

$$P(\alpha,\beta) = \lim_{N\to\infty} P_N\left(\left[\frac{\alpha}{\rho_N},\ \frac{\beta}{\rho_N}\right]\right)$$

for some appropriate scaling $\rho_N\sim N$. One finds (and here RH techniques play a key role) that

$$P(\alpha,\beta) = \det(1-K_s)\quad,\quad s=\beta-\alpha \tag{46}$$

where K_s has a kernel

$$K_s(x, y) = \frac{\sin(x - y)}{\pi(x - y)} \qquad \text{acting on} \qquad L^2(0, 2s).$$

Clearly $K_s(x, y) = \dfrac{e^{ix}\, e^{-iy} - e^{-ix}\, e^{iy}}{2\pi\, i(x - y)}$ is an integrable operator. The asymptotics of $P(\alpha, \beta)$ can then be evaluated asymptotically with great precision as $s \to \infty$, by applying the nonlinear steepest descent method for RHP's to the RHP associated with the integrable operator K_s, as in the case for K_t in (44) et seq.

Thus RMT is integrable in the sense that a key statistic, the gap probability in the bulk scaling limit, is an integrable system in the sense of (14a,b):

$$\text{Scaled gap probability } \; P_{(\alpha,\beta)} \xrightarrow{\varphi} K_s(x, y) \in \text{ Integrable operators}$$

$$\varphi^{-1} \quad \text{is then evaluated via the formula } \; \det(1 - K_s)$$

$$\text{which can be controlled precisely as } \; s \to \infty.$$

The situation is similar for many other key statistics in RMT. It turns out that $P_{(\alpha,\beta)}$ solves the Painlevé V equation as a function of $s = \beta - \alpha$ (this is a famous, result of Jimbo et al. [25]). But the Painlevé V equation is a classically integrable Hamiltonian system which is also integrable in the sense of (14a,b). Indeed it is a consequence of the seminal work of the Japanese School of Sato et al. that all the Painlevé equations can be solved via associated RHP's (the RHP for Painlevé II in particular was also found independently by Flaschka and Newell), and hence are integrable in the sense of (14a,b) and amenable to nonlinear steepest descent asymptotic analysis, as described, for example, in the text, Painlevé Transcendents by Fokas et al. [18].

There is another perspective one can take on RMT as an integrable system. The above point of view is that RMT is integrable because key statistics are described by functions which are solutions of classically integrable Hamiltonian systems. But this point of view is unsatisfactory in that it attributes integrability in one area (RMT) to integrability in another (Hamiltonian systems). Is there a notion of integrability for stochastic systems that is intrinsic? In dynamics the simplest integrable system is free motion

$$\dot{x} = y, \quad \dot{y} = 0 \qquad \Longrightarrow \qquad x(t) = x_0 + y_0\, t \quad , \quad y(t) = y_0. \tag{47}$$

Perhaps the simplest stochastic system is a collection of coins flipped independently. Now, we suggest, just as an integrable Hamiltonian system becomes (47) in new variables, the analogous property for a stochastic system should be that, in the appropriate variables, it is integrable if it just a product of independent spin flips.

Consider the scaled gap probability,

$$P_{(\alpha,\beta)} = \text{Prob} \{ \text{ no eigenvalues in } (\alpha, \beta)\} = \det(1 - K_s) \tag{48}$$

But as the operator K_s is trace-class and $0 \le K_s < 1$, it follows that

$$P_{\alpha,\beta} = \prod_{i=1}^{\infty} (1 - \lambda_i) \tag{49}$$

where $0 \le \lambda_i < 1$ are the eigenvalues of K_s. Now imagine we have a collection of boxes, $B_1, B_2, \ldots$. With each box we have a coin: With probability λ_i a ball is placed in box B_i, or equivalently, with probability $1 - \lambda_i$ there is no ball placed in B_i. The coins are independent. Thus we see that the probability that there are no eigenvalue in (α, β), is the same as the probability of no balls being placed in all the boxes!

This is an intrinsic probabilistic view of RMT integrability. It applies to many other stochastic systems. For example, consider Ulam's longest increasing subsequence problem:

Let $\pi = \pi(1)\,\pi(2), \ldots \pi(N)$ *be a permutation in the symmetric group* S_N. *If*

$$i_1 < i_2 < \cdots < i_k \qquad \text{and} \qquad \pi(i_1) < \cdots < \pi(i_k) \tag{50}$$

we say that

$$\pi(i_1)\;\pi(i_2)\;\ldots, \pi(i_k) \tag{51}$$

is an increasing subsequence for π *of length* k. *Let* $\ell_N(\pi)$ *denote the greatest length of any increasing subsequence for* π, *e.g. for* $N = 6$, $\pi = 31{,}5624 \in S_6$ *has* $\ell_6(\pi) = 3$ *and* 356, 254 *and* 156 *are all longest increasing subsequences for* π. *Equip* S_N *with uniform measure. Thus for* $n \le N$.

$$\begin{aligned} q_{n,N} &\equiv \text{Prob } (\ell_N \le n) \\ &= \frac{\#\,\{\pi : \ell_N(\pi) \le n\}}{N!} \end{aligned} \tag{52}$$

Question How does $q_{n,N}$ behave as $n,\ N \to \infty$?

Theorem 1 (Baik et al. [1]) *Let* $t \in \mathbb{R}$ *be given. Then*

$$F(t) \equiv \lim_{N\to\infty} \textit{Prob} \left(\ell_N \le 2\sqrt{N} + t\,N^{1/6}\right) \tag{53}$$

exists and is given by $e^{-\int_t^\infty (x-t)\,u^2(x)\,dx}$ *where* $u(x)$ *is the (unique) Hastings-McLeod solution of the Painlevé II equation*

$$u'' = 2u^3 + xu \tag{54}$$

normalized such that

$$u(x) \sim Ai(x) = \textit{Airy function}, \quad \textit{as } x \to +\infty$$

(The original proof of this Theorem used RHP/steepest descent methods. The proof was later simplified by Borodin, Olshanski and Okounkov using the so-called Borodin-Okounkov-Case-Geronimo formula.)

Some observations:

(i) As Painlevé II is classically integrable, we see that the map

$$q_{n,N} \quad \xrightarrow{\varphi} \quad u^2(t) = -\frac{d^2}{dx^2} \log F(x)$$

transforms Ulam's longest increasing subsequence problem into an integrable system whose behavior is known with precision. There are many other classical integrable systems associated with $q_{n,N}$ but that is another story (see Baik et al. [2]).

(ii) The distribution $F(t) = e^{-\int_t^\infty (x,t)\, u^2(x)\, dx}$ is the famous Tracy-Widom distribution for the largest eigenvalue $\lambda_{\max}$ of a random Hermitian matrix in the edge-scaling limit. In other words, the length of the longest increasing subsequence behaves like the largest eigenvalue of a random Hermitian matrix. More broadly, what we are seeing here is an example of how RMT plays the role of a stochastic special function theory describing a stochastic problem from some other a priori unrelated area. This is no different, in principle, from the way the trigonometric functions describe the behavior of the simple harmonic oscillator. RMT is a very versatile tool in our IM toolbox—tiling problems, random particle systems, random growth models, the Riemann zeta function, ..., all the way back to Wigner, who introduced RMT as a model for the scattering resonances of neutrons off a large nucleus, are all problems whose solution can be expressed in terms of RMT.

(iii) $F(t)$ can also be written as

$$F(t) = \det(1 - A_t) \tag{55}$$

where A_t is a particular trace class integrable operator, the Airy operator, with $0 \le A_t < 1$. Thus $F(t) = \prod_{i=1}^{\infty} \left(1 - \tilde{\lambda}_i(t)\right)$ where $\{\tilde{\lambda}_i(t)\}$ are the eigenvalues of A_t. We conclude that $F(t)$, the (limiting) distribution for the length ℓ_N of the longest increasing subsequence, corresponds to an integrable system in the above intrinsic probabilistic sense.

(iv) It is of considerable interest to note that in recent work Gavrylenko and Lisovyy [20] have shown that the isomonodromic tau function for general Fuchsian systems can be expressed, up to an explicit elementary function, as a Fredholm determinant of the form $\det(1 - K)$ for some suitable trace class