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# Advances in Soft Computing

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# Interval / Probabilistic Uncertainty and Non-classical Logics

Springer

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# Preface

*Large-scale data processing is important.* Most successful applications of modern science and engineering, from discovering the human genome to predicting weather to controlling space missions, involve processing large amounts of data and large knowledge bases. The corresponding large-scale data and knowledge processing requires intensive use of computers.

*Computers are based on processing exact data values and truth values from the traditional 2-value logic.* The ability of computers to perform fast data and knowledge processing is based on the hardware support for super-fast elementary computer operations, such as performing arithmetic operations with (exactly known) numbers and performing logical operations with binary ("true"-"false") logical values.

*In practice, we need to go beyond exact data values and truth values from the traditional 2-value logic.* In practical applications, we need to go beyond such operations.

*Input is only known with uncertainty.* Let us first illustrate this need on the example of operations with numbers. Hardware-supported computer operations (implicitly) assume that we know the exact values of the input quantities. In reality, the input data usually comes from measurements. Measurements are never 100% accurate. Due to such factors as imperfection of measurement instruments and impossibility to reduce noise level to 0, the measured value $\widetilde{x}$ of each input quantity is, in general, different from the (unknown) actual value $x$ of this quantity. It is therefore necessary to find out how this input uncertainty $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x \neq 0$ affects the results of data processing.

*Probabilistic approach to uncertainty.* The need to take into account the uncertainty of input data has been known for centuries. In the early 19 century, Gauss developed basic statistical techniques for processing such uncertainty. These techniques are based on the assumption that we know the probabilities of different values of the measurement error $\Delta x$. Usually, we assume that the distribution is normal (Gaussian), with 0 mean and a known standard deviation $\sigma$. Such *probabilistic* methods are actively used in engineering and scientific practice.

*Interval approach to uncertainty.* There are practical situations in which we do not know the probabilities of different values of the measurement error $\Delta x$. In many such situations, we only know the upper bound $\Delta$ on the (absolute value of the) measurement error, i.e., the value $\Delta$ for which $|\Delta x| \le \Delta$. In such situations, after we perform the measurement and get the measured value $\widetilde{x}$, the only information that we have about the actual (unknown) value $x$ is that this value belongs to the interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$.

Techniques for processing data under such *interval* uncertainty can be traced back to ancient scientists such as Archimedes. Their active development started by T. Sunaga (Japan), by M. Warmus (Poland), and especially by R. Moore (USA) in the 1950s, when the arrival of modern computers led to the practical need for such development.

*Need to combine probabilistic and interval uncertainty.* At present:

- we have well-developed techniques for handling situations in which we know the exact probability distribution for the measurement error $\Delta x$, and
- we have well-developed techniques for handling situations in which we have no information about the probabilities – and we only know the upper bound on the measurement error.

In real life, we often encounter intermediate situations in which we have *partial* information about the probabilities.

This information is frequently described in interval-related terms: e.g., instead of knowing the exact value $p$ of the probability, we may only know the interval $[\underline{p}, \overline{p}]$ of possible values of this probability. To handle such situation, it is desirable to combine probabilistic and interval approaches to uncertainty.

Several formalisms have been developed for such combination, such as imprecise probabilities, Dempster-Shafer approach, approaches related to rough sets, and many others.

*First objective of the workshop.* One of the main objectives of this workshop was to bring together researchers working on interval, probabilistic, and combined methods, so as to promote collaboration and further applications.

*Need for non-classical logics.* Another aspect in which we need to go beyond hardware-supported computer operations is logic.

In the computer, only operations from the traditional 2-valued ("true"-"false") logic are supported. However, in practice, usually, experts are not 100% sure about the truth of the statement included in the knowledge bases.

An important aspect of this uncertainty is that the experts' statements can be *vague* (*fuzzy*). For example, an expert can be 100% sure that the temperature in a certain region will be high, but he or she is not 100% sure whether this "high" necessarily means greater than 40°.

*Fuzzy logic and type-2 fuzzy logic.* Several logical techniques have been proposed to provide a more adequate description of expert knowledge. Historically, one of the first such techniques was the (standard) fuzzy logic, in which we use numbers from the interval $[0, 1]$ to describe the expert's degree of certainty in a statement:

- 1 means 100% certainty,
- 0 means no certainty at all, and
- intermediate values represent different grades of uncertainty.

An even more adequate representation of the expert's uncertainty comes from type-2 fuzzy logic, in which we take into account that just like an expert usually cannot describe his or her knowledge about some quantity by a single number, this same expert cannot describe his or her certainty by a single number: this certainty can be described by an interval of possible values, or, even more generally, by a fuzzy subset of the interval $[0, 1]$.

*Probabilistic logics.* In fuzzy logic, degrees from the interval $[0, 1]$ represent subjective degrees. In some cases, we can describe these degrees in a more objective way: e.g., in the weather example, as a probability that in the past, in similar situations, the temperature was above 40°. Such cases are handled by *probabilistic* logic.

*Algebraic approach to logic.* Yet another alternative comes from the fact that the most natural way for an expert to describe his/her knowledge is by using words from natural language. So, instead of quantizing these values, we may want to describe possible values of certainty as the set of such words and word combinations, and define appropriate "and"- and "or"-operations on this set – which would make it a logic.

This approach was one of the motivations behind the development of *algebraic* logics, i.e., logics described not by a specific implementation but rather by the properties of the corresponding logical operations.

*Modal logics.* For an individual event, such as temperature exceeding 40°, it is reasonable to ask whether this event will happen or not – and what is the expert's confidence that this event will happen. In practice, expert statements usually refer not to individual events, but rather to *repeating* events. For such events, it is also reasonable to ask whether it is *possible* that the event will happen, whether it is *necessary* that this event will happen – and if yes, to what degree.

Such statements about possibility and necessity form *modal logic*, another non-classical logic actively used in processing knowledge.

*Other non-classical logics.* There exist many other non-classical logics, e.g., logics used to describe induction (making generalizations based on several facts) and abduction (making conclusions about the causes of an observed event).

*Need to combine uncertainty analysis with non-classical logics.* In many practical problems, we need to process both measurement data and expert knowledge.

- We have already mentioned that to adequately process measurement data, we need to take into account probabilistic and interval uncertainty – and combine these two types of uncertainty.
- We have also mentioned that to adequately describe expert knowledge, we need to use various non-classical logic techniques – and sometimes we need to combine different non-classical logic methods.

It is therefore desirable to combine uncertainty analysis with non-classical logic.

Such combination was the main objective of this workshop – and of these proceedings.

*Objectives of the workshop.* Specifically, the main objectives of the workshop were:

- to bring together researchers working on uncertainty formalisms in information and knowledge systems;
- to attract researchers working in social sciences (economics, business, and environmental sciences) who are interested in applying uncertainty-related tools and techniques;
- to promote the cross-fertilization between the fundamental ideas connected with various approaches used in the study of non-classical logics;
- to bring together researchers from various fields on non-classical logics and applications in order to foster collaboration and further research, and
- to present and discuss open research problems and challenges.

Papers presented in these Proceedings describe different aspects of these problems.

We hope that this workshop will lead to a boost in the much-needed collaboration between the uncertainty analysis communities and the non-classical logic communities.

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- last but not the least, Professor Janusz Kacprzyk, the series editor, for his encouragement and support.

Let the collaborations continue and blossom!

JAIST, Japan
March 2008

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# Part I

# Keynote Addresses

# An Algebraic Approach to Substructural Logics – An Overview

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We will give a state-of-the-art survey of the study of substructural logics. Originally, substructural logics were introduced as logics which, when formulated as Gentzen-style systems, lack some of the three basic *structural rules*, i.e. *contraction*, *weakening* and *exchange*. For example, relevance logics and linear logic lack the weakening rule, many-valued logics, fuzzy logics and linear logic lack the contraction rule, and hence all of them can be regarded as substructural logics. These logics have been studied extensively and various theories have been developed for their investigation. However their study has been carried out independently, mainly due to the different motivations behind them, avoiding comparisons between different substructural logics.

On the other hand, the general study of substructural logics has a comparative character, focusing on the absence or presence of structural rules. As such, at least in the initial stages of research, it was a study on how structural rules affect logical properties. This naturally led to a syntactic or proof-theoretic approach yielding deep results about properties of particular logics, provided that they can be formalized in systems, like cut-free Gentzen calculi. An obvious limitation of this study comes from the fact that not all logics have such a formulation.

Semantical methods, in contrast, provide a powerful tool for analyzing substructural logics from a more uniform perspective. Both Kripke-style semantics and algebraic semantics for some particular subclasses of substructural logics, e.g. relevant logics, were already introduced in the 70s and 80s and were studied to a certain extent.

An algebraic study of substructural logics for the last decade, based mainly on algebraic logic and universal algebra, has brought us a new perspective on substructural logics, which came from the observation that these logics all share the *residuation property*. Though this is usually not noticed, it is revealed explicitly in a sequent formulation by the use of extralogical symbols, denoted by commas. Precisely speaking, consider the following equivalence, concerning implication $\rightarrow$, that holds in most of sequent systems: *A formula* $\gamma$ *follows from formulas* $\alpha$ *and* $\beta$ *if and only if the implication* $\alpha \rightarrow \gamma$ *follows from* $\beta$ *alone.*

Here 'follows from' is given by the particular substructural logic, and is usually denoted by $\Rightarrow$. Thus, the above equivalence can be restated as:

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$$\alpha, \beta \Rightarrow \gamma \text{ is provable} \quad \text{iff} \quad \beta \Rightarrow \alpha \rightarrow \gamma \text{ is provable,}$$

where provability is taken with respect to a sequent calculus for the particular substructural logic. If we replace the auxiliary symbol 'comma' by a new logical connective $\cdot$, called *fusion*, we have:

$$\alpha \cdot \beta \Rightarrow \gamma \text{ is provable} \quad \text{iff} \quad \beta \Rightarrow \alpha \rightarrow \gamma \text{ is provable.}$$

Algebraically, this can be expressed as

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a \rightarrow c$$

which is known as the *law of residuation* in residuated (ordered) structures.

Thus, this observation naturally leads us to the thesis that *substructural logics are exactly logics of residuated structures.* This will explain why substructural logics, especially when formulated in Gentzen-style sequent systems, encompass most of the interesting classes of non-classical logics. For, implication, admittedly the most important logical connective, can be understood as the *residual* of fusion, a connective that behaves like a semigroup or groupoid operation. From a mathematical point of view, it is easier to discuss the latter than the former, just like developing a theory of multiplication of numbers is easier than developing a theory of division.

This is a starting point of our algebraic approach to substructural logics. Due to significant advances in the study of both residuated lattices and the abstract algebraic logic in the recent years, the research field is developing rapidly. Recent research reveals us that there are strong interplays between algebra and logic, and even between algebraic methods and proof-theoretic ones. These facts will provide us a deeper understanding of the subject and suggest us new directions of research.

The source of the present abstract is the book [1], jointly written with Galatos, Jipsen and Kowalski, where these topics are extensively discussed.

## Reference

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# On Modeling of Uncertainty Measures and Observed Processes

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**Summary.** This paper is about a short survey of some basic uncertainty measures in systems analysis arising from coarse data, together with new modeling results on upper semicontinuous random processes, viewed as random fuzzy sets. Specifically, we present the most general mathematical framework for analyzing coarse data, such as random fuzzy data, which arise often in information systems. Our approach is based upon the theory of continuous lattices. This probabilistic analysis is also useful for investigating upper semicontinuous random functions in stochastic optimization problems.

## 1 Introduction

Empirical data are necessary in almost all fields of science, especially in decision-making where theoretical criteria need to be validated from observed data. A typical situation is in mathematical finance where investment decisions are based upon risk measures (e.g. Levy, 2006 [8]). Future returns $X, Y$ on two different prospects should be compared for selection. A partial order relation on random variables is defined in the spirit of Von Neumann and Morgenstern's expected utility theory as $X \succeq Y$ ($X$ is preferred to $Y$) if and only if for all increasing function $u : \mathbb{R} \to \mathbb{R}$, $Eu(X) \geq Eu(Y)$. A characterization of $\succeq$ turns out to be $F(\cdot) \leq G(\cdot)$ where $F$, $G$ are distribution functions of $X, Y$, respectively. This is a *theory* for risk assessment which is based upon distribution functions of random variables involved. But we never know these distributions! Thus, in order to make decisions (i.e. choosing investment prospects) we need to use empirical data to check whether this stochastic dominance order is valid. This is a hypothesis testing problem. In other words, statistics is needed to support decision-making. But statistical inference procedures depend heavily on which *type* of available data. For example, if available data are only observed within bounds. Such type of imprecise data is referred to as *coarse data* (i.e. data with low quality, e.g. Heitjan and Rubin, 1991 [7]), such as missing outcomes, censored data, partially observed data, interval data on outcomes or covariates are available, rather than point measurements, hidden Markov data in bioinformatics (e.g., Cappé et al., 2005 [1]), indirect observed data (e.g., in auction theory, Paarsch and Hong, 2006 [14]).

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It is precisely the type of observed data which dictates inference procedures. Thus, when random data are imprecise, such as only known to lie within bounds, we face interval statistics where interval computations should be called upon, say, to extend standard statistical procedures (e.g. estimation and testing) to the new type of data. More generally, the theory of random sets, e.g., Matheron, 1975 [9]; Nguyen, 2006 [11], is necessary for inference with random set data. In intelligent systems building, we encounter perception-based information (Zadeh, 2002 [17]) which is fuzzy in nature (see also e.g., Nguyen and Wu, 2006 [12]). For each type of coarse data we need an appropriate mathematical modeling for the observed process in order to carry out any inference procedures, such as model building and forecasting.

In this paper, we focus on the case of coarse data which are both random and fuzzy, as in coarsening schemes of human perception-based information gathering processes. We provide a general and rigorous mathematical model for random fuzzy sets, extending Matheron's theory of random closed sets.

In analyzing coarse data, we come across various types on uncertainty, a survey of basic aspects of modeling will be given first.

## 2 Some Uncertainty Measures Derived from Coarse Data

First, let's look at a standard situation where coarse data are *set-valued observations*. While set-valued observations, i.e. outcomes of random experiments or records of natural phenomena, have different interpretations, depending on the goals of the analysis, such as tumor growth patterns in medical statistics, shape analysis, the specific situation related to coarse data is this. Let $X$ be a random vector of interest. Either by performing a random experiment or observing $X$ in a sample data $X_1, X_2, \ldots, X_n$, to discover, say, the distribution of $X$, we are unable to observe or measure this sample with accuracy. Instead, what we observe is a collection of sets $S_1, S_2, \ldots, S_n$ which contain the sample, i.e. $X_i \in S_i$, $i = 1, 2, \ldots, n$. The statistical problem is the same, but instead of using $X_1, X_2, \ldots, X_n$, we only have at our disposal the coarse sample $S_1, S_2, \ldots, S_n$. This clearly is a generalization of multivariate statistical analysis. In order to analyze the set-valued observations, we need to model the observation process. Since probability theory provides us with a fairly general setting, namely random elements in general measurable spaces of arbitrary nature, such as metric spaces, we can simply view $S_1, S_2, \ldots, S_n$ as a random sample from a *random set* $S$ which contains $X$ almost surely, i.e. $X$ is an almost sure selector of $S$, or the other way around, $S$ is a coarsening of $X$. Random set models for coarse data turn out to be useful in exhibiting various uncertainty measures in artificial intelligence.

### 2.1 Belief Functions

Consider the case where $S$ is a coarsening of $X$ on a finite set $U$, to avoid topological details. Let $A \subseteq U$ be an event. $A$ is said to occur if $X(\omega) \in A$. But if we cannot observe $X(\omega)$, but only $S(\omega)$, then clearly we are even uncertain about the occurrence of $A$. If $S(\omega) \subseteq A$, then clearly, $A$ occurs. So, from a "pessimistic" viewpoint, we *quantify* our degrees of belief in the occurrence of an event $A$ by $P(S \subseteq A)$ which is less than the actual probability that $A$ occurs, namely $P(X \in A)$ , since $X$ is an a.s. selector of $S$ . This is in fact the starting point for the now well-known Dempster-Shafer theory of evidence or of belief functions (e.g. Shafer, 1976 [15]).

Let $F : 2^U \to [0,1]$, where $2^U$ denotes the power set of $U$, be defined by $F(A) = P(S \subseteq A)$, which is the distribution function of the random set $S$, in the sense that $F$ determines the probability law of $S$. Indeed, the set-function $F$ satisfies the following basic properties:

(i) $F(\varnothing) = 0, F(U) = 1$
(ii) For any $n \geq 1$, and $A_1, A_2, ..., A_n$,

$$F(\bigcup_{i=1}^{n} A_i) \geq \sum_{\varnothing \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} F(\bigcap_{i \in I} A_i)$$

Clearly, (i) follows from the fact that $S$ is a non-empty random set with values in $2^U$. As for (ii), this is a weakening of Poincare's equality for probability measures. Observe that since $U$ is finite,

$$F(A) = P(S \subseteq A) = \sum_{B \subseteq A} f(B) \text{ where we set } f(B) = P(S = B)$$

Let $J(B) = \{i \in \{1, 2, ..., n\} : B \subseteq A_i\}$. Clearly, we have

$$\{B : J(B) \neq \varnothing\} \subseteq \{B : B \subseteq \cup_{i=1}^{n} A_i\}$$

Now

$$\begin{aligned} P(\{B : J(B) \neq \varnothing\}) &= \sum_{B \subseteq U, J(B) \neq \varnothing} f(B) \\ &= \sum_{B \subseteq U, J(B) \neq \varnothing} f(B) [\sum_{\varnothing \neq I \subseteq J(B)} (-1)^{|I|+1}] \\ &= \sum_{\varnothing \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} \sum_{B \subseteq \cap_{i \in I} A_i} f(B) \\ &= \sum_{\varnothing \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} F(\cap_{i \in I} A_i) \end{aligned}$$

To see that the axiomatic theory of belief functions is precisely the axiomatization of distributions of random sets, exactly like the case of random variables, it suffices to show the converse. Let $F : 2^U \to [0, 1]$ such that

(i) $F(\varnothing) = 0, F(U) = 1$
(ii) For any $n \geq 1$, and $A_1, A_2, ..., A_n$,

$$F(\cup_{i=1}^{n} A_i) \geq \sum_{\varnothing \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} F(\cap_{i \in I} A_i)$$

Then there exist a probability space $(\Omega, \mathcal{A}, P)$ and a non-empty random set $S : \Omega \to 2^U$ such that $F(A) = P(S \subseteq A)$.

Indeed, by Möbius inversion, we have

$$f(A) = \sum_{B \subset A} (-1)^{|A \setminus B|} F(B)$$

which is a bona fide probability density function on $2^U$.

### 2.2 Possibility Measures

Again, let $S$ be a coarsening of $X$ on a finite set $U$. If $S(\omega) \cap A \neq \varnothing$, then all we can say is that "*it is possible that A occurs*". A plausible way to *quantify* these degrees of possibility is to take $P(S(\omega) \cap A \neq \varnothing)$.

First, this seems to be consistent with the common sense that possibilities are larger than probabilities since possibilities tend to represent an "optimistic attitude" as opposed to beliefs.

This is indeed the case since, as an a.s. selector, we clearly have $\{X \in A\} \subseteq \{S \cap A \neq \varnothing\}$, and hence $P(\{X \in A\}) \leq P(\{S \cap A \neq \varnothing\})$.

Now observe that the set-function $T(A) = P(\{S \cap A \neq \varnothing\})$ is dual to the belief function $F$ via $T(A) = 1 - F(A^c)$, the monotonicity of infinite order of $F$ above implies the alternating of infinite order of $T$, namely

$$T(\bigcap_{i=1}^{n} A_i) \leq \sum_{\varnothing \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} T(\bigcup_{i \in I} A_i)$$

which still characterizes the distribution of a random set. However, not all such $T$ can be used to model possibility measures, since possibility measures need to be truth-functional. According to Zadeh [16], a subjective concept of "possibility distributions" is primitive: $\pi : U \to [0,1]$, just like a membership function of a fuzzy concept. From $\pi$, possibilities of events could be derived. Now, in our coarsening scheme, there exists a *canonical random set* which does just that! This is completely analogous to the situation in *survey sampling* in applied statistics! The canonical random set has its roots in an early work of Goodman [5] concerning relations between fuzzy sets and random sets.

From $T(A) = P(\{S \cap A \neq \varnothing\})$, we see that when $A = \{u\}, T(\{u\}) = P(u \in S) = \pi(u)$, the covering function of the random set $S$, where $\pi : U = \{u_1, ..., u_k\} \to [0,1]$. Given $\pi$, there exist many random sets $S$ admitting $\pi(\cdot)$ as their common covering function. Indeed, let $V_i(\omega) = S(\omega)(u_i),\ i = 1, 2, \ldots, k$, where, again, for $A \subseteq U$, we write $A(u)$ for the value of the indicator function of the set $A$ at $u$. Each $V_i$ is a $\{0,1\}$-valued random variable with

$$P(V_i = 1) = P(u_i \in S) = \pi(u_i)$$

The distribution of the random vector $V = (V_1, V_2, \ldots, V_k)$ is completely determined by $P_S$ and vice versa. Indeed, for any $x = (x_1, x_2, \ldots, x_k) \in \{0,1\}^k$, we have $P(V = x) = P(S = B)$, where $B = \{u_i \in U : x_i = 1\}$. The distribution function of $V_i$ is

$$F_i(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - \pi(u_i) & \text{if } 0 \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

Thus, given the marginals $F_i,\ i = 1, 2, \ldots, k$, the joint distribution function $F$ of $V$ is the form

$$F(y_1, y_2, \ldots, y_k) = C(F_1(y_1), F_2(y_2), \ldots, F_k(y_k))$$

where $C$ is an *$k$-copula*, according to Sklar's theorem (e.g., Nelsen, 1999 [10]).

For $C(y_1, y_2, \ldots, y_k) = \prod_{i=1}^{k} y_i$ , we obtain the well-known Poisson sampling design (in sampling from a finite population) with density (of the random set $S$, the probability sampling plan):

$$f(A) = \prod_{j \in A} \pi(j) \prod_{j \in A^c} (1 - \pi(j))$$

For $C(y_1, y_2, \ldots, y_k) = \bigwedge y_j$, we get

$$f(A) = \sum_{B \subseteq A} (-1)^{|A-B|}[1 - \max\{\pi(j) : j \in B^c\}]$$

which is the density of the following nested random set (canonical):

Let $\alpha : \Omega \to [0, 1]$ be a random variable, uniformly distributed. Define $S(\omega) = \{u \in U : \pi(u) \ge \alpha(\omega)\}$. Then $\pi(u) = P(u \in S)$ and moreover,

$$P(S \cap B \neq \emptyset) = P\left(\omega : \alpha(\omega) \le \max_{u \in B} \pi(u)\right) = \max_{u \in B} \pi(u)$$

Thus, using this canonical random set (a coarsening scheme) we arrive at Zadeh's axioms for possibility measures.

## 3 Canonical Borel-$\sigma$ Fields and Continuous Lattices

In standard statistics, we model observed data as random variables, vectors or random functions by specifying canonical $\sigma$-fields on their range spaces. These are measurable mappings, defined on an abstract probability space $(\Omega, \mathcal{A}, P)$, with values in $\mathbb{R}$, $\mathbb{R}^d$, $C[0, 1]$ (space of continuous functions defined on $[0, 1]$), respectively. The canonical $\sigma$-field on $\mathbb{R}^d$ is taken to be the $\sigma$-field generated by the ordinary topology of $\mathbb{R}^d$, and that of $C[0, 1]$ is the one generated by the topology of the sup-norm. When observed data are discrete sets, closed sets (of some topological space) or upper semicontinuous functions, it is natural to inquire about canonical topologies on these spaces of sets, in some unified manner, in order to consider associated Borel $\sigma$-fields to model rigorously these random elements.

So far, as far as we know, $\sigma$-fields on these spaces are defined in some ad-hoc manner. Discrete random sets (e.g., on $\mathbb{Z}^2$) are defined in Goutsias [6], random closed sets are defined by using the hit-or-miss topology in Matheron [9], and compact random fuzzy sets are defined in, e.g., Gil et al. [4]. Now observe that these new types of observed data have natural order structures, and as such, we should look for topologies generated by such natural order structures. It turns out that these order structures are all of a very special types if we look at them in the right "angle"! Specifically, they form continuous lattices, and as such, canonical topologies on them should provide desirable $\sigma$-fields for modeling these observed data as random elements appropriately.

For ease of reading, let us recall the essentials of continuous lattices. For further details on continuous lattices, we refer the reader to Gierz et al. [2, 3]. It is known that there is a canonical Hausdorff and compact topology (called the Lawson topology) on every continuous lattice, and the space of closed sets of a Hausdorff and locally compact space is a continuous lattice. We give here essential background details for LCHS (locally compact, Hausdorff and second countable) space $X$ establishing that the space $\mathcal{F}$ is a compact, Hausdorff and second countable space whose Lawson topology coincides with the hit-or-miss topology. As we will see in the next section, the space $USC(X)$ of upper semi-continuous functions is also a continuous lattice, and hence its Lawson topology is a natural topology to consider.

Recall that if $(L, \leq)$ is a poset, then $x$ is said to be *way below* $y$, denoted as $x \ll y$, iff for all directed sets $D \subseteq L$ for which $\sup D$ exists, the relation $y \leq \sup D$ always implies $\exists d \in D$ such that $x \leq d$. Note that in a complete lattice, $x \ll y$ iff for any $A \subseteq L$, $y \leq \sup A$ implies the existence of a finite subset $B \subseteq A$ such that $x \leq \sup B$.

A lattice $(L, \leq)$ is called a continuous lattice if $L$ is complete and satisfies the *axiom of approximation:*

$$x = \sup \Downarrow x, \text{ where } \Downarrow x = \{u \in L : u \ll x\} \text{ for all } x \in L.$$

On the continuous lattice $L$, the Lawson topology has as subbase the sets of the form

$$\Uparrow x = \{y \in L : x \ll y\}$$

or

$$L \setminus \uparrow x = \{y \in L : x \nleq y\}, x \in L$$

The sets $\Uparrow x$ form a base for the Scott topology.

## 4 Discrete Random Sets

In image processing using random set approach instead of random fields (Goutsias, 1997 [6]), it is necessary to provide a rigorous foundation for discrete random sets, e.g., random sets on $\mathbb{Z}^2$. Now closed sets of discrete spaces, with their discrete topologies, are power sets which turn out to be continuous lattices under set inclusion. As such, there exist canonical $\sigma$-fields for defining discrete random sets. The $\sigma$-field proposed in Goutsias [6] for the discrete space $\mathbb{N}$, for example, is precisely the canonical $\sigma$-field generated by the Lawson topology on the continuous lattice $(2^{\mathbb{N}}, \subseteq)$.

Indeed, consider the LCHS space $(\mathbb{N}, 2^{\mathbb{N}})$ with the discrete topology where $2^{\mathbb{N}}$ is the space of closed sets of $\mathbb{N}$. We see that $(2^{\mathbb{N}}, \subseteq)$ is a continuous lattice. This is so because any $B \subseteq \mathbb{N}$ can be written as $B = \cup\{A \subseteq \mathbb{N}\colon A \ll B\}$ which is obvious since for any $b \in B$, $\{b\} \ll B$ and $B = \cup_{b\in B}\{b\}$. As such, we look at its Lawson topology. Now observe that if $A \subseteq \mathbb{N}$ is an infinite subset, then

$$\Uparrow A = \{B \subseteq \mathbb{N} : A \ll B\} = \varnothing$$

Next, if $A$ is finite, then

$$\Uparrow A = \{B \subseteq \mathbb{N} : A \subseteq B\} = \cap_{x\in A}\{B : \{x\} \subseteq B\}$$

Thus, the sets of the forms $\{B \subseteq \mathbb{N} : \{x\} \subseteq B\}$, $x \in \mathbb{N}$, form a base for the Scott topology. The other type of sets in the subbase of the Lawson topology is $\{B : A \nsubseteq B\}$, $A \subseteq \mathbb{N}$. The $\sigma$-field $\lambda(2^{\mathbb{N}})$ generated by the Lawson topology is also generated by $\{B \subseteq \mathbb{N} : A \subseteq B\}, A \subseteq \mathbb{N}$, since $\sigma$-fields are closed under set complement. Also, if $A \subseteq \mathbb{N}$, we have

$$\{B \subseteq \mathbb{N} : A \subseteq B\} = \cap_{x\in A}\{B : \{x\} \subseteq B\}$$

and by closure under countable intersections, $\lambda(2^{\mathbb{N}})$ is seen to be generated by $\{B \subseteq \mathbb{N} : \{x\} \subseteq B\} = \{B \subseteq \mathbb{N} : x \in B\}$, $x \in \mathbb{N}$. Thus, $\lambda(2^{\mathbb{N}})$ is generated by $\{B \subseteq \mathbb{N} : F \cap B \neq \varnothing\}$ $(= \cup_{x\in F}\ \{B \subseteq \mathbb{N} : x \in B\})$ for all finite $F$, i.e. $\lambda(2^{\mathbb{N}})$ coincides with Goutsias's $\sigma$-field for discrete random sets.

## 5 The Space of Closed Sets as a Continuous Lattice

In viewing the space $\mathcal{F}(X)$ of closed sets of a LCHS space $X$ as a continuous lattice with respect to the partial order $\supseteq$, we show that the hit-or-miss topology coincides with the Lawson topology on $\mathcal{F}(X)$. Moreover, since $X$ is second countable, $\mathcal{F}(X)$ is a compact, Hausdorff and second countable topological space, and hence metrizable.

First, note that $(\mathcal{F}(X), \subseteq)$ is a complete lattice but not continuous in general, we have

**Proposition 1.** $\bigwedge\{F_i : i \in I\} = \bigcap\{F_i : i \in I\}$, *and* $\bigvee\{F_i : i \in I\} =$ *the closure of* $\bigcup\{F_i : i \in I\}$.

To see that, take $X = \mathbb{R}$, we notice that if $A \ll \mathbb{R}$, then for any subset $B$ of $A$, i.e $B \subseteq A$, we also have $B \ll \mathbb{R}$ (just use the equivalent condition of the way-below relation). Then any singleton closed set, e.g. $\{0\}$, is not way-below $\mathbb{R}$. Indeed,

$$\bigvee_{n\in\mathbb{N}}\{(-\infty, -1/n] \cup [1/n, \infty)\} = \mathbb{R}$$

but we can not find any finite subset $A$ of $\{(-\infty,-1/n]\cup[1/n,\infty)\}_{n\in\mathbb{N}}$ such that $\{0\}\subseteq\bigvee A$. Therefore, the only closed set that is way-below $\mathbb{R}$ is the empty set. Then

$$\sup\{A\in\mathcal{F}(\mathbb{R}):A\ll\mathbb{R}\}=\sup\{\emptyset\}=\emptyset\neq\mathbb{R}$$

However, for locally compact $X$, $(\mathcal{F}(X),\supseteq)$ is a continuous lattice. Indeed, for any $F\in\mathcal{F}(X)$, it is enough to show that $F\leq\sup\{A\in\mathcal{F}(X):A\ll F\}$, i.e. $F\supseteq\bigcap\{A\in\mathcal{F}(X):A\ll F\}$ or $F^c\subseteq\bigcup\{A^c\in\mathcal{F}(X):A\ll F\}$. For any $x\in F^c$ : open, since $X$ is locally compact, there exists a compact set $Q_x\subseteq F^c$ such that its interior $W_x$ containing $x$. Let $A=W_x^c\in\mathcal{F}(X)$, then $A^c=W_x\subseteq Q_x\subseteq F^c$. It follows that $A\ll F$, and therefore, $x\in\cup\{A^c\in\mathcal{F}(X):A\ll F\}$. For more details, see Nguyen and Tran [13].

The following result shows that the hit-or-miss topology coincides with the Lawson topology on $(\mathcal{F}(X),\supseteq)$.

**Proposition 2.** *The Lawson topology of $\mathcal{F}(X)^{op}$, denoted by $\tau_F$, has a subbase consisting of sets of the following form*

$$\{F\in\mathcal{F}(X):F\cap K=\emptyset\}\ \textit{and}\ \{F\in\mathcal{F}(X):F\cap U\neq\emptyset\}$$

*where $K\in\mathcal{K}$ and $U\in\mathcal{G}$.*

*Proof.* We only need to verify that for any $A\in\mathcal{F}(X)$,

$$\mathcal{F}(X)\backslash\uparrow A=\{F\in\mathcal{F}(X):F\cap A^c\neq\emptyset\}$$

and then we just let $U=A^c$.

Indeed, since $\uparrow A=\{F\in\mathcal{F}(X):A\leq F\}=\{F\in\mathcal{F}(X):F\subseteq A\}$, we have

$$\mathcal{F}(X)\backslash\uparrow A=\{F\in\mathcal{F}(X):F\nsubseteq A\}=\{F\in\mathcal{F}(X):F\cap A^c\neq\emptyset\}$$

In fact, $\{F\in\mathcal{F}(X):F\cap K=\emptyset\}_{K\in\mathcal{K}}$ is closed under finite intersection, so the Lawson topology of $\mathcal{F}(X)^{op}$ has as a base the sets of the form

$$\{F\in\mathcal{F}(X):F\cap K=\emptyset\text{ and }F\cap U_i\neq\emptyset,i=1,\ldots,n\}$$

where $K\in\mathcal{K}$ and $U_i\in\mathcal{G}$.

Moreover, for a LCHS space $(X,\mathcal{G})$, the space $\mathcal{F}(X)^{op}$ is compact, Hausdorff and second countable (and hence metrizable).

## 6 The Space of USC Functions as a Continuous Lattice

As emphasized by Zadeh [17], most of the information used by humans in control and decisions are based upon perception. Now perception-based information are imprecise and uncertain, and as such, can be modeled by probability theory and fuzzy sets theory.

The process underlying human perception-based gathering can be modeled as random fuzzy sets of a particular type, namely random fuzzy sets taking values in some fuzzy partition of the measurement space.

Again, $X$ is LCHS and $USC(X)$ is the space of all usc functions from $X$ to $[0,1]$, i.e. fuzzy subsets of $X$ generalizing closed sets of $X$. The Lawson topology on it will provide a canonical $\sigma$-field for defining random fuzzy sets as random elements extending Matheron's theory of random closed sets.

Similarly to the special case of closed sets, $USC(X)$ is a complete lattice but not continuous with the point-wise order $\leq$, i.e. $f \leq g$ iff $f(x) \leq g(x), \forall x \in X$, and

$$\bigwedge_{j\in J} f_j = \inf_{j\in J} f_j, \text{ where } f_j \in USC(X),\ j \in J$$

Let $f = \inf_{j\in J} f_j$, to show $USC(X)$ is a complete lattice it is enough to show that $f \in USC(X)$, i.e. for any $r \in [0,1]$, $\{x : f(x) < r\}$ is open.

Indeed,

$$\{x : f(x) < r\} = \bigcup_{j\in J} \{x : f_j(x) < r\}$$

and since each $\{x : f_j(x) < r\}$ is open, $\{x : f(x) < r\}$ is also open.

*Note 1.* For any $f \in USC(X)$,

$$f = \inf_{r,K \text{ (compact)}} \{g_{r,K} : f(y) < r, \forall y \in K\}$$

where $g_{r,K}(x) = r$ if $x \in \overset{\circ}{K}$ and $= 1$ otherwise.

The proof of the following results can be found in Nguyen and Tran [13].

**Theorem 1.** *$L = (USC(X), \leq^{op})$ is a continuous lattice, where $f \leq^{op} g$ iff $f(x) \geq g(x), \forall x \in X$.*

*Remark 1.* For any $f, g \in L$, then $g \ll f$ implies $\forall x \in X, \exists r, K$ such that $x \in \overset{\circ}{K}$ and $f(y) < r \leq g(y), \forall y \in K$.

**Theorem 2.** *For any $r \in (0,1]$ and $K$(compact) $\subseteq X$, we have*

$$\{f \in L : f(y) < r,\ \forall y \in K\} = \bigcup_{\overset{\circ}{K}_i \supseteq K} \{f \in L : g_{r,K_i} \ll f\}$$

*where $g_{r,K_i}$ is defined as above.*

**Theorem 3.** *The Scott topology $\tau(L)$ has as a subbase the sets $\{f : f(y) < r,\ \forall y \in K\}$, where $r \in (0,1]$ and $K$(compact) $\subseteq X$. In other words, the Scott topology $\tau(L)$ has as a base the sets*

$$\bigcap_{i=1}^{n} \{f : f(y) < r_i,\ \forall y \in K_i\}$$

*where $r_i \in (0,1]$, $K_i$(compact) $\subseteq X$, and $n \in \mathbb{N}$.*

**Corollary 1.** *The Lawson topology $\Lambda(L)$ has as a subbase the sets $\{f : f(y) < r, \forall y \in K\}$, where $r \in (0,1]$ and $K(compact) \subseteq X$ together with the sets $\{f : \exists x \in X$ such that $g(x) < f(x)\}$, where $g \in L$.*

*Remark 2.* It is known that $(L, \sigma(L))$ is second countable iff $(L, \Lambda(L))$ is second countable (Gierz et al. [3]).

Thus, to show that $(L, \Lambda(L))$ is second countable, it suffices to show that $(L, \sigma(L))$ has a countable base. See Nguyen and Tran [13] for details.

*Remark 3.* In view of the above results, by a *random fuzzy (closed) set* on a LCSHS space $X$, we mean a random element with values in the measurable space $(USC(X), \sigma(\Lambda))$, where $\sigma(\Lambda)$ is the Borel $\sigma$-field associated with the Lawson topology of the continuous lattice $USC(X)$ (with reverse order $\geq$). With the Lawson topology, $USC(X)$ is a compact, Hausdorff and second countable (hence metrizable). This falls neatly in the framework of separable metric spaces in probability theory.

## 7 Concluding Remarks

Empirical science is relied upon available data. Data exhibit error in a variety of forms. Measurement error in covariates is a well-known phenomenon in classical statistics. Research efforts are directed to providing robust statistical procedures in the case of low quality of data such as this. In complex systems, coarse data present several new aspects of "error" which need to be modeled for information processing. This paper focused on modeling of uncertainties in observed coarse data as well as the data themselves. Although data are typically modeled as random outcomes of phenomena, different types of uncertainty arise of which some are related to probabilistic uncertainty (belief function, possibility measures) , and others are of a different nature (e.g. fuzziness). Specifying various uncertainty measures involved and modeling of complex observed data form a firm step toward developing robust inference procedures for applications.

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