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Princeton, 1986

Now, $W(\chi) = \chi(\delta\varphi)\, W'(\chi\,;\,\delta,\varphi)$ by the definition. Hence it follows from Proposition 2, §3, II, that

$$|W(\chi)| = |\chi(\delta\varphi)|\,|W'(\chi\,;\,\delta,\varphi)| = 1,$$
$$\overline{W(\chi)} = \overline{\chi}(\delta\varphi)\, W'(\overline{\chi}\,;\,\delta,\varphi) = W(\overline{\chi}),$$

so that

$$W(\chi)\,W(\overline{\chi}) = 1.$$

On the other hand, if $\chi = 1$, then $W(1) = 1$. Hence we obtain from (1) that

$$\xi(s\,;\,\chi) = W(\chi)\big(W(\overline{\chi})\,\eta(s\,;\,\chi) + \eta(1-s\,;\,\overline{\chi}) + \tfrac{\varepsilon v}{s(s-1)}\big)$$
$$= W(\chi)\,\xi(1-s\,;\,\overline{\chi}),$$

namely,

$$\xi(s\,;\,\chi) = W(\chi)\,\xi(1-s\,;\,\overline{\chi}).$$

By Proposition 1, §4, we have

$$L(s\,;\,\chi) = A(s\,;\,\chi)^{-1}\,\gamma(s\,;\,\chi)^{-1}\,\xi(s\,;\,\chi). \tag{2}$$

for $Re(s) > 1$. Here $A(s\,;\,\chi)^{-1}$ is a function of the form e^{as+b}, $a, b =$ const. and $\gamma(s\,;\,\chi)^{-1}$ is a product of functions of the form $\Gamma(cs+d)^{-1}$. Since $\Gamma(s)^{-1}$ is holomorphic on the entire s-plane, $A(s\,;\,\chi)^{-1}\gamma(s\,;\,\chi)^{-1}$ is also holomorphic for arbitrary s. It then follows from the above that the L-function $L(s\,;\,\chi)$, which was originally defined for s with $Re(s) > 1$, is a meromorphic function of s on the entire s-plane, satisfying (2) for arbitrary s. If $\chi \neq 1$, then $\xi(s\,;\,\chi)$ is an entire function of s. Hence $L(s\,;\,\chi)$ is also an entire function of s. Let $\chi \equiv 1$. $\zeta_F(s) = L(s\,;\,1)$ is holomorphic at $s \neq 0, 1$, and since

$$\gamma(s\,;\,1)^{-1} = \Gamma(\tfrac{s}{2})^{-r_1}\,\Gamma(s)^{-r_2}, \qquad r_1 + r_2 > 0,$$

has a zero at $s = 0$, $\zeta_F(s)$ is still holomorphic at $s = 0$. At $s = 1$, $\xi(s\,;\,1)$ has a simple pole with residue $v = \mu_{\bar{J}_1}(\bar{J}_1)$, and

$$A(1,1)^{-1} = \frac{\pi^{\frac{r_1}{2}}}{\sqrt{d}}, \quad \gamma(1\,;\,1)^{-1} = \Gamma(\tfrac{1}{2})^{-r_1} = \pi^{-\frac{r_1}{2}}.$$

Hence $\zeta_F(s)$ has a simple pole at $s = 1$ with residue $\frac{v}{\sqrt{d}}$.

We now summarize our results as follows:

<u>Theorem 1</u>. The L-function $L(s\,;\,\chi)$ for a Hecke character χ of F, which was originally defined for s with $Re(s) > 1$, is a meromorphic function of s on the entire s-plane. If $\chi \neq 1$, then $L(s\,;\,\chi)$ is holomorphic everywhere (i.e., an entire function of s). If $\chi = 1$, then $\zeta_F(s) = L(s\,;\,1)$ has a unique simple pole at $s = 1$ with residue $\frac{v}{\sqrt{d}}$, $v = \mu_{\bar{J}_1}(\bar{J}_1) < +\infty$. Let

$$\xi(s\,;\,\chi) = A(s\,;\,\chi)\,\gamma(s\,;\,\chi)\,L(s\,;\,\chi), \qquad s \in \mathbb{C},$$

with $A(s\,;\,\chi)$ and $\gamma(s\,;\,\chi)$ in §3. Then $\xi(s\,;\,\chi)$ satisfies the function equation

$$\xi(s;\chi) = W(\chi)\,\xi(1-s;\bar{\chi}), \qquad s \in \mathbb{C},$$

where $W(\chi)$ is a constant, depending only upon χ, such that

$$|W(\chi)| = 1, \quad \overline{W(\chi)} = W(\bar{\chi}).$$

Now, since $m_k, m_k' \geq 0$ and s_k are real,

$$\gamma(s;\chi) = \prod_{k=1}^{r} \Gamma\left(\tfrac{e_k}{2}\left(s + \tfrac{m_k + m_k'}{2} + i s_k\right)\right)$$

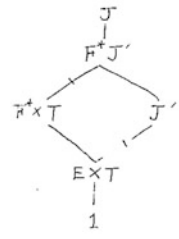
does not vanish for $Re(s) > 1$. Hence $\xi(s;\chi) = A(s;\chi)\gamma(s;\chi)L(s;\chi)$ also has no zero for $Re(s) > 1$. By the functional equation, we then see that $\xi(s;\chi)$ has no zero for $Re(s) < 0$. Hence the zeros of $L(s;\chi)$ in the domain $Re(s) < 0$ are obtained from those of $\gamma(s;\chi)^{-1}$, and they can be easily determined because $\Gamma(s)^{-1}$ has only simple zeros at $s = 0, -1, -2, \cdots$. Thus all "non-trivial" zeros of $L(s;\chi)$ are in the critical strip $\{ s \mid 0 \leq Re(s) \leq 1 \}$. The generalized Riemann hypothesis states that all such zeros (and $0 < Re(s) < 1$) are on the straight line $Re(s) = \frac{1}{2}$.

In general, a locally compact group is compact if and only if it has a finite total Haar measure. In the above, we have shown that $\mu_{\bar{J}_1}(\bar{J}_1) < +\infty$. Hence:

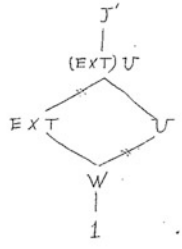
<u>Theorem 2</u>. The group $\bar{J}_1 = J_1 / F^*$ is a compact group.

We shall next show that the compactness of $\bar{J}_1$ implies two fundametal theorems on algebraic number fields, namely, the finiteness of class numbers and Dirichlet's unit theorem.

(A)



(B)



Let $J' = U_0 \times J_\infty$ and U be as before. Consider the diagrams in the above. Since $T \subset J_\infty' \subset J'$, we have $(F^* \times T)J' = F^*J'$, and $(F^* \times T) \cap J' = (F^* \cap J') \times T = E \times T$, where $E = F^* \cap J'$ is the group of units of F. Since $U \subset J_1$, $J = J_1 \times T$, $(E \times T) \cap U = E \cap U = W$,

Foreword

In 1964, Kenkichi Iwasawa gave a course of lectures at Princeton University on the adelic approach to Hecke's L-functions. The present book carefully reproduces Iwasawa's own beautifully handwritten notes used for the course, and follows faithfully his terminology.

Hecke's proof, for any number field, of the analytic continuation and functional equation of the abelian L-series, and more generally of his L-functions with Hecke characters, is of fundamental importance in algebraic number theory. Moreover, thanks to the theory of complex multiplication, it also establishes the analytic continuation and functional equation of the complex L-series of abelian varieties with complex multiplication. The modern adelic approach to Hecke's complicated classical theory was discovered independently by Iwasawa and Tate around 1950, and marked the beginning of the whole modern adelic approach to automorphic forms and L-series. While Tate's thesis at Princeton University in 1950 was finally published in 1967 in the volume *Algebraic Number Theory* edited by Cassels and Fröhlich, no detailed account of Iwasawa's work has previously appeared, beyond a very brief note in the Proceedings of the International Congress of Mathematicians in 1950, and a short letter to Dieudonné (in *Adv. Studies Pure Math.* 21, 1992). The lectures presented in this volume at last provide a detailed account of Iwasawa's work.

After two preliminary chapters on the basic local and global theory of number fields, and the theory of Haar measure on the group of idèles, Chap. 3 of the book establishes the basic expression, due to Iwasawa and Tate, for the complex L-series of the Hecke L-series $L(s, \chi)$ attached to an arbitrary Hecke character χ of a number field F as an integral over the idèle group J of F of an idelic theta function (see Sect. 3.4). Iwasawa then goes on to prove the analytic continuation and functional equation from this expression. Not only are his proofs both beautiful and fully detailed, but he also carefully explains the method in the simplest case of the Riemann zeta function. He then goes on to establish Dirichlet's formula for the residue at $s = 1$ of the complex zeta function of F, pointing out that an elegant argument involving the compactness of the idèle class group of F also gives a non-classical proof of the finiteness of the class number and the unit theorem.

In the final chapter, Iwasawa succinctly explains the link between the adelic approach and the classical theory. He then goes on to give detailed proofs of key classical results on the distribution of prime ideals, and on the class number formulae for cyclotomic fields.

This volume provides an elegant and detailed account of questions which are of seminal importance for modern number theory, and it covers material which is not treated as fully or as elegantly in other basic texts on algebraic number theory. We believe that it will provide an ideal text for future courses on this central part of number theory. Finally, we warmly thank Takahiro Kitajima for his accurate conversion of Iwasawa's handwritten notes into LaTeX, and Rei Otsuki for his help with proofreading.

Cambridge, UK — John Coates
Yokohama, Japan — Masato Kurihara
February 2019

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