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Varsha Daftardar-Gejji
Editor

Fractional Calculus and Fractional Differential Equations

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Fractional Calculus and Fractional Differential Equations

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Foreword

It gives me immense pleasure to write the foreword for the volume edited by Prof. Varsha Daftardar-Gejji with contributions from eminent researchers in the fields of fractional calculus (FC) and fractional differential equations (FDEs). These are the most important and prominent areas for research which have emerged as interdisciplinary branch of mathematical, physical, biological sciences and engineering. This book provides a systematic, logical development of modern topics through the articles by eminent scientists and active researchers working in this area all over the globe.

Fractional calculus has a history of more than 300 years, while modelling of various phenomena in terms of fractional differential equations has gained momentum since the last two decades or so. There is an upsurge of research articles in the areas of FC and FDEs. This book is appealing and unique in this context as it encompasses numerical analysis of fractional differential equations, dynamics and stability analysis of fractional differential equations involving delay, variable-order fractional operators along with chapters on engineering applications. Moreover, the fractional analogues of classical Poisson processes, analysis of fractional differential equations using inequalities and comparison theorems are dealt with in a concise manner in this book.

Bologna, Italy

Francesco Mainardi
University of Bologna

Preface

Fractional calculus (FC) and fractional differential equations (FDEs) have emerged as the most important and prominent areas of interdisciplinary interest in recent years. FC has a history of more than 300 years, yet its applicability in different domains has been realised only recently. In the last three decades, the subject witnessed exponential growth and a number of researchers around the globe are actively working on this topic. The Department of Mathematics at Savitribai Phule Pune University (SPPU) organised a national workshop on fractional calculus in 2012, which was the first workshop in India that exclusively focussed on fractional calculus. This workshop attracted researchers of pure and applied mathematics, statisticians, physicists and engineers from all over India, working in fractional calculus and related areas. Deliberations in that workshop have been appeared earlier as a book titled *Fractional Calculus: Theory and Applications* which was very well received.

As a continuation of this, in 2017, we organised a national conference on fractional differential equations bringing together researchers in FDEs for academic exchange of ideas through discussions. Many active scientists from all parts of the country participated in this conference. It covered a significant range of topics motivating us to take up this endeavour. The present book comprises excellent contributions by the resource persons in this conference besides invited contributions from experts abroad, who willingly contributed. This book gives a panoramic overview of the latest developments and is expected to help new researchers entering this vast field.

The book comprises eight chapters which cover numerical analysis of FDEs, fractional Poisson processes, variable-order fractional operators, fractional-order delay differential equations and related phenomena including chaos, impulsive FDEs, inequalities and comparison theorems in FDEs. Moreover, artificial neural network and FDEs are also discussed by a group of engineers. New transform methods such as Sumudu transform methods are presented, and their utility for solving fractional partial differential equations (PDEs) is discussed.

The book is written keeping young researchers in mind who are planning to embark upon the research problems in FC and FDEs and related topics. There are many aspects that are still open for pursuing further research. If this book motivates some readers to venture into these areas, the aim of the endeavour will be fulfilled.

I am very grateful to all the researchers who have made wonderful contributions to this volume. My sincere thanks to Springer India Pvt. Ltd. for publishing this beautiful book. I also take this opportunity to thank the authorities of SPPU and my colleagues at the Department of Mathematics. Last but not least, my sincere thanks to my parents, husband and children for their unfailing support throughout.

Pune, India Varsha Daftardar-Gejji

Contents

About the Editor

Varsha Daftardar-Gejji is Professor at the Department of Mathematics, Savitribai Phule Pune University, India. She completed her Ph.D. at Pune University, India. She has developed original methods for solving fractional differential equations that have become widely popular. Her noteworthy contributions include analysis of fractional differential equations and developing theories of fractional-ordered dynamical systems and related phenomena such as chaos. She is the editor of the book *Fractional Calculus: Theory and Applications* and has co-authored the book *Differential Equations* (Schaum's Outline Series). She has published more than 65 papers in reputed international journals in areas of fractional calculus, fractional differential equations and general relativity.

Numerics of Fractional Differential Equations

Varsha Daftardar-Gejji

Abstract Fractional calculus has become a basic tool for modeling phenomena involving memory. However, due to the non-local nature of fractional derivatives, the computations involved in solving a fractional differential equations (FDEs) are tedious and time consuming. Developing numerical and analytical methods for solving nonlinear FDEs has been a subject of intense research at present. In the present article, we review some of the existing numerical methods for solving FDEs and some new methods developed by our group recently. We also perform their comparative study.

1 Introduction

Fractional calculus (FC) is emerging as an unavoidable tool to model many phenomena in Science and Engineering [1, 2]. Fractional differential equations (FDEs) play a pivotal role in formulating processes involving memory effects. This realization is rather recent and during the past 3–4 decades there is an upsurge of intense activity exploring various aspects of FC and FDEs. Compared to integer-order differential equations, the FDEs open up great opportunities for modeling and simulations of multi-physics phenomena, e.g., seamless transition from wave propagation to diffusion, or from local to non-local dynamics. Due to the extra free parameter order, fractional-order based methods provide an additional degree of freedom in optimization performance. Not surprisingly, many fractional-order based methods have been used in image processing [3], image denoising, cryptography, controls, and many engineering applications very successfully.

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There exist many inequivalent definitions of fractional derivatives albeit Riemann–Liouville and Caputo derivatives are most popular. Analysis of FDEs involving these derivatives has been studied extensively in the literature [4–6]. Various analytical methods have been developed for solving nonlinear fractional differential equations such as Adomian Decomposition Method [7], New Iterative Method [8], Homotopy perturbation method, and so on. In these decomposition methods, solutions are obtained without discretizing the equations or approximating the operators. As these decomposition methods yield local solutions around initial conditions, for studying long-time behavior of the solutions of FDEs one has to resort to numerical methods. An important objective for developing new numerical methods is to study fractional-ordered dynamical systems and related phenomena such as bifurcations and chaos. For simulation work in fractional-ordered dynamical systems, accurate and time-efficient numerical methods are required. Due to the nonlocal nature of fractional derivatives, FDEs are computationally expensive to solve. So developing time-efficient, accurate, and stable numerical methods for FDEs is currently an active area of research.

Present article intends to give an overview of the numerical methods that are currently used in the literature. Section 2 gives basics and preliminaries. In Sect. 3, fractional Adams predictor–corrector method (FAM) has been presented. Section 4 deals with the new predictor–corrector method developed by Daftardar-Gejji et al. [9]. In Sect. 5, predictor–corrector method introduced by Jhinga and Daftardar-Gejji has been introduced along with its error estimate [10]. In Sect. 6, some illustrative examples have been presented which are solved by all the three methods, and a comparative study is made in the context of time taken, accuracy, and performance of the method for very small values of the order of the derivative. Finally, conclusions are drawn in the last section.

2 Fractional Calculus: Preliminaries

2.1 Definitions

Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f(t) \in C[a, b]$ is defined as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds. \tag{1}$$

Caputo fractional derivative of order $\alpha > 0$ of a function $f \in C^m[a, b]$, $m \in \mathbb{N}$ is defined as

$$^cD_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left[\int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds \right] = I_a^{m-\alpha} D^m f(t), \quad m-1 < \alpha < m, \tag{2}$$

where $D^m f(t) = \frac{d^m f(t)}{dt^m}$, $^cD_a^m f(t) = D^m f(t)$.

2.2 Properties of the Fractional Derivatives and Integrals

1. Let $f \in C^m[a,b],\ m-1<\beta\leq m,\ m\in\mathbb{N}$ and $\alpha>0$. Then
   a. $I_a^{\alpha}({}^cD_a^{\beta}f(t)) = {}^cD_a^{\beta-\alpha}f(t),$ if $\alpha<\beta$.
   b. $I_a^{\beta}({}^cD_a^{\beta}f(t)) = f(t) - \sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{\Gamma(k+1)}(t-a)^k$.
2. For $\alpha,\beta>0$ and $f(t)$ sufficiently smooth,
   a. if $\alpha\in\mathbb{N}$, then

$$ {}^cD_a^{\beta}(I_a^{\alpha}f(t)) = I_a^{(\alpha-\beta)}f(t). \tag{3}$$

   b. For $\alpha<\beta, m-1\leq\alpha<m, n-1\leq\beta<n$,

$$ {}^cD_a^{\beta}(I_a^{\alpha}f(t)) = {}^cD_a^{\beta-\alpha}f(t) + \sum_{k=0}^{n-m}\frac{f^{(k)}(a)}{\Gamma(k+1+\alpha-\beta)}(t-a)^{k+\alpha-\beta}. \tag{4}$$

3. For $\alpha>0, n\in\mathbb{N}$,

$$ {}^cD_a^{\alpha}(D_a^{n}f(t)) = {}^cD_a^{n+\alpha}f(t). \tag{5}$$

2.3 DGJ Method

Daftardar-Gejji and Jafari [8] introduced a new decomposition method (DGJ method) for solving functional equations of the form

$$ y = f + N(y) \tag{6}$$

where f is a known function and $N(y)$ is a nonlinear operator from a Banach space $B\to B$.

Equation (6) represents a variety of problems such as nonlinear ordinary differential equations, integral equations, fractional differential equations, partial differential equations, and systems of them.

In this method, we assume that solution y of Eq. (6) is of the form:

$$ y = \sum_{i=0}^{\infty} y_i. \tag{7}$$

The nonlinear operator is decomposed as

$$ N\left(\sum_{i=0}^{\infty}y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{k=0}^{i}y_k\right) - N\left(\sum_{k=0}^{i-1}y_k\right)\right\} \tag{8}$$

$$= \sum_{i=0}^{\infty} G_i, \tag{9}$$

where $G_0 = N(y_0)$ and $G_i = \left\{N\left(\sum_{k=0}^{i} y_k\right) - N\left(\sum_{k=0}^{i-1} y_k\right)\right\},\ i \geq 1.$

Equation (6) takes the form

$$\sum_{i=0}^{\infty} y_i = f + \sum_{i=0}^{\infty} G_i. \tag{10}$$

$y_i,\ i = 0, 1, \ldots$ are then obtained by the following recurrence relation:

$$\begin{aligned} y_0 &= f, \\ y_1 &= G_0, \\ y_2 &= G_1, \\ &\vdots \\ y_i &= G_{i-1}, \\ &\vdots \end{aligned} \tag{11}$$

Then

$$(y_1 + y_2 + \cdots + y_i) = N(y_0 + y_1 + \cdots + y_{i-1}),\ i = 1, 2, \ldots,$$

and

$$y = f + \sum_{i=1}^{\infty} y_i = f + N\left(\sum_{i=0}^{\infty} y_i\right).$$

The k-term approximation is obtained by summing up first k-terms of (11) and is defined as

$$y = \sum_{i=0}^{k-1} y_i. \tag{12}$$

Bhalekar and Daftardar-Gejji [11] have done the convergence analysis of this method. Theorems regarding convergence of DGJ method are stated below [11].

Theorem 1 *If N is C^{∞} in a neighborhood of y_0 and*

$$||N^{(n)}(y_0)|| = \sup\left\{N^{(n)}(y_0)(h_1, h_2, \ldots, h_n) : ||h_i|| \leq 1, 1 \leq i \leq n\right\} \leq L,$$

for some real number $L > 0$ and for any n and $||y_i|| \leq M < \frac{1}{e}, i = 1, 2, \ldots,$ then $\sum_{i=0}^{\infty} G_i$ is absolutely convergent and moreover,

$$||G_n|| \leq L M^n e^{n-1}(e-1),\ n = 1, 2, \ldots. \tag{13}$$

Theorem 2 *If N is C^∞ and $||N^{(n)}(y_0)|| \leq L < 1/e$, $\forall n$, then the series $\sum_{i=0}^{\infty} G_i$ is absolutely convergent.*

3 Fractional Adams Method (FAM)

Consider the initial value problem (IVP) for $0 < \alpha < 1$:

$$ {}^cD_0^\alpha x(t) = f(t, x(t)), \; x(0) = x_0, \tag{14}$$

where ${}^cD_0^\alpha$, denotes Caputo derivative and $f : [0, T] \times \mathbb{D} \longrightarrow \mathbb{R}, \; \mathbb{D} \subseteq \mathbb{R}$. For solving Eq. (14) on $[0, T]$, the interval is divided into l subintervals.

Let $h = \frac{T}{l}$, $t_n = nh$, $n = 0, 1, 2, \ldots, l \in \mathbb{Z}^+$. Then

$$x(t_n) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau. \tag{15}$$

Consider an equispaced grid $t_j = t_0 + jh$, with step length h. Let x_j denote the approximate solution at t_j and $x(t_j)$ denotes the exact solution of the IVP (14) at t_j. Further denote $f_j = f(t_j, x_j)$.

$$\begin{aligned}
I^\alpha f(t_n, x(t_n)) &= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(s, x(s)) ds \\
&\approx \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(t_k, x(t_k)) ds \\
&= h^\alpha \sum_{k=0}^{n-1} b_{n-k-1} f(t_k, x(t_k)), \\
&\text{where } b_k = \frac{1}{\Gamma(\alpha+1)} [(k+1)^\alpha - k^\alpha].
\end{aligned}$$

$$\text{Hence } x_n = x_0 + h^\alpha \sum_{k=0}^{n-1} b_{n-k-1} f_k. \tag{16}$$

Equation (16) is referred as fractional rectangle rule.

Implicit Adams quadrature method (using trapezoidal rule) gives the following formula. On each subinterval $[t_k, t_{k+1}]$, the function $f(t)$ is approximated by straight line

$$\begin{aligned}
\tilde{f}(t, x(t)) \mid_{[t_k, t_{k+1}]} = \frac{t_{k+1} - t}{t_{k+1} - t_k} f(t_k, x(t_k)) + \\
\frac{t - t_k}{t_{k+1} - t_k} f(t_{k+1}, x(t_{k+1})).
\end{aligned}$$

In view of this approximation

$$I_0^\alpha f(t_n, x(t_n)) \approx \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - t)^{\alpha-1} \tilde{f}(t, x(t)) \,|_{[t_k, t_{k+1}]} \, dt$$

$$= h^\alpha \sum_{k=0}^{n} a_{n-k} f(t_k, x(t_k)), \text{ where}$$

$$a_j = \begin{cases} \dfrac{1}{\Gamma(\alpha+2)} & \text{if} \quad j = 0, \\ \dfrac{(j-1)^{\alpha+1} - 2j^{\alpha+1} + (j+1)^{\alpha+1}}{\Gamma(\alpha+2)} & \text{if } j = 1, \ldots, n-1, \\ \dfrac{(n-1)^{\alpha+1} - n^\alpha(n-\alpha-1)}{\Gamma(\alpha+2)} & \text{if} \quad j = n. \end{cases}$$

$$\text{Hence } x_n = x_0 + h^\alpha a_n f_0 + h^\alpha \sum_{j=1}^{n} a_{n-j} f(t_j, x_j). \tag{17}$$

Equation (17) is referred as fractional trapezoidal rule.

Thus, fractional rectangle rule and fractional trapezoidal rule form a predictor–corrector algorithm. A preliminary approximation x_n^p (predictor) is made using Eq. (16), which is substituted in Eq. (17) to give a corrector. This method is also known as fractional Adams method [12], and used for simulations of FDEs extensively.

$$x_n^p = x_0 + h^\alpha \sum_{j=0}^{n-1} b_{n-j-1} f(t_j, x_j), \tag{18}$$

$$x_n^c = x_0 + h^\alpha a_n f_0 + h^\alpha \sum_{j=1}^{n-1} a_{n-j} f(t_j, x_j) + h^\alpha a_0 f(t_n, x_n^p). \tag{19}$$

Order of the method is said to be p when the error can be shown to have $O(h^p)$ as $h \to 0$ for step length $h > 0$. Order of the method is often regarded as a benchmark for comparing methods.

The error in the FAM [12] behaves as $\text{Max}_{j=0,1,\ldots,n} \left| x(t_j) - x_j \right| = O(h^p)$, where $p = \min\{2, 1+\alpha\}$.

4 New Predictor–Corrector Method (NPCM)

Though FAM is extensively used in the literature, for carrying out simulations pertaining to fractional-ordered dynamical systems, one needs more time-efficient numerical methods as solving FDEs involves memory effects. In pursuance to this, Daftardar-Gejji et al. [9] have proposed a new predictor–corrector method (NPCM). This method is developed as a combination of fractional trapezoidal rule and DGJ decomposition. We describe this method below.

Consider the initial value problem given in Eq. (14)

$$ {}^{c}D_0^{\alpha} x(t) = f(t, x(t)), x(0) = x_0, 0 < \alpha < 1. $$

Equation (15) can be discretized as follows.

$$ \begin{aligned} x(t_n) &= x(0) + h^{\alpha} \sum_{j=0}^{n} a_{n-j} f(t_j, x_j) \\ &= x(0) + h^{\alpha} \sum_{j=0}^{n-1} a_{n-j} f(t_j, x_j) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_n, x_n). \end{aligned} \tag{20} $$

The solution of Eq. (20) can be approximated by DGJ method, where

$$ N(x(t_n)) = \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_n, x_n). \tag{21} $$

We apply DGJ method to get approximate value of x_1, as follows:

$$ x(t_1) = x_1 = x_0 + h^{\alpha} a_1 f(t_0, x_0) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_1, x_1), \tag{22} $$

$$ x_{1,0} = x_0 + h^{\alpha} a_1 f(t_0, x_0), $$

$$ x_{1,1} = N(x_{1,0}) = \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_1, x_{1,0}), $$

$$ x_{1,2} = N(x_{1,0} + x_{1,1}) - N(x_{1,0}). $$

The three-term approximation of $x_1 \approx x_{1,0} + x_{1,1} + x_{1,2} = x_{1,0} + N(x_{1,0} + x_{1,1})$. This gives a new predictor–corrector formula as follows:

$$ \begin{aligned} y_1^p &= x_{1,0}, \qquad z_1^p = N(x_{1,0}), \\ x_1^c &= y_1^p + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_1, y_1^p + z_1^p). \end{aligned} $$

$x(t_2), x(t_3), \ldots$ can be obtained similarly.

Daftardar-Gejji et al. [9] have proposed this new predictor–corrector method (NPCM), which is derived by combining fractional trapezoidal formula and DGJ method [8] and it leads to the following formula:

$$y_n^p = x_0 + h^\alpha \sum_{j=0}^{n-1} a_{n-j} f(t_j, x_j),$$

$$z_n^p = \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_n, y_n^p),$$

$$x_n^c = y_n^p + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_n, y_n^p + z_n^p),$$

where

$$a_j = \begin{cases} \dfrac{1}{\Gamma(\alpha+2)} & \text{if} \quad j = 0, \\ \dfrac{(j-1)^{\alpha+1} - 2j^{\alpha+1} + (j+1)^{\alpha+1}}{\Gamma(\alpha+2)} & \text{if} \quad j = 1, \ldots, n-1, \\ \dfrac{(n-1)^{\alpha+1} - n^\alpha(n-\alpha-1)}{\Gamma(\alpha+2)} & \text{if} \quad j = n. \end{cases}$$

Here y_n^p and z_n^p are called as predictors and x_n^c is the corrector. Here x_j denotes the approximate value of solution of Eq. (20) at $t = t_j$. This is called three-step iterative method for solving nonlinear equation (20).

Error Estimation in NPCM

Let ${}^cD^\alpha x(t) \in C^2[0, T],\ T > 0$, then $\max_{0 \le j \le l} \mid x(t_j) - x_j \mid = O(h^2)$.
Comment: For $0 < \alpha < 1$, the error estimate for the case ${}^cD^\alpha x(t) \in C^2[0, T]$ in the FAM is of the order $O(h^{1+\alpha})$, whereas for NPCM $O(h^2)$. Hence NPCM in this case gives more accuracy.

4.1 Stability Regions

It is further noted that both FAM and three-term NPCM are strongly stable methods. Comparison of the stability regions of NPCM and FAM is given below [13]. It should be noted that the NPCM is more stable than FAM (Figs. 1, 2, 3 and 4).

S_1: Stability region of FAM, S_2: Stability region of NPCM

4.2 NPCM for System of FDEs

The NPCM can be generalized for solving following system of fractional differential equations. Consider the following system of FDEs, for $\alpha_i > 0,\ 1 \le i \le r$:

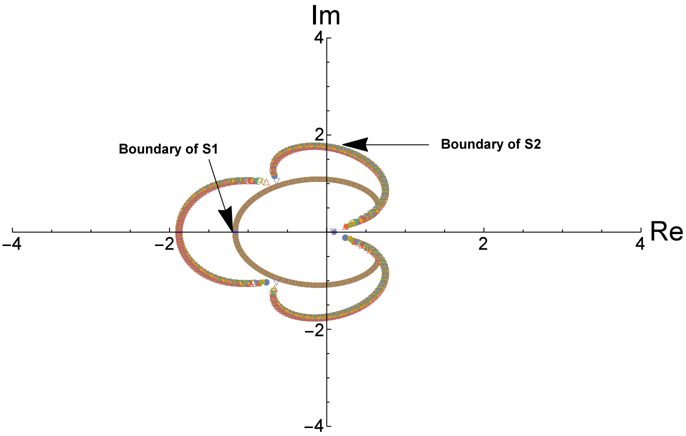

Fig. 1 $\alpha = 0.3$

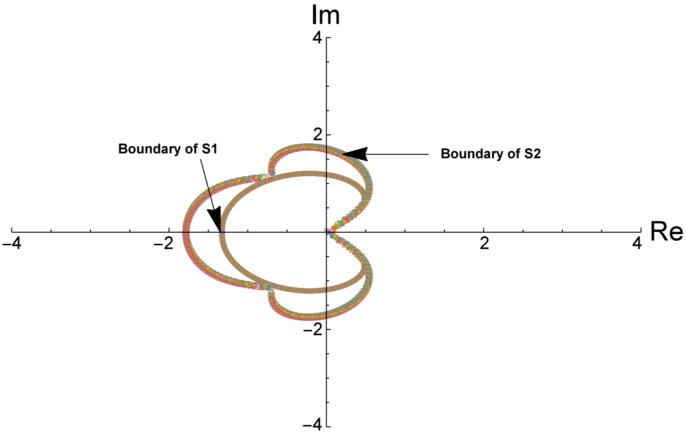

Fig. 2 $\alpha = 0.5$

$$\begin{aligned}
{}^{c}D_0^{\alpha_1}u_1(t) &= f_1(t,\bar{u}(t)),\ u_1^{(k_1)}(0) = u_{10}^{(k_1)},\ k_1 = 0,1,2,\ldots,\lceil\alpha_1\rceil - 1,\\
{}^{c}D_0^{\alpha_2}u_2(t) &= f_2(t,\bar{u}(t)),\ u_2^{(k_2)}(0) = u_{20}^{(k_2)},\ k_2 = 0,1,2,\ldots,\lceil\alpha_2\rceil - 1,\\
&\vdots\\
{}^{c}D_0^{\alpha_r}u_r(t) &= f_r(t,\bar{u}(t)),\ u_r^{(k_r)}(0) = u_{r0}^{(k_r)},\ k_r = 0,1,2,\ldots,\lceil\alpha_r\rceil - 1.
\end{aligned} \tag{23}$$