

Maria Gorelik
Vladimir Hinich
Anna Melnikov
Editors

Representations and Nilpotent Orbits of Lie Algebraic Systems

In Honour of the 75th Birthday of Tony
Joseph

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Maria Gorelik
Department of Mathematics
Weizmann Institute of Science
Rehovot, Israel

Vladimir Hinich
Department of Mathematics
University of Haifa
Haifa, Israel

Anna Melnikov
Department of Mathematics
University of Haifa
Haifa, Israel

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Preface

Lie theory, inaugurated through the fundamental work of Sophus Lie during the late nineteenth century, has proved central in many areas of mathematics and theoretical physics. Sophus Lie's formulation was originally in the language of analysis and geometry; however, by now, a vast algebraic counterpart of the theory has been developed. As in algebraic geometry, the deepest and most far-reaching results in Lie theory nearly always come about when geometric and algebraic techniques are combined.

A core part of Lie theory is the structure and representation theory of complex semisimple Lie algebras and Lie groups, which is an exemplary harmonious field in modern mathematics. It has deep ties to physics, and many areas of mathematics, such as combinatorics, category theory, and others. This field has inspired many generalizations, among them the representation theories of affine Lie algebras, vertex operator algebras, locally finite Lie algebras, Lie superalgebras, etc. This volume originates from a pair of sister conferences titled “Algebraic Modes of Representations” held in Israel in July 2017. The first conference took place at the Weizmann Institute of Science, Rehovot, July 16–18, and the second conference took place at the University of Haifa, July 19–23. Both conferences were dedicated to the 75th birthday of Anthony Joseph, who has been one of the leading figures in Lie Theory from the 1970s until today. The conferences were supported by the United States–Israel Binational Science Foundation and the Chorafas Institute for Scientific Exchange (Weizmann part) and by the Israel Science Foundation (Haifa part).

Joseph has had a fundamental influence on both classical representation theory and quantized representation theory. A detailed description of his work in both areas has been given in the articles by W. McGovern and D. Farkash–G. Letzter in the volume “Studies in Lie theory,” *Progress in Mathematics*, vol. 243, Birkhauser. Concerning Joseph's contribution to classical representation theory, it is impossible not to mention his classification of primitive ideals of the universal enveloping algebra of $\mathfrak{sl}(n)$. The essential new ingredient here is the introduction of a partition of the Weyl group into left cells, corresponding to the Robinson map from the symmetric group to the standard Young tableaux. Joseph further extended this result

to other simple Lie algebras using similar techniques, and this has since then become a powerful tool in Lie theory.

As for quantized representation theory, Joseph's monograph "Quantum Groups and Their Primitive Ideals," *Ergebnisse der Mathematik und Ihrer Grenzgebiete*, 3rd series, vol. 29, has had a fundamental influence over the field since its appearance in 1995.

The present volume contains 14 original papers covering a broad spectrum of current aspects of Lie theory. The areas discussed include primitive ideals, invariant theory, geometry of Lie group actions, crystals, quantum affine algebras, Yangians, categorification, and vertex algebras.

The authors of this volume are happy to dedicate their works to Anthony Joseph.

Rehovot, Israel
Haifa, Israel
Haifa, Israel

Maria Gorelik
Vladimir Hinich
Anna Melnikov

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Singular Support of a Vertex Algebra and the Arc Space of Its Associated Scheme



Tomoyuki Arakawa and Andrew R. Linshaw

Dedicated to Professor Anthony Joseph on his seventy-fifth birthday

Abstract Attached to a vertex algebra $\mathcal{V}$ are two geometric objects. The *associated scheme* of $\mathcal{V}$ is the spectrum of Zhu's Poisson algebra $R_{\mathcal{V}}$. The *singular support* of $\mathcal{V}$ is the spectrum of the associated graded algebra $\text{gr}(\mathcal{V})$ with respect to Li's canonical decreasing filtration. There is a closed embedding from the singular support to the arc space of the associated scheme, which is an isomorphism in many interesting cases. In this note we give an example of a non-quasi-lisse vertex algebra whose associated scheme is reduced, for which the isomorphism is not true as schemes but true as varieties.

MSC: 17B69, 81R10, 14E18

1 Introduction

Attached to a vertex algebra $\mathcal{V}$ are two geometric objects. The *associated scheme* $\tilde{X}_{\mathcal{V}}$ of $\mathcal{V}$ is the spectrum of commutative algebra $R_{\mathcal{V}}$, which is an affine Poisson scheme of finite type.¹ The *singular support* $\text{SS}(\mathcal{V})$ of $\mathcal{V}$ is the spectrum of the

¹Provided that $\mathcal{V}$ is finitely strongly generated.

T. Arakawa (✉)

Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan
e-mail: arakawa@kurims.kyoto-u.ac.jp

A. R. Linshaw

University of Denver, Denver, CO, USA
e-mail: andrew.linshaw@du.edu

associated graded algebra $\text{gr}(\mathcal{V})$ with respect to Li's canonical decreasing filtration, which is a *vertex Poisson* scheme of infinite type.² There is a closed embedding

$$\Phi : \text{SS}(\mathcal{V}) \hookrightarrow (\tilde{X}_{\mathcal{V}})_{\infty}$$

from the singular support to the arc space $\tilde{X}_{\infty}$ of the associated scheme, which is an isomorphism in many interesting cases.

Originally Zhu [29] introduced the algebra $R_{\mathcal{V}}$ to define a certain finiteness condition on a vertex algebra. Recall that a vertex algebra $\mathcal{V}$ is called *lisce* (or C_2 -cofinite) if $\dim \tilde{X}_{\mathcal{V}} = 0$. Using the map Φ one can show that this condition is equivalent to that $\dim \text{SS}(\mathcal{V}) = 0$, and hence, the lisse condition is a natural finiteness condition [3]. It is known that lisse vertex (operator) algebras have many nice properties, such as modular invariance property of characters [23, 29], and this condition has been assumed in many significant theories of vertex (operator) algebras. However, recently non-lisse vertex algebras have caught a lot of attention due to the *Higgs branch conjecture* by Beem and Rastelli [10], which states that the reduced scheme $X_{\mathcal{V}}$ of $\tilde{X}_{\mathcal{V}}$ should be isomorphic to the Higgs branch of a four-dimensional $N = 2$ superconformal field theory $\mathcal{T}$ if $\mathcal{V}$ obtained from $\mathcal{T}$ by the correspondence discovered by [9], see the survey articles [4, 5] and the references therein.

It is natural to ask whether the map Φ is always an isomorphism, and if not, whether Φ defines an isomorphism as varieties. Very recently counterexamples to the first question were found by van Ekeren and Heluani [16] in the case that $\mathcal{V}$ is lisse in their study of chiral homology of elliptic curves. It was also shown recently in [8] that the map Φ defines an isomorphism as varieties if $\mathcal{V}$ is *quasi-lisse*, that is, the Poisson variety $X_{\mathcal{V}}$ has finitely many symplectic leaves. In this note we give an example of a non-quasi-lisse vertex algebra whose associated scheme is reduced, for which Φ is not an isomorphism of schemes, but still defines an isomorphism of varieties. We remark that by tensoring one of the lisse examples in [16] with any non-quasi-lisse vertex algebra, one can trivially obtain a non-quasi-lisse example. However, all such examples have the property that the associated scheme is nonreduced.

2 Vertex Algebras

We assume that the reader is familiar with vertex algebras, which have been discussed from various points of view in the literature [11, 17–19]. Given an element a in a vertex algebra $\mathcal{V}$, the field associated to a via the state-field correspondence is denoted by

²Unless $\mathcal{V}$ is finite-dimensional.

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in \text{End}(\mathcal{V})[[z, z^{-1}]].$$

Throughout this paper, we shall identify $\mathcal{V}$ with the corresponding space of fields. Given $a, b \in \mathcal{V}$, the *operators product expansion* (OPE) formula is given by

$$a(z)b(w) \sim \sum_{n \geq 0} (a_{(n)}b)(w) (z-w)^{-n-1}.$$

Here $(a_{(n)}b)(w) = \text{Res}_z[a(z), b(w)](z-w)^n$ where

$$[a(z), b(w)] = a(z)b(w) - (-1)^{|a||b|}b(w)a(z),$$

and $\sim$ means equal modulo terms which are regular at $z = w$. The *normally ordered product* : $a(z)b(z)$: is defined to be

$$a(z)_- b(z)_+ + (-1)^{|a||b|} b(z)_+ a(z)_-,$$

where

$$a(z)_- = \sum_{n < 0} a(n) z^{-n-1}, \quad a(z)_+ = \sum_{n \geq 0} a(n) z^{-n-1}.$$

We usually omit the formal variable z and write : $a(z)b(z)$: = : ab :, when no confusion can arise. For $a_1, \dots, a_k \in \mathcal{V}$, the iterated normally ordered product is defined inductively by

$$: a_1 a_2 \cdots a_k : = : a_1 (: a_2 \cdots a_k :) :. \quad (2.1)$$

A subset $S = \{a_i \mid i \in I\}$ of $\mathcal{V}$ is said to *strongly generate* $\mathcal{V}$, if $\mathcal{V}$ is spanned by the set of normally ordered monomials

$$: \partial^{k_1} a_{i_1} \cdots \partial^{k_m} a_{i_m} :, \quad i_1, \dots, i_m \in I, \quad k_1, \dots, k_m \geq 0.$$

If S is an ordered strong generating set $\{\alpha^1, \alpha^2, \dots\}$, we say that S *freely generates* $\mathcal{V}$, if $\mathcal{V}$ has a PBW basis consisting of

$$\begin{aligned} & : \partial^{k_1^1} \alpha^{i_1} \cdots \partial^{k_{r_1}^1} \alpha^{i_1} \partial^{k_1^2} \alpha^{i_2} \cdots \partial^{k_{r_2}^2} \alpha^{i_2} \cdots \partial^{k_1^n} \alpha^{i_n} \cdots \partial^{k_{r_n}^n} \alpha^{i_n} :, \quad 1 \leq i_1 < \cdots < i_n, \\ & k_1^1 \geq k_2^1 \geq \cdots \geq k_{r_1}^1, \quad k_1^2 \geq k_2^2 \geq \cdots \geq k_{r_2}^2, \quad \dots, \quad k_1^n \geq k_2^n \geq \cdots \geq k_{r_n}^n, \\ & k_1^t > k_2^t > \cdots > k_{r_t}^t \end{aligned} \quad (2.2)$$

whenever α^{i_t} is odd.

In particular, the monomials (2.2) are linearly independent, so there are no nontrivial normally ordered polynomial relations among the generators and their derivatives.

$\beta\gamma$ -System The $\beta\gamma$ -system $\mathcal{S}$ is freely generated by even fields β, γ satisfying

$$\begin{aligned}\beta(z)\gamma(w) &\sim (z-w)^{-1}, & \gamma(z)\beta(w) &\sim -(z-w)^{-1}, \\ \beta(z)\beta(w) &\sim 0, & \gamma(z)\gamma(w) &\sim 0.\end{aligned}\tag{2.3}$$

It has Virasoro element $L^{\mathcal{S}} = \frac{1}{2}(\beta\partial\gamma : - : \partial\beta\gamma :)$ of central charge $c = -1$, under which β, γ are primary of weight $\frac{1}{2}$.

$\mathcal{W}_3$ -Algebra The $\mathcal{W}_3$ -algebra $\mathcal{W}_3^c$ with central charge c was introduced by Zamolodchikov [28]. It is an extension of the Virasoro algebra, and is freely generated by a Virasoro field L and an even weight 3 primary field W . In fact, $\mathcal{W}_3^c$ is isomorphic to the principal $\mathcal{W}$ -algebra $\mathcal{W}^k(\mathfrak{sl}_3, f_{\text{prin}})$ where $c = 2 - \frac{24(k+2)^2}{k+3}$. For generic values of c , $\mathcal{W}_3^c$ is simple, but for certain special values it has a nontrivial ideal. In this paper, we only need the case $c = -2$, which is nongeneric. We shall denote the simple graded quotient of $\mathcal{W}_3^{-2}$ by $\mathcal{W}$ for the rest of the paper. Since $\mathcal{W}_3^{-2}$ has a nontrivial ideal, $\mathcal{W}$ is strongly but not freely generated by L, W .

There is a useful embedding $i : \mathcal{W} \rightarrow \mathcal{S}$ due to Wang [26], given by

$$\begin{aligned}L &\mapsto \frac{1}{2} : \beta\beta\gamma\gamma : + : \beta(\partial\gamma) : - : (\partial\beta)\gamma :, \\ W &\mapsto \frac{1}{4\sqrt{2}} \left(2 : \beta^3\gamma^3 : + 9 : \beta^2(\partial\gamma)\gamma : + 3 : \beta\partial^2\gamma : - 9 : (\partial\beta)\beta\gamma^2 : - 12\partial\beta(\partial\gamma) : + 3 : (\partial^2\beta)\gamma : \right),\end{aligned}\tag{2.4}$$

and we shall identify $\mathcal{W}$ with its image in $\mathcal{S}$. In fact, $\mathcal{W}$ is precisely the subalgebra of $\mathcal{S}$ that commutes with the Heisenberg algebra generated by $\beta\gamma$. Note that W is normalized so that it satisfies

$$\begin{aligned}W(z)W(w) &\sim -\frac{9}{8}(z-w)^{-6} + \frac{27}{8}L(w)(z-w)^{-4} + \frac{27}{16}\partial L(w)(z-w)^{-3} \\ &+ \left(\frac{9}{2} : LL : - \frac{27}{32}\partial^2 L \right)(w)(z-w)^{-2} + \left(\frac{9}{2} : (\partial L)L : - \frac{3}{16}\partial^3 L \right)(w)(z-w)^{-1}.\end{aligned}$$

This normalization is nonstandard but convenient for our purposes.

Zhu's Commutative Algebra and the Associated Variety Given a vertex algebra $\mathcal{V}$, define

$$C(\mathcal{V}) = \text{Span}\{a_{(-2)}b \mid a, b \in \mathcal{V}\}, \quad R_{\mathcal{V}} = \mathcal{V}/C(\mathcal{V}).\tag{2.5}$$

It is well known that $R_{\mathcal{V}}$ is a commutative, associative algebra with product induced by the normally ordered product [29]. Also, if $\mathcal{V}$ is graded by conformal weight, $R_{\mathcal{V}}$ inherits this grading. Define the associated scheme

$$\tilde{X}_{\mathcal{V}} = \text{Spec}(R_{\mathcal{V}}), \quad (2.6)$$

and the associated variety

$$X_{\mathcal{V}} = \text{Specm}(R_{\mathcal{V}}) = (\tilde{X}_{\mathcal{V}})_{\text{red}}. \quad (2.7)$$

Here $(\tilde{X}_{\mathcal{V}})_{\text{red}}$ denotes the reduced scheme of $\tilde{X}_{\mathcal{V}}$. If $\{\alpha_i \mid i \in I\}$ is a strong generating set for $\mathcal{V}$, the images of these fields in $R_{\mathcal{V}}$ will generate $R_{\mathcal{V}}$ as a ring. In particular, $R_{\mathcal{V}}$ is finitely generated if and only if $\mathcal{V}$ is strongly finitely generated.

Since the $\beta\gamma$ -system $\mathcal{S}$ is freely generated by β, γ , $R_{\mathcal{S}} \cong \mathbb{C}[b, g]$, where b, g denote the images of β, γ in $R_{\mathcal{S}}$. On the other hand, since $\mathcal{W}$ is not freely generated by L, W , the structure of $R_{\mathcal{W}}$ is more complicated.

Lemma 2.1 *Let ℓ, w denote the images of L, W in $R_{\mathcal{W}}$. Then $R_{\mathcal{W}} \cong \mathbb{C}[\ell, w]/\langle w^2 - \ell^3 \rangle$.*

Proof Since $\mathcal{W}$ is strongly generated by L, W , $R_{\mathcal{W}}$ is generated by ℓ, w , so $R_{\mathcal{W}} \cong \mathbb{C}[\ell, w]/I$ for some ideal I . By Lemma 2.1 of [27], we have the following normally ordered relation in $\mathcal{W}$ at weight 6:

$$: W^2 : - : L^3 : - \frac{7}{8} : (\partial^2 L)L : - \frac{19}{32} : (\partial L)^2 : = 0. \quad (2.8)$$

Note that (2.8) differs slightly from the formula in [27] because our normalization of W is different. It follows that $w^2 - \ell^3 \in I$.

To see that $I \subseteq \langle w^2 - \ell^3 \rangle$, let $p = p(\ell, w) \in I$. Without loss of generality, we may assume p is homogeneous of weight d . It must come from a normally ordered polynomial relation

$$P = P(L, \partial L, \dots, W, \partial W, \dots) = 0$$

of weight d in $\mathcal{W}$ among L, W and their derivatives. The monomials of p correspond to the normally ordered monomials of P which do not lie in $C(\mathcal{W})$, and have the form

$$: L^i W^j :, \quad 2i + 3j = d. \quad (2.9)$$

Using (2.8) repeatedly, we can rewrite this relation in the form

$$P' = P'(L, \partial L, \dots, W, \partial W, \dots) = 0,$$

where all terms of the form (2.9) that appear either have $j = 0$ or $j = 1$. In fact, since P' is homogeneous of weight d , we must have $j = 0$ if d is even, and $j = 1$ if d is odd, so only one such term can appear. If this term appears with nonzero coefficient, as a normally ordered polynomial in β, γ and their derivatives, it will contribute the term $:\beta^{2i+3j}\gamma^{2i+3j}:$, which cannot be canceled. This contradicts $P' = 0$, so each monomial in P' must lie in $C(\mathcal{W})$. Equivalently, $p \in \langle w^2 - \ell^3 \rangle$. $\square$

3 Jet Schemes and Arc Spaces

We recall some basic facts about jet schemes, following the notation in [15]. Let X be an irreducible scheme over $\mathbb{C}$ of finite type. The first jet scheme X_1 is the total tangent space of X , and for $m > 1$ the jet schemes X_m are higher-order generalizations which are determined by their functor of points. Given a $\mathbb{C}$ -algebra A , we have a bijection

$$\mathrm{Hom}(\mathrm{Spec}(A), X_m) \cong \mathrm{Hom}(\mathrm{Spec}(A[t]/\langle t^{m+1} \rangle), X).$$

Thus the $\mathbb{C}$ -valued points of X_m correspond to the $\mathbb{C}[t]/\langle t^{m+1} \rangle$ -valued points of X . For $p > m$, we have projections $\pi_{p,m} : X_p \rightarrow X_m$ and $\pi_{p,m} \circ \pi_{q,p} = \pi_{q,m}$ when $q > p > m$. The assignment $X \mapsto X_m$ is functorial, and a morphism $f : X \rightarrow Y$ induces $f_m : X_m \rightarrow Y_m$ for all $m \geq 1$. If X is nonsingular, X_m is irreducible and nonsingular for all m . If X, Y are nonsingular and $f : X \rightarrow Y$ is a smooth surjection, f_m is surjective for all m .

For an affine scheme $X = \mathrm{Spec}(R)$ where $R = \mathbb{C}[y_1, \dots, y_r]/\langle f_1, \dots, f_k \rangle$, X_m is also affine and we can give explicit equations for X_m as follows. Define variables $y_1^{(i)}, \dots, y_r^{(i)}$ for $i = 0, \dots, m$, and define a derivation D by

$$D(y_j^{(i)}) = \begin{cases} y_j^{(i+1)} & 0 \leq i < m, \\ 0 & i = m, \end{cases} \quad (3.1)$$

which specifies its action on all of $\mathbb{C}[y_1^{(i)}, \dots, y_r^{(i)}]$, for $0 \leq i \leq m$. In particular, $f_\ell^{(i)} = D^i(f_\ell)$ is a well-defined polynomial in $\mathbb{C}[y_1^{(i)}, \dots, y_r^{(i)}]$. Letting

$$R_m = \mathbb{C}[y_1^{(i)}, \dots, y_r^{(i)}]/\langle f_1^{(i)}, \dots, f_k^{(i)} \rangle,$$

we have $X_m \cong \mathrm{Spec}(R_m)$. By identifying y_j with $y_j^{(0)}$, we may identify R with a subalgebra of R_m . There is a $\mathbb{Z}_{\geq 0}$ -grading on R_m which we call *height*, given by

$$R_m = \bigoplus_{n \geq 0} R_m[n], \quad \mathrm{ht}(y_j^{(i)}) = i. \quad (3.2)$$

For all m , $R_m[0] = R$ and $R_m[n]$ is an R -module.

Given a scheme X , define

$$X_\infty = \varprojlim X_m, \quad (3.3)$$

which is known as the *arc space* of X . For a $\mathbb{C}$ -algebra A , we have a bijection

$$\mathrm{Hom}(\mathrm{Spec}(A), X_\infty) \cong \mathrm{Hom}(\mathrm{Spec}(A[[t]]), X),$$

so the $\mathbb{C}$ -valued points of X_∞ correspond to the $\mathbb{C}[[t]]$ -valued points of X . If $X = \mathrm{Spec}(R)$ as above,

$$X_\infty \cong \mathrm{Spec}(R_\infty), \text{ where } R_\infty = \mathbb{C}[y_1^{(i)}, \dots, y_r^{(i)}] / \langle f_1^{(i)}, \dots, f_k^{(i)} \rangle.$$

Here $i \geq 0$, and $D(y_j^{(i)}) = y_j^{(i+1)}$ for all i .

By a theorem of Kolchin [21], X_∞ is irreducible if X is irreducible. However, even if X is irreducible and reduced, X_∞ need not be reduced. The following result is due to Sebag (see Example 8 of [24], as well as more general results in [25]), but we include a proof for the benefit of the reader.

Lemma 3.1 *For $X = \mathrm{Spec}(\mathbb{C}[\ell, w] / \langle w^2 - \ell^3 \rangle) = \tilde{X}_{\mathcal{W}}$, X_∞ is not reduced.*

Proof We have

$$X_\infty \cong \mathrm{Spec}(R_\infty), \quad R_\infty = \mathbb{C}[\ell^{(0)}, \ell^{(1)}, \dots, w^{(0)}, w^{(1)}, \dots] / \langle f^{(0)}, f^{(1)}, \dots \rangle, \quad (3.4)$$

where $f^{(0)} = (\ell^{(0)})^3 - (w^{(0)})^2$. Consider the element

$$r_1 = 3\ell^{(1)}w^{(0)} - 2\ell^{(0)}w^{(1)} \in \mathbb{C}[\ell^{(0)}, \ell^{(1)}, \dots, w^{(0)}, w^{(1)}, \dots]. \quad (3.5)$$

First, $r_1 \notin \langle f^{(0)}, f^{(1)}, \dots \rangle$ since no element of this ideal has leading term of degree 2. However, $(r_1)^3 \in \langle f^{(0)}, f^{(1)}, \dots \rangle$; a calculation shows that

$$\begin{aligned} (r_1)^3 = & \left(-81\ell^{(0)}\ell^{(1)}\ell^{(2)}w^{(0)} - \frac{27}{2}(\ell^{(0)})^2\ell^{(3)}w^{(0)} + 18(\ell^{(0)})^2\ell^{(2)}w^{(1)} - 4(w^{(1)})^3 + 15w^{(0)}w^{(1)}w^{(2)} \right. \\ & \left. + 9(w^{(0)})^2w^{(3)} \right) f^{(0)} + \left(\frac{9}{2}(\ell^{(0)})^2\ell^{(2)}w^{(0)} + 12(\ell^{(0)})^2\ell^{(1)}w^{(1)} - 7w^{(0)}(w^{(1)})^2 - 3(w^{(0)})^2w^{(2)} \right) f^{(1)} \\ & + \left(-\frac{9}{2}(\ell^{(0)})^2\ell^{(1)}w^{(0)} - 6(\ell^{(0)})^3w^{(1)} + 9(w^{(0)})^2w^{(1)} \right) f^{(2)} + \left(\frac{9}{2}(\ell^{(0)})^3w^{(0)} - \frac{9}{2}(w^{(0)})^3 \right) f^{(3)}. \end{aligned} \quad (3.6)$$

Therefore regarded as an element of R_∞ , $r_1 \neq 0$ but $(r_1)^3 = 0$. $\square$

It is well known that in characteristic zero, for any affine scheme X , the nilradical $\mathcal{N} \subseteq \mathcal{O}(X_\infty)$ is a differential ideal; in other words, $D(\mathcal{N}) \subseteq \mathcal{N}$. A natural question (see [20]) is whether $\mathcal{N}$ is finitely generated as a differential ideal, and whether an explicit generating set can be found. In general, $\mathcal{N}$ need not be finitely generated; this was shown for $X = \mathrm{Spec}(\mathbb{C}[x, y] / \langle xy \rangle)$ in [12]. In the example

$X = \text{Spec}(\mathbb{C}[\ell, w]/\langle w^2 - \ell^3 \rangle)$, a calculation shows that in addition to r_1 ,

$$r_2 = (w^{(1)})^2 - \frac{9}{4}\ell^{(0)}(\ell^{(1)})^2 \quad (3.7)$$

does not lie in $\langle f^{(0)}, f^{(1)}, \dots \rangle$, but $(r_2)^3$ does. So r_2 is another nontrivial element of $\mathcal{N}$. We expect that $\mathcal{N}$ is generated as a differential ideal by r_1 and r_2 .

The following characterization of $\mathcal{N}$ in this example will also be useful to us.

Lemma 3.2 *Let $X = \text{Spec}(\mathbb{C}[\ell, w]/\langle w^2 - \ell^3 \rangle)$ and let t be a coordinate function on $\mathbb{C}$. Consider the map*

$$\mathbb{C} \rightarrow X, \quad t \mapsto (t^2, t^3), \quad (3.8)$$

and the induced homomorphism

$$\varphi : \mathcal{O}(X_\infty) \rightarrow \mathcal{O}(\mathbb{C}_\infty), \quad \ell^{(0)} \mapsto (t^{(0)})^2, \quad w^{(0)} \mapsto (t^{(0)})^3. \quad (3.9)$$

Then $\mathcal{N} = \ker(\varphi)$.

Proof Since (3.8) is birational, the map $\mathbb{C}_\infty \rightarrow X_\infty$ on arc spaces induced by (3.8) is dominant, see Proposition 3.2 of [15]. Therefore $\ker(\varphi) \subseteq \mathcal{N}$. On the other hand, $\mathcal{N} \subseteq \ker(\varphi)$ since $\mathcal{O}(\mathbb{C}_\infty) \cong \mathbb{C}[t^{(0)}, t^{(1)}, \dots]$, which is an integral domain. $\square$

4 Li's Filtration and Singular Support

For any vertex algebra $\mathcal{V}$, we have Li's canonical decreasing filtration

$$F^0(\mathcal{V}) \supseteq F^1(\mathcal{V}) \supseteq \dots,$$

where $F^p(\mathcal{V})$ is spanned by elements of the form

$$: \partial^{n_1} a^1 \partial^{n_2} a^2 \dots \partial^{n_r} a^r :,$$

where $a^1, \dots, a^r \in \mathcal{V}$, $n_i \geq 0$, and $n_1 + \dots + n_r \geq p$ [22]. Clearly $\mathcal{V} = F^0(\mathcal{V})$ and $\partial F^i(\mathcal{V}) \subseteq F^{i+1}(\mathcal{V})$. Set

$$\text{gr}(\mathcal{V}) = \bigoplus_{p \geq 0} F^p(\mathcal{V})/F^{p+1}(\mathcal{V}),$$

and for $p \geq 0$ let

$$\sigma_p : F^p(\mathcal{V}) \rightarrow F^p(\mathcal{V})/F^{p+1}(\mathcal{V}) \subseteq \text{gr}(\mathcal{V})$$

be the projection. Note that $\text{gr}(\mathcal{V})$ is a graded commutative algebra with product

$$\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(a_{(-1)}b),$$

for $a \in F^p(\mathcal{V})$ and $b \in F^q(\mathcal{V})$. We say that the subspace $F^p(\mathcal{V})/F^{p+1}(\mathcal{V})$ has height p . Note that $\text{gr}(\mathcal{V})$ has a differential ∂ defined by

$$\partial(\sigma_p(a)) = \sigma_{p+1}(\partial a),$$

for $a \in F^p(\mathcal{V})$. Finally, $\text{gr}(\mathcal{V})$ has the structure of a Poisson vertex algebra [22]; for $n \geq 0$, we define

$$\sigma_p(a)_{(n)}\sigma_q(b) = \sigma_{p+q-n}a_{(n)}b.$$

Zhu's commutative algebra $R_{\mathcal{V}}$ is isomorphic to the subalgebra $F^0(\mathcal{V})/F^1(\mathcal{V}) \subseteq \text{gr}(\mathcal{V})$, since $F^1(\mathcal{V})$ coincides with the space $C(\mathcal{V})$ defined by (2.5). Moreover, $\text{gr}(\mathcal{V})$ is generated by $R_{\mathcal{V}}$ as a differential graded commutative algebra [22]. Since $\tilde{X}_{\mathcal{V}} = \text{Spec}(R_{\mathcal{V}})$, there is always a surjective homomorphism of differential graded rings

$$\Phi_{\mathcal{V}} : \mathcal{O}((\tilde{X}_{\mathcal{V}})_{\infty}) \rightarrow \text{gr}(\mathcal{V}), \quad (4.1)$$

where the grading on $\mathcal{O}((\tilde{X}_{\mathcal{V}})_{\infty})$ is given by (3.2). Define the *singular support*

$$\text{SS}(\mathcal{V}) = \text{Spec}(\text{gr}(\mathcal{V})), \quad (4.2)$$

which is then a subscheme of $(\tilde{X}_{\mathcal{V}})_{\infty}$. A natural question which was raised by Arakawa and Moreau [7] is whether the map (4.1) is always an isomorphism. This is true in many examples and it was recently shown in [8] to hold as varieties when $\mathcal{V}$ is quasi-lisse, that is, if $X_{\mathcal{V}}$ has finitely many symplectic leaves, see [6] for the details. We note that the vertex algebra $\mathcal{W}$ is not quasi-lisse.

5 Main Result

Theorem 5.1 *For the vertex algebra $\mathcal{W}$, the map $\Phi_{\mathcal{W}} : \mathcal{O}((\tilde{X}_{\mathcal{W}})_{\infty}) \rightarrow \text{gr}(\mathcal{W})$ is not injective, so $(\tilde{X}_{\mathcal{W}})_{\infty}$ and $\text{SS}(\mathcal{W})$ are not isomorphic as schemes.*

Proof As before, we use the notation

$$\mathcal{O}((\tilde{X}_{\mathcal{W}})_{\infty}) \cong R_{\infty} = \mathbb{C}[\ell^{(0)}, \ell^{(1)}, \dots, w^{(0)}, w^{(1)}, \dots] / \langle f^{(0)}, f^{(1)}, \dots \rangle.$$

We use the same notation $\partial^i L$, $\partial^i W$ to denote the images of the fields $\partial^i L$, $\partial^i W \in \mathcal{W}$ in the subspace $F^i(\mathcal{W})/F^{i+1}(\mathcal{W})$ of $\text{gr}(\mathcal{W})$. We therefore may identify $\text{gr}(\mathcal{W})$ with a quotient of the polynomial ring $\mathbb{C}[L, \partial L, \dots, W, \partial W, \dots]$. In this notation, $\Phi_{\mathcal{W}}(\ell^{(0)}) = L$ and $\Phi_{\mathcal{W}}(w^{(0)}) = W$.

We will show that the nilpotent elements r_1 and r_2 in $\mathcal{O}((\tilde{X}_{\mathcal{W}})_{\infty})$ given by (3.5) and (3.7) lie in $\ker(\Phi_{\mathcal{W}})$. By Lemma 2.1 of [27], we have the following relation in $\mathcal{W}$ at weight 6:

$$3 : (\partial L)W : -2 : L(\partial W) : + \frac{1}{4} \partial^3 W = 0.$$

Therefore in $F^1(\mathcal{W})/F^2(\mathcal{W})$, we have the relation

$$3(\partial L)W - 2L\partial W = 0.$$

Since $\Phi_{\mathcal{W}}(r_1) = 3(\partial L)W - 2L\partial W$, $r_1 \in \ker(\Phi_{\mathcal{W}})$.

Similarly, in $\mathcal{W}$ we have the following relation in weight 8:

$$: (\partial W)^2 : - \frac{9}{4} : (\partial L)^2 L : - \frac{3}{16} : (\partial^4 L)L : - \frac{3}{8} : (\partial^3 L)(\partial L) : - \frac{9}{32} : (\partial^2 L)^2 : + \frac{1}{160} \partial^6 L = 0,$$

so in $F^2(\mathcal{W})/F^3(\mathcal{W})$ we have the relation $(\partial W)^2 - \frac{9}{4}(\partial L)^2 L = 0$, and $r_2 \in \ker(\Phi_{\mathcal{W}})$. $\square$

Theorem 5.2 *Even though $(\tilde{X}_{\mathcal{W}})_{\infty}$ and $SS(\mathcal{W})$ differ as schemes, the map of varieties*

$$SS(\mathcal{W})_{red} \rightarrow ((\tilde{X}_{\mathcal{W}})_{\infty})_{red}$$

induced by $\Phi_{\mathcal{W}}$ is an isomorphism.

Proof It suffices to show that the map $\varphi : \mathcal{O}((\tilde{X}_{\mathcal{W}})_{\infty}) \rightarrow \mathcal{O}(\mathbb{C}_{\infty})$ given by (3.9) factors through the map $\Phi_{\mathcal{W}} : \mathcal{O}((\tilde{X}_{\mathcal{W}})_{\infty}) \rightarrow \text{gr}(\mathcal{W})$, since $\ker(\varphi) = \mathcal{N}$. First, the embedding $i : \mathcal{W} \rightarrow \mathcal{S}$ given by (2.4) induces a map

$$\text{gr}(i) : \text{gr}(\mathcal{W}) \rightarrow \text{gr}(\mathcal{S}).$$

Identifying $\text{gr}(\mathcal{S})$ with $\mathbb{C}[\beta, \partial\beta, \dots, \gamma, \partial\gamma, \dots]$, this map is given on generators by

$$\text{gr}(i)(L) = (\beta\gamma)^2, \quad \text{gr}(i)(W) = (\beta\gamma)^3.$$

We also have an injective map of differential graded algebras

$$\psi : \mathcal{O}(\mathbb{C}_{\infty}) \rightarrow \text{gr}(\mathcal{S}),$$

defined on the generator $t^{(0)}$ of $\mathcal{O}(\mathbb{C}_{\infty})$ by $\psi(t^{(0)}) = \beta\gamma$. Since

$$\Phi_{\mathcal{W}}(\ell^{(0)}) = (\beta\gamma)^2 = \text{gr}(i)(L) = \psi((t^{(0)})^2), \quad \Phi_{\mathcal{W}}(w^{(0)}) = (\beta\gamma)^3 = \text{gr}(i)(W) = \psi((t^{(0)})^3),$$

and L, W generate $\text{gr}(\mathcal{W})$ as a differential algebra, it is clear that

$$\text{gr}(i)(\text{gr}(\mathcal{W})) = \psi(A) \cong A = \varphi(\mathcal{O}((\tilde{X}_{\mathcal{W}})_{\infty})),$$

where $A \subseteq \mathcal{O}(\mathbb{C}_{\infty})$ is the subalgebra generated by $(t^{(0)})^2, (t^{(0)})^3$, and their derivatives. This completes the proof. $\square$

In this example, we expect that $\text{gr}(i) : \text{gr}(\mathcal{W}) \rightarrow \text{gr}(\mathcal{S})$ is injective, so that $\text{gr}(\mathcal{W}) \cong A$, and in particular is reduced. However, we caution the reader that the associated graded functor is not left exact in general.

6 Failure of Associated Graded Functor to be Left Exact

Here we give an example of a simple vertex algebra $\mathcal{V}$ which has a free field realization $i : \mathcal{V} \rightarrow \mathcal{H}$ where $\mathcal{H}$ is the Heisenberg algebra, such that the induced map $\text{gr}(i) : \text{gr}(\mathcal{V}) \rightarrow \text{gr}(\mathcal{H})$ is not injective.

First, $\mathcal{H}$ is generated by an even field α satisfying

$$\alpha(z)\alpha(w) \sim (z - w)^{-2},$$

and has Virasoro element $L = \frac{1}{2} : \alpha\alpha :$ of central charge $c = 1$. There is an action of $\mathbb{Z}_2$ sending $\alpha \mapsto -\alpha$ which preserves L , and we consider the orbifold

$$\mathcal{V} = \mathcal{H}^{\mathbb{Z}_2}.$$

By a result of Dong and Nagatomo [14], $\mathcal{V}$ is strongly generated by L together with a unique up to scalar weight 4 field primary field

$$W^4 = -\frac{1}{6\sqrt{6}} : \alpha^4 : - \frac{1}{4\sqrt{6}} : (\partial\alpha)^2 : + \frac{1}{6\sqrt{6}} : (\partial^2\alpha)\alpha :,$$

which is normalized so that it satisfies

$$W^4(z)W^4(w) \sim \frac{1}{4}(z - w)^{-8} + \dots$$

One can check by direct calculation that $\mathcal{V}$ is isomorphic to the simple, principal $\mathcal{W}$ -algebra of $\mathfrak{sp}_4$ with central charge $c = 1$. It is convenient to replace W^4 with the field

$$W = \frac{35}{132} : (\partial^2\alpha)\alpha : = \frac{35\sqrt{2/3}}{33} W^4 + \frac{70}{297} : L^2 : + \frac{35}{396} \partial^2 L,$$

which is not primary. A calculation shows that we have the following nontrivial relations in $\mathcal{V}$ at weights 8 and 10, respectively.

$$\begin{aligned} & :W^2 : - :L^2W : + \frac{35}{132} :(\partial^2L)L^2 : - \frac{35}{264} :(\partial L)^2L : + \frac{13265}{69696} :(\partial^4L)L : \\ & + \frac{19495}{139392} :(\partial^3L)\partial L : - \frac{59}{88} :(\partial^2L)W : - \frac{497}{352} :(\partial L)\partial W : - \frac{181}{528} :L\partial^2W : \\ & - \frac{139}{2112}\partial^4W + \frac{10955}{557568}\partial^6L = 0, \end{aligned} \quad (6.1)$$

$$\begin{aligned} & :L^3W : + \frac{4455}{1024} :(\partial W)\partial W : - \frac{35}{132} :(\partial^2L)L^3 : + \frac{35}{264} :(\partial L)^2L^2 : + \frac{347}{256} :(\partial L)^2W : - \\ & \frac{1069}{256} :(\partial L)L\partial W : - \frac{49}{16} :L^2\partial^2W : + \frac{385}{576} :(\partial^4L)L^2 : + \frac{48965}{101376} :(\partial^3L)(\partial L)L : - \\ & \frac{35}{44} :(\partial^2L)^2L : + \frac{35}{88} :(\partial^2L)(\partial L)^2 : - \frac{1687}{1536} :(\partial^4L)W : - \frac{5939}{3072} :(\partial^3L)(\partial W) : - \\ & \frac{247}{256} :(\partial^2L)(\partial^2W) : - \frac{10927}{6144} :(\partial L)(\partial^3W) : + \frac{779}{1536} :L\partial^4W : + \frac{3899}{36864} :(\partial^6L)L : + \\ & \frac{102851}{270336} :(\partial^5L)\partial L : + \frac{7525}{67584} :(\partial^4L)\partial^2L : + \frac{659645}{4866048} :(\partial^3L)^2 : \\ & + \frac{68311}{6488064}\partial^8L - \frac{3187}{49152}\partial^6W = 0. \end{aligned} \quad (6.2)$$

Lemma 6.1 *Let ℓ, w denote the images of L, W in $R_{\mathcal{V}}$. Then*

$$R_{\mathcal{V}} \cong \mathbb{C}[\ell, w]/I$$

where I is the ideal generated by $w(w - \ell^2)$ and ℓ^3w . In particular, $\tilde{X}_{\mathcal{V}} = \text{Spec}(R_{\mathcal{V}})$ is irreducible of dimension one, but is not reduced.

Proof Since $\mathcal{V}$ is strongly generated by L, W , it follows from (6.1) and (6.2) that $R_{\mathcal{V}} \cong \mathbb{C}[\ell, w]/I$ for some ideal I which contains $w(w - \ell^2)$ and ℓ^3w . The proof that I is generated by these two elements is similar to the proof of Lemma 2.1, and is omitted. Since I is contained in the ideal $\langle w \rangle$, the map $\mathbb{C}[\ell] \rightarrow R_{\mathcal{V}}$ is injective, and $R_{\mathcal{V}}$ has Krull dimension 1. Since w is a nontrivial nilpotent element of $R_{\mathcal{V}}$, $\tilde{X}_{\mathcal{V}}$ is not reduced. Finally, it is easy to see that the nilradical $\mathcal{N}$ of $R_{\mathcal{V}}$ is generated by w , so $\mathcal{N}$ is prime and $\tilde{X}_{\mathcal{V}}$ is irreducible. $\square$

Corollary 6.2 *Let $i : \mathcal{V} \rightarrow \mathcal{H}$ be the inclusion. Since $\text{gr}(\mathcal{H})$ is the polynomial ring $\mathbb{C}[\alpha, \partial\alpha, \dots]$, the induced map $\text{gr}(i) : \text{gr}(\mathcal{V}) \rightarrow \text{gr}(\mathcal{H})$ is not injective.*

In fact, it is easy to verify that the image of $\text{gr}(i)$ is just the differential polynomial algebra generated by $\text{gr}(i)(L) = \frac{1}{2}\alpha^2$. Finally, we remark that as in our main