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Vladimir A. Garanzha
Lennard Kamenski · Hang Si *Editors*

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Numerical Geometry, Grid Generation and Scientific Computing

Proceedings of the 9th International Conference, NUMGRID 2018 / Voronoi 150, Celebrating the 150th Anniversary of G.F. Voronoi, Moscow, Russia, December 2018

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Dedicated to
Georgy Feodosevich Voronoi (1868–1908)
on the Occasion
of His 150th Birthday

Foreword

It is with great pleasure that I write a few lines to introduce the proceedings of the NUMGRID Voronoi 2018 conference. This year is the 150th anniversary of the birth of Georgy Voronoi. After a century and a half, his work is still the base of many researches and has received a renovated attention in particular in the field of mesh generation. Voronoi graph and its dual, the Delaunay tessellation, are the base for the most popular and effective tetrahedral and polyhedral mesh generators.

In addition, recently a number of researchers have also attempted a direct usage of the Voronoi graph ideas to construct a polyhedral mesh, avoiding the usage of the Delaunay tessellation. Mesh optimization is also greatly affected by concepts that have their foundation in Voronoi's work.

Most academic and industrial simulation packages offer, one way or another, a mesh generator that exploits the power of Voronoi graph. This should make us consider how the advances in mathematics, achieved in an "era" when computers were not yet available, continue to affect our life with unexpected fruits.

Computational Fluid Dynamics and Stress Analysis, the two main areas where simulation and therefore mesh generation play a strong role, are responsible for reduced cost and time-to-market of many objects that we commonly use in our day-to-day life. Cars, airplanes, engines, turbomachines, ships, roads, bridges, buildings, etc. can now be designed and tested at a pace that was unreachable only a few decades ago.

It is worth noting that Voronoi graph has been used in other research and application areas such as medicine, chemistry, biology, logistics, and operations research as a demonstration that a seminal idea can in time generate very fruitful consequences. Therefore I believe that the NUMGRID VORONOI conferences are a positive effort in the direction of increasing the knowledge and the growth of Voronoi's work that can still stimulate innovative solutions, balancing the algorithmic advances and their practical applications.

Rome, Italy — Stefano Paoletti
April 2019

Preface

This volume presents a selection of papers presented at the 9th International Conference on Numerical Geometry, Grid Generation, and Scientific Computing celebrating the 150th anniversary of Georgy F. Voronoi (NUMGRID 2018/Voronoi 150), held on December 3–5, 2019, at the Dorodnicyn Computing Center of the Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences in Moscow, Russia. The conference is biannual (since 2002) and it is one of the well-known international conferences in the area of mesh generation. The main topic of this conference, grid (mesh) generation, is about how to create a geometric discretization of a given domain. It is an indispensable tool for solving field problems in nearly all areas of applied mathematics. The background of grid generation is highly interdisciplinary and involves mathematics, computer science, and engineering.

The objective of this book is to provide a good balance between engineering algorithms and mathematical foundations. The book includes an overview of the current progress in numerical geometry, grid generation and adaptation in terms of mathematical foundations, algorithm and software development, and applications. In focus are the Voronoi-Delaunay theory and algorithms for tilings and partitions, mesh deformation and optimization, equidistribution principle, error analysis, discrete differential geometry, duality in mathematical programming and numerical geometry, mesh-based optimization and optimal control methods, iterative solvers for variational problems, as well as algorithm and software development. The applications of the discussed methods are multidisciplinary and include problems from mathematics, physics, biology, chemistry, material science, and engineering. The presented 22 papers were selected from 38 submissions. The main section criteria are based on the recommendations of anonymous peer reviews from experts of the corresponding fields. All accepted papers are revised according to the comments of reviewers and the program committee.

The organizers would like to thank all who submitted papers and all who helped to evaluate the contributions by providing reviews for the submissions. The reviewers' names are acknowledged in the following pages. The organizers would like to thank all participants of NUMGRID for making it a successful and interesting experience.

Moscow, Russia — Vladimir A. Garanzha
Berlin, Germany — Lennard Kamenski
Berlin, Germany — Hang Si
April 2019

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Part I
Voronoi Meshing: Theory, Algorithms, and Applications

Secondary Power Diagram, Dual of Secondary Polytope

Na Lei, Wei Chen, Zhongxuan Luo, Hang Si, and Xianfeng Gu

Abstract An ingenious construction of Gel'fand et al. (Discriminants, Resultants, and Multidimensional Determinants. Birkhäuser, Basel, 1994) geometrizes the triangulations of a point configuration, such that all coherent triangulations form a convex polytope, the so-called secondary polytope. The secondary polytope can be treated as a weighted Delaunay triangulation in the space of all possible coherent triangulations. Naturally, it should have a dual diagram. In this work, we explicitly construct the secondary power diagram, which is the power diagram of the space of all possible power diagrams with non-empty boundary cells. Secondary power diagram gives an alternative proof for the classical secondary polytope theorem based on Alexandrov theorem. Furthermore, secondary power diagram theory shows one can transform a non-degenerated coherent triangulation to another non-degenerated coherent triangulation by a sequence of bistellar modifications, such that all the intermediate triangulations are non-degenerated and coherent.

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1 Introduction

An ingenious construction of Gel'fand et al. [1–3] geometrizes the triangulations of a point configuration, such that all coherent triangulations form a convex polytope, the so-called secondary polytope.

The secondary polytope can be treated as a weighted Delaunay triangulation in the space of all possible triangulations. Naturally, it should have a dual diagram. In this work, we explicitly construct the secondary power diagram, which is the power diagram of the space of all possible power diagrams.

A more precise description is as follows: fix a point configuration, the secondary polytope is the "triangulation" of the space of all coherent triangulations, the secondary power diagram is the "power diagram" of the space of all power diagrams with non-empty boundary cells.

Secondary power diagram gives an alternative proof for the classical secondary polytope theorem based on Alexandrov theorem. Furthermore, secondary power diagram theory shows one can transform a non-degenerated coherent triangulation to another non-degenerated coherent triangulation by a sequence of bistellar modifications, such that all the intermediate triangulations are non-degenerated and coherent.

1.1 Basic Terminologies

1.1.1 Secondary Polytope

A *point configuration* is a finite set of distinct points $Y = \{y_1, y_2, \ldots, y_k\}$ in the n-dimensional Euclidean space $\mathbb{R}^n$. The *convex hull* of Y is denoted as $Conv(Y)$. We call y_i a *boundary point* if y_i is on the convex hull $Conv(Y)$, and *interior point* if y_i is in the interior volume bounded by $Conv(Y)$.

Let T be a triangulation of the interior volume bounded by $Conv(Y)$ with vertices in Y, denoted as a triangulation of $(Y, Conv(Y))$. All boundary points are vertices of T, but some interior points may be not. The *characteristic vector* of T is $\psi_T = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\lambda_i = \sum_{y_i \sim \sigma} vol(\sigma)$, where σ is a simplex in T. The convex hull of all the characteristic vectors of all possible triangulations of $(Y, Conv(Y))$ is denoted as $\Sigma(Y)$, and called *secondary polytope* of Y. All the triangulations corresponding to the characteristic vectors on the secondary polytope are called *coherent triangulation*. If a coherent triangulation misses some points in Y, then it is called *degenerated*.

Let Δ be a simplicial complex, $F \in \Delta$ be a face and $G \subseteq F$. If F is a finite set, we write $\bar{F} := \{G \subseteq F\}$ to denote the simplex on F and $\partial \bar{F} := \{G \subsetneq F\}$ to denote the boundary complex of the simplex on F. The *star* of F in Δ and the *link* of F in Δ both describe the local structure of Δ around F:

$$st_\Delta(F) := \{G \in \Delta | F \cup G \in \Delta\}, \quad lk_\Delta(F) := \{G \in st_\Delta(F) | F \cap G = \emptyset\}.$$

If Γ and Δ are simplicial complexes on disjoint vertex sets, their *join* is the simplicial complex:

$$\Gamma * \Delta := \{F \cup G | F \in \Delta, G \in \Gamma\}.$$

Note that $st_{\Delta}(F) = \bar{F} * lk_{\Delta}(F)$, and the *deletion* of F from Δ is defined as:

$$\Delta \backslash F = \{G \in \Delta | F \nsubseteq G\}.$$

Let Δ be a simplicial complex, and assume that $A \in \Delta$, $B \notin \Delta$ and $lk_{\Delta}(A) = \partial \bar{B}$, so that $st_{\Delta}(A) = \bar{A} * \partial \bar{B}$. Then the process of removing $\bar{A} * \partial \bar{B}$ and replacing it with $\partial \bar{A} * \bar{B}$ is called a *bistellar transformation* or a *bistellar flip* [4]:

$$\Delta \rightarrow \Delta \backslash (\bar{A} * \partial \bar{B}) \cup (\partial \bar{A} * \bar{B}).$$

Roughly speaking, a bistellar transformation is a local topological operation that modifies the configuration of a small set of adjacent d-simplices. For instance, in three dimensions, a bistellar transformation can modify two tetrahedra to three, three to two, one to four, or four to one, as shown in Fig. 1.

Furthermore, each edge on the secondary polytope $\Sigma(Y)$ represents a bistellar transformation from one coherent triangulation to another one. All the coherent triangulations can be transformed by bistellar transformations by traversing the edges of $\Sigma(Y)$ [5, 6].

1.1.2 Primary Power Diagram

Given the *powers* $R = \{r_1, r_2, \ldots, r_k\}$, $r_i \in \mathbb{R}$, the *power distance* is defined as $pow(x, y_i) = |x - y_i|^2 - r_i^2$. The *nearest power diagram* $D(R)$ is a cell

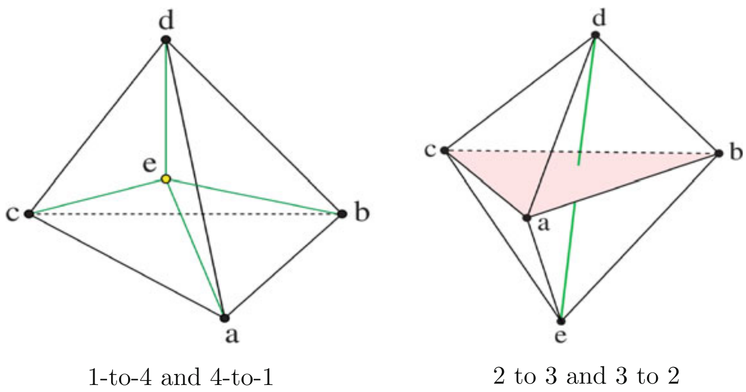


Fig. 1 The four types of bistellar transformation in three dimensions

decomposition of $\mathbb{R}^n$ $D(R) = \bigcup_{i=1}^{k} W_i(R)$, where each nearest power cell is defined as

$$W_i(R) = \left\{x \in \mathbb{R}^n \mid pow(x, y_i) \leq pow(x, y_j), 1 \leq j \leq k\right\}.$$

We call the infinite cells as the *boundary cells*. The infinite cells correspond to the boundary points in Y. Some $W_i(R)$'s might be empty. Let $\Omega \subset \mathbb{R}^n$ be a convex compact domain, the volume of the intersection between $W_i(R)$ and Ω is denoted as

$$w_i(R) := \mathrm{vol}(W_i(R) \cap \Omega),$$

it is obvious that $\sum_{i=1}^{k} w_i(R) = \mathrm{vol}(\Omega)$. Some $w_i(R)$'s might be 0's.

The *nearest weighted Delaunay triangulation* $T(R)$ of $(Y, Conv(Y))$ is the Poincaré dual of the nearest power diagram: each simplex in the weighted triangulation with vertices $\{y_{i_1}, y_{i_2}, \ldots, y_{i_m}\}$ corresponds to the intersection of cells in the power diagram $W_{i_1} \cap W_{i_2} \cdots \cap W_{i_m}$. Figure 2 shows the duality between a 2-d power diagram and the weighted triangulation, the power is illustrated as red circles with radii r_i's.

Power diagrams and weighted Delaunay triangulations are closely related to convex polytopes. For each point $y_i \in Y$, we construct a plane

$$\pi_i(x) = \langle x, y_i \rangle - h_i, \quad h_i = \frac{1}{2}|y_i|^2 - r_i^2.$$

The heights of all the planes are represented as a *height vector* $h = (h_1, h_2, \ldots, h_k)$. The *upper envelope* of the planes is denoted as $Env(\{\pi_i\})$, or simply $Env(h)$, which is the graph of the function

$$u_h(x) := \max_{i=1}^{k}\{\pi_i(x)\}.$$

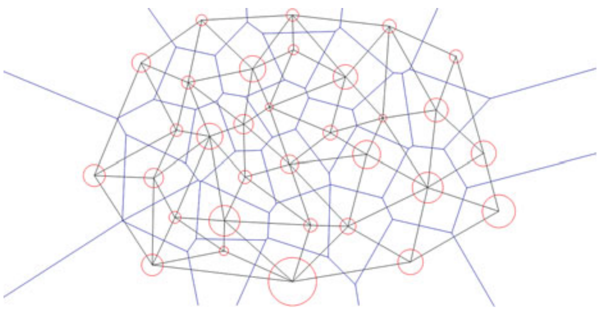


Fig. 2 Power diagram (blue) and its dual weighted Delaunay triangulation (black), the power weights equal to the square of radii (red circles)

Each plane π_i has a dual point $\pi_i^* = (y_i, h_i)$, the convex hull is denoted as $Conv(\{\pi_i^*\})$, or simply $Conv(h)$. For the upper envelope $Env(h)$, the *Legendre dual* of $u_h(x)$ is

$$u_h^*(y) = \sup_{x \in \Omega}\{\langle x, y\rangle - u_h(x)\}.$$

The graph of $u_h^*(y)$ is the lower part of the convex hull. Figure 3 shows the Legendre dual relation between the u_h and u_h^*, namely the upper envelope $Env(h)$ and the lower part of the convex hull $Conv(h)$.

Because $pow(x, y_i) \leq pow(x, y_j)$ is equivalent to $\pi_i(x) \geq \pi_j(x)$, the projection of $Env(h)$ is exactly the same power diagram $D(R)$, hence we also denote the power diagram as $D(h)$. The projection of the lower part of the convex hull $Conv(h)$ is the weighted Delaunay triangulation $T(R)$, also denoted as $T(h)$. $D(h)$ and $T(h)$ are Poincaré dual to each other. Figure 3 shows the relations among upper envelope $Env(h)$, the lower convex hull $Conv(h)$, the nearest power diagram $D(h)$, and the weighted Delaunay triangulation $T(h)$.

Similarly, we can define *furthest power diagram*, where each cell is defined as

$$W_i(R) = \left\{x \in \mathbb{R}^n \mid pow(x, y_i) \geq pow(x, y_j), 1 \leq j \leq k\right\}.$$

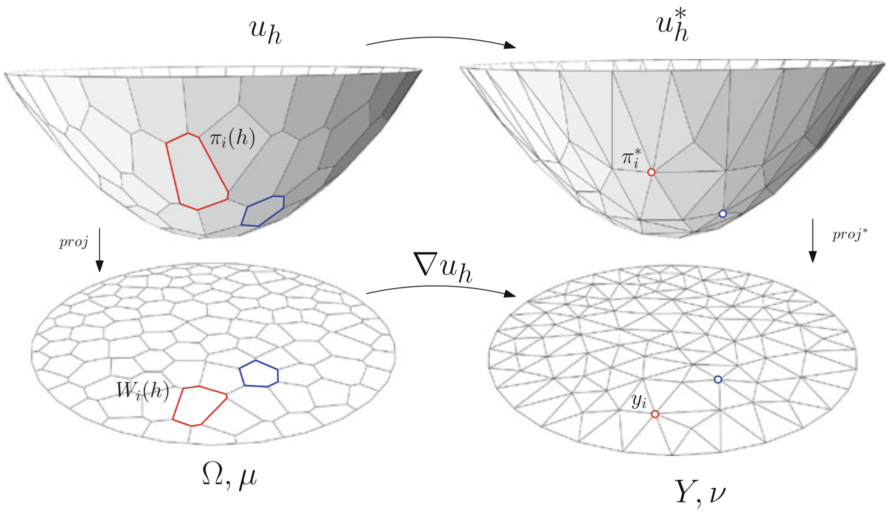
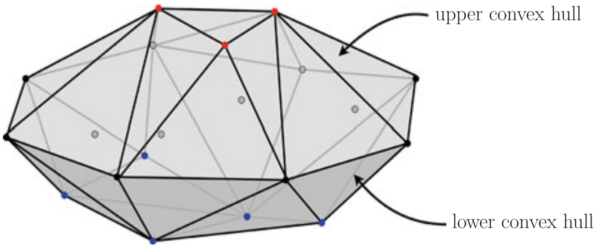


Fig. 3 The upper envelope u_h, Legendre dual to the lower convex hull u_h^*. The upper envelope projects to the power diagram $D(h)$, the lower convex hull projects to the weighted Delaunay triangulation $T(h)$

Fig. 4 The lower and upper parts of a convex hull



It is the projection of the *lower envelope*,

$$u_h(x) := \min_{i=1}^{k}\{\pi_i(x)\},$$

which is dual to the upper part of the convex hull $Conv(h)$. The lower and upper convex hull (see Fig. 4) share the boundary points in Y, the nearest and furthest power diagrams share the infinite cells. The projection of the lower part of the convex hull is the weighted Delaunay triangulation, in which all the vertices are the boundary points in Y.

In the following discussion, by power diagram we refer to the nearest power diagram, unless stated explicitly otherwise. All the theories holds for both nearest and furthest power diagrams.

1.1.3 Alexandrov Power Diagram Space

Alexandrov Theorem 13 shows that, given a set of positive numbers $\nu = \{\nu_1, \nu_2, \ldots, \nu_k\}$, satisfying $\sum_{i=1}^{k} \nu_i = vol(\Omega)$, there exists a power diagram $D(h)$ with the height vector h, such that $w_i(h) = \nu_i, \forall i$. Furthermore, such h is unique upto adding a constant $(c, c, \ldots, c)$.

Definition 1 (Alexandrov Power Diagram) Fixing a convex domain $\Omega \subset \mathbb{R}^n$ and a small positive number $\varepsilon > 0$, a power diagram $D(h)$ is called an Alexandrov power diagram with respect to (Ω, ε) if its cell volumes satisfy the following conditions:

1. For each boundary point $y_i \in Y$, the boundary cell volume $w_i(h) > \epsilon$;
2. for each interior point $y_j \in Y$, the interior cell volume $w_j(h) > 0$;

If the equality holds, then the diagram is called a *generalized Alexandrov power diagram*.

Definition 2 (Alexandrov Power Diagram Space) The space of all Alexandrov power diagrams with respect to (Ω, ε) parameterized by the height vectors is called the *Alexandrov power diagram space* of Y with respect to (Ω, ε), and denoted as

$$\mathscr{H}_{\Omega}^{\varepsilon}(Y) := \{h|D(h)\ is\ an\ Alexandrov\ power\ diagram\ w.r.t.\ (\Omega, \varepsilon)\}. \tag{1}$$

We will show that the $\mathscr{H}_{\Omega}^{\varepsilon}(Y)$ is a $k-1$ dimensional convex domain in $\mathbb{R}^k$. Its closure $\overline{\mathscr{H}}^{\varepsilon}_{\Omega}(Y)$ is called the *generalized Alexandrov power diagram space* of Y with respect to (Ω, ε). Note that any height vector h in $\mathscr{H}_{\Omega}^{\varepsilon}$ generates a non-degenerated coherent triangulation $T(h)$ via nearest power diagram; some height vectors on the boundary of $\overline{\mathscr{H}}^{\varepsilon}_{\Omega}$ generate degenerated coherent triangulations via either nearest or furthest power diagrams.

1.2 Main Result: Secondary Power Diagram

The secondary polytope $\Sigma(Y)$ is the polytope of all (coherent) triangulations of $(Y, Conv(Y))$. Similarly, in this work, we construct the *secondary power diagram* $\Pi(Y)$, which is the power diagram of the space of all (generalized) Alexandrov power diagrams $\overline{\mathscr{H}}^{\varepsilon}_{\Omega}(Y)$. Then $\Sigma(Y)$ and $\Pi(Y)$ are Poincaré dual to each other. Similar to the primary power diagram, we will construct the upper envelope $Env(Y)$ corresponding to the power diagram $\Pi(Y)$, and whose Legendre dual convex hull coincides with $\Sigma(Y)$.

Definition 3 (Secondary Power Diagram) The secondary power diagram is defined as

$$\Pi(Y) : \overline{\mathscr{H}}^{\varepsilon}_{\Omega}(Y) = \bigcup_{T\in\Sigma(Y)} D_T(Y),\ D_T(Y) := \left\{h \in \overline{\mathscr{H}}^{\varepsilon}_{\Omega}(Y)|T(h) = T\right\}, \tag{2}$$

where each cell $D_T(Y)$ is given by all the generalized Alexandrov power diagrams, whose dual weighted Delaunay triangulation $T(h)$ are T.

We show that all the cells $D_T(Y)$ are non-empty convex cones.

The principles of the secondary power diagram $\Pi(Y)$ are exactly the same as those of the primary power diagram. $\Pi(Y)$ is induced by the upper envelope of a set of hyper-planes $\pi_T(h)$, $Env(\{\pi_T(h), T \in \Sigma(Y)\})$, which is the graph of the piecewise linear function

$$U(Y) := \max_{T\in\Sigma(Y)} \{\pi_T(h)\},$$

as well as the lower envelope, which is the graph of the function $\min_{T\in\Sigma(Y)} \{\pi_T(h)\}$.

The hyper-plane $\pi_T(h)$ has explicit geometric meaning. As shown in Fig. 5, let $T \in \Sigma(Y)$ be a triangulation, choose one simplex $\sigma \in T$ with vertices $\{y_{i_0}, y_{i_1}, \ldots, y_{i_n}\}$, construct a simplex $\tilde{\sigma}$ with points $\tilde{y}_{i_l} = (y_{i_l}, h_{i_l}) \in \mathbb{R}^{n+1}$. The

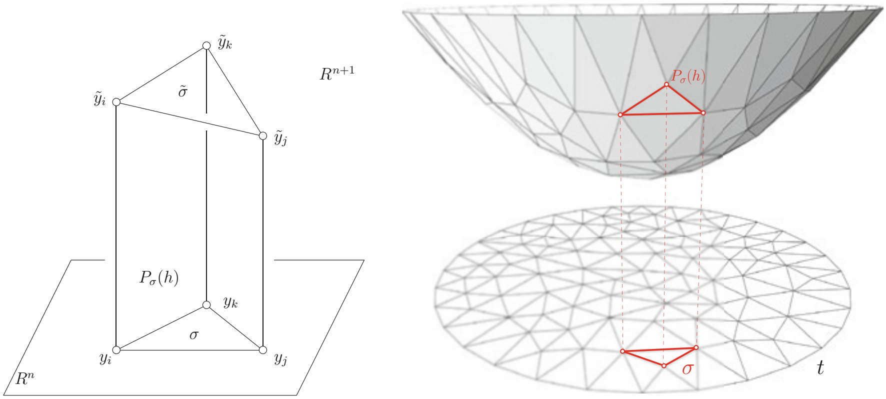


Fig. 5 The prism $P_\sigma(h)$ constructed from one simplex σ in the triangulation $t \in \mathscr{T}(Y)$. The volume $\pi_t(h)$ of the union of the prisms

simplices σ and $\tilde{\sigma}$ bound a prism $P_\sigma(h)$. The summation of the volumes of all such prisms is

$$\pi_T(h) = \sum_{\sigma \in T} \text{vol } P_\sigma(h) = \frac{1}{n+1} \langle \psi_T, h \rangle, \tag{3}$$

where ψ_T is the characteristic vector of T. Hence the volume function $\pi_T(h)$ is a linear function of h, which is the supporting plane of the upper envelope $Env(Y)$.

If all the heights of the supporting planes $\pi_T(h)$ of $Env(Y)$ are 0's, therefore the dual of each hyperplane $\pi_T(h)$ is $(\psi_T, 0)$. Then the graph of the Legendre dual $U^*(Y)$ coincides with the secondary polytope $\Sigma(Y)$ itself.

Suppose Ω contains the origin of $\mathbb{R}^k$, $\lambda\Omega$ $(\lambda > 0)$ represents the scaling of Ω by factor λ. Then $\mathscr{H}_{\lambda\Omega}^{\lambda^n \varepsilon} = \lambda \mathscr{H}_{\Omega}^{\varepsilon}$, and their secondary power diagram are exactly the same. When λ goes to infinity, $\mathscr{H}_{\lambda^n \Omega}^{\varepsilon}$ covers the whole space $\mathbb{R}^k$, each power cell is a cone. $\Pi(Y)$ gives the complete fan structure.

Furthermore, our secondary power diagram theorem shows that one can transform one non-degenerated coherent triangulation to another non-degenerated coherent triangulation by bistellar transformations as defined in Theorem 11. All the intermediate triangulates are non-degenerated coherent as well.

1.3 Contributions

The main contributions of the current work are as follows:

- This work proposes the concept of secondary power diagram $\Pi(Y)$, and gives an explicit geometric construction: $\Pi(Y)$ is the power diagram of all possible power diagrams of a fixed point configuration.
- The secondary power diagram theory can reproduce secondary polytope theory based on Alexandrov theorem.
- The secondary power diagram theory shows all the non-degenerated coherent triangulations are connected by bistellar transformations.

The work is organized as follows: Sect. 2 reviews the theory of secondary polytope; Sect. 3 explains Alexandrov theorem and a variational approach for constructing Alexandrov polytope; the main theorems of secondary power diagram are proven in Sect. 4; the work is concluded in Sect. 6. In appendix, detailed proofs are given, as well as a symbol list for the major concepts in this work.

2 Secondary Polytope

In this section, we briefly recall the basic concepts and theorems of Gel'fand's second polytope theory, details can be found in [3, 7].

Let Y be a *point configuration*, a finite set of distinct points in $\mathbb{R}^n$, $Conv(Y)$ is the convex hull of Y. A triangulation T of $(Y, Conv(Y))$ decomposes the interior volume bounded by $Conv(Y)$ into simplices with vertices in Y. Some $y_i \in Y$ may not appear as a vertex of a simplex.

A *circuit* Z is obtained by adding one point to the set of vertices of a simplex. There is a unique affine relation among the elements of a circuit, up to a real multiple:

$$\sum_{\omega \in Z} c_\omega \cdot \omega = 0, \sum c_\omega = 0.$$

Let $Z_+ := \{\omega \in Z | c_\omega > 0\}$ and $Z_- := \{\omega \in Z | c_\omega < 0\}$. The convex hull of Z, $Conv(Z)$, has exactly two triangulations,

$$T_+ = \bigcup_{\omega \in Z_+} Conv(Z - \{\omega\}); \quad T_- = \bigcup_{\omega \in Z_-} Conv(Z - \{\omega\}).$$

Given a triangulation T, a *piecewise linear* function $g : Conv(Y) \to \mathbb{R}$ is affine-linear on every simplex of T. Furthermore, g is *concave*, if for any $x, y \in \Omega$ $g(tx + (1-t)y) \geq tg(x) + (1-t)g(y)$.

Definition 4 (Coherent Triangulation) A triangulation T of $(Y, Conv(Y))$ is called *coherent* if there exists a concave piecewise linear function whose domains of linearity are precisely (maximal) simplices of T.

Denote by $\mathbb{R}^Y$ the space of all functions $Y \to \mathbb{R}$. Given a function $\psi \in \mathbb{R}^Y$, we can linearly extend it to a piecewise linear function $g_{\psi,T} : Conv(Y) \to \mathbb{R}$ by linearly interpolating ψ inside each simplex.

Definition 5 (Cone) Let T be a triangulation of $(Y, Conv(Y))$. We shall denote by $C(T)$ the cone in $\mathbb{R}^Y$ consisting of functions $\psi : Y \to \mathbb{R}$ with the following two properties:

- The function $g_{\psi,T} : Conv(Y) \to \mathbb{R}$ is concave.
- For any $\omega \in Y$ which is not a vertex of any simplex from T, we have $g_{\psi,T}(\omega) \geq \psi(\omega)$.

A triangulation T is coherent if and only if the interior of $C(T)$ is non-empty. Moreover, ψ lies in the interior of $C(T)$ if and only if T can be obtained from the projection of $Conv(\{(\omega, \psi(\omega)), \omega \in Y\})$.

Definition 6 (Fan) A fan in $\mathbb{R}^k$ is a finite collection $\mathscr{F}$ of convex polyhedral cones, such that

- Every face of every cone from $\mathscr{F}$ belongs to $\mathscr{F}$
- The intersection of any two cones from $\mathscr{F}$ is a face of both of them.

If the cones from $\mathscr{F}$ cover the whole space, then the fan $\mathscr{F}$ is called complete.

Lemma 7 *Let Y and $Conv(Y)$ be fixed. The cones $C(T)$ for all the coherent triangulations of $(Y, Conv(Y))$ together with all faces of these cones form a complete fan in $\mathbb{R}^Y$, which is called the secondary fan of Y.*

Let T be a triangulation of $(Y, Conv(Y))$. The *characteristic function* of T, $\varphi_T : Y \to \mathbb{R}$, is defined as follows:

$$\varphi_T(\omega) = \sum_{\omega \in Vert(\sigma)} Vol(\sigma) \tag{4}$$

where the summation is over all (maximal) simplices of T for which ω is a vertex. If ω is not a vertex of any simplex of T, then $\varphi_T(\omega) = 0$.

Definition 8 (Secondary Polytope) The secondary polytope $\Sigma(Y)$ is the convex hull in the space $\mathbb{R}^Y$ of the vectors ψ_T for all the triangulations T of $(Y, Conv(Y))$.

The *normal cone* $N_{\varphi_T}\Sigma(Y)$ consists of all linear forms ψ on $\mathbb{R}^Y$ such that

$$\psi(\varphi_T) = \max_{\varphi \in \Sigma(Y)} \psi(\varphi)$$

The point φ_T is a vertex of $\Sigma(Y)$ if and only if the interior of this cone is non-empty.

Theorem 9 *It holds:*

(a) *The secondary polytope $\Sigma(Y)$ has dimension $k-n-1$ where $k=\#(Y)$.*
(b) *Vertices of $\Sigma(Y)$ are precisely the characteristic functions got for all coherent triangulations T of $(Y, Conv(Y))$. If T is a coherent triangulation of $(Y, Conv(Y))$ then $\varphi_T \neq \varphi_{T'}$ for any other triangulation T' of $(Y, Conv(Y))$.*
(c) *For any triangulation T (coherent or not) the normal cone $N_{\varphi_T}\Sigma(Y)$ coincides with the cone $C(T) \subset \mathbb{R}^Y$.*

Definition 10 Let T be a triangulation of $(Y, Conv(Y))$, and let $Z \subset Y$ be a circuit. We say that T is supported on Z if the following conditions hold:

- There are no vertices of T inside $Conv(Z)$ except for the elements of Z itself.
- The polytope $Conv(Z)$ is a union of the faces of the simplices of T.
- Let $Conv(l)$ and $Conv(l')$ be two simplices (of maximal dimension) of one of the two possible triangulations of $Conv(Z)$. Then, for every subset $F \subset A - Z$, the simplex $Conv(I \cup F)$ appears in T if and only if $Conv(l' \cup F)$ appears.

Let T be a triangulation supported on a circuit Z. Then T induces one of two possible triangulations on $Conv(Z)$, say T_+. We let $sz(T)$ denote the new triangulation of $(Y, Conv(Y))$ that is obtained from T by taking away all the simplices of the form $Conv(l \cup F)$ with $Conv(l) \in T_+$ and adding the simplices of the form $Conv(l' \cup F)$ with $Conv(l') \in T_-$ and the same F. We say that $sz(T)$ is obtained from T by the modification along Z. It is clear that $sz(T)$ is also supported on Z, and $sz(sz(T)) = T$.

Theorem 11 *Let T and T' be two coherent triangulations of (Q, A). The vertices $\varphi_T, \varphi_{T'} \in \Sigma(A)$ are joined by an edge if and only if there is a circuit $Z \subset A$ such that T and T' are both supported on Z and obtained from each other by the modification along Z.*

This type of modification is also called *bistellar transformation*.

3 Convex Geometry

In this section, we briefly recall the basic concepts and theorems of Minkowski and Alexandrov theory in convex geometry, which can be described by Monge-Ampere equation and closely related to power diagram and weighted Delaunay triangulation. This intrinsic connection gives the theoretic tool to study the Alexandrov polytope space. Details can be found in [8, 9].