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Sergey Leble

Waveguide Propagation of Nonlinear Waves

Impact of Inhomogeneity and Accompanying Effects

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Impact of Inhomogeneity and Accompanying Effects

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To my wife
Anna Perelomova

Dal centro al cerchio, e si dal cerchio al centro movesi l'acqua in un ritondo vaso, secondo ch' é percosso fuori o dentro…

Dante Alighieri, LA DIVINA COMMEDIA, Paradiso, Cante XIV

Paradiso, Cante XIV

The water in a rounded dish vibrates from the centre to the rim, or from the rim to the centre, depending on how it is struck, from inside or out.

From centre unto rim, from rim to centre, In a round vase the water moves itself, As from without 'tis struck or from within.

by Henry Wadsworth Longfellow

Thought after thought, wave after wave - Two manifestations of the elements of one: In the heart of a small, in the boundless sea, Here - in conclusion, there - in the open, - The same everlasting surf and hang up, The same all ghost is alarmingly empty.

F. I. Tyutchev. "Thought after thought, wave after wave ..."

Preface

This book is a direct continuation and development of my book [2], published in 1988, taking into account the results in [3–5]. Generally, the route to any explicit formula that describes the evolution of a disturbance in a medium must involve a number of crucial simplifications [1]. These procedures carry such names as derivation or heuristic arguments. The results are often very impressive and are beginning to write their own story. The celebrated examples obtained by a linearized statement of the problem can be found in any textbook on mathematical physics. These are the D'Alembert (wave, string) equation, the Laplace–Poisson equation, and the heat (diffusion) equation. Derivations of these equations are a necessary feature of such textbooks. When we carry out this procedure, we neglect some terms, but we rarely see attempts to justify them mathematically.

Recently, a boom in mathematical physics has seen the development of standard 'minimal' nonlinear equations, such as so-called integrable equations. The list of such equations already includes many entries and is still growing. The best known among these are the Korteweg–de Vries (KdV) and nonlinear Schrödinger (NS) equations. These describe one-dimensional wave packets in the long wavelength limit in the KdV case and very close to a (carrier) frequency in the NS case. Considering the form of these equations, it is clear that their one-component and one-dimensional nature requires a lot of explanation to embed the resulting solutions in a general statement of the problem based on the original multicomponent and multidimensional description.

There are other important features in the form of these equations. Both contain only the first derivative in time, which means that they consider only one direction of wave propagation, and this is the principal difference from the above-mentioned D'Alembert equation. It is clear that, considering the initial stages in the evolution of a 1D perturbation of small amplitude, we observe propagation in both directions. This directly indicates the complex content of the potential development of the general initial perturbation, involving a kind of superposition to be taken into account in the mathematical description. This phenomenon has been well studied in the theory of 1D string evolution, but some effort is needed to translate the idea into multicomponent field language. In this book, we suggest a systematic realization of such a program that brings out all the ingredients and stages involved in transforming the 'minimal' equations into general statements of problems of hydro- and electrodynamics. We consider mainly two kinds of problems: initial (Cauchy) and boundary regime problems.

We develop a method for separating disturbances in a medium into components. Hydrodynamic systems are locally split into coupled nonlinear equations of interacting modes. Linearization provides independent modes with a specific evolution. The corresponding projection operators can be used to formulate initial value problems for each mode and introduce a physical basis for following interacting disturbances. The one-dimensional problem for exponential stratification is examined in detail as an example. Entropy and directed components (modes) are introduced, and interaction equations are derived. The weak nonlinearity/dispersion account introduces Burgers/KdV-like systems for directed waves interacting with a mean flow.

In the general three-dimensional case, we derive five eigenvectors of a linear thermoviscous flow over a homogeneous background for the quasiplane geometry of the flow. The corresponding projectors are calculated and applied to get the nonlinear evolution equations for the interacting vortical and acoustic modes. We specify the equation for the streaming caused by an arbitrary acoustic wave. We examine the correspondence with known results on streaming caused by a quasiperiodic source. The acoustic radiation force is calculated for a monopolar source [5]. In a further application of the projection method, we treat the propagation and interaction of acoustic waves in a medium stratified by gravity.

The same method is applied systematically to electromagnetic disturbances of a medium with given dispersion and magnetic properties. The separation of a disturbance is provided on the basis of a linear dispersion relation that is introduced either in the frequency domain for a boundary regime or in the wavevector domain for a Cauchy problem [3, 4]. The problem is in a sense 'diagonalized' in a linear version and partially diagonalized in the nonlinear regime. The linear and nonlinear problems are considered in terms of polarized directed wave propagation and interaction in a dispersive medium.

Compared to the previous book [2], we pay more attention to the kinetic description of wave propagation and its effects. Certain simplifications of the Boltzmann equation are used in the case of wave phenomena in rarefied gases, in the so-called Knudsen regime, with a transition to generalized hydrodynamics. We use kinetic equations like Vlasov's in combination with Maxwell's for a plasma and specify a direct route to a wavepacket description. The Kolmogorov kinetic equation provides an immediate tool for studying charge transfer along a nanowaveguide with various kinds of resistivity impact.

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Chapter 1
Introduction

1.1 General Remarks

We shall first make some remarks about the history, physical applications, and general context of waves and waveguide propagation, developing what was said in [1]. This book presents the nonlinear evolution equations and the theory of wave interactions in waveguides (quasi-waveguides) that result from nonlinearity and an inhomogeneity in the propagation medium. The theoretical description of finite amplitude wave dynamics is relevant to problems in mathematical physics as well as geophysical hydrodynamics [2, 3].

The development of the theory of wave motion was initially motivated by the need for a description of surface waves [4]. Surface wave propagation, however, may be considered as a limiting degenerate case of waveguide propagation [5]. Within this framework, the system can be described by choosing one or more coordinates in such a way that the dependence of dynamical variables on them may have a wave structure but no propagation. These coordinates are referred to as transverse, while the remaining ones will be called longitudinal.

The theory of nonlinear waves was first developed in the previous century by Scott-Russell, Riemann, Korteweg, de Vries, Boussinesq, and Stokes [6]. These early papers introduced the fundamental ideas of 1D shock waves (Riemann), solitary waves (Scott Russell), counteracting nonlinearity and dispersion effects (Stokes), and the distinction between waves moving in opposite directions (Boussinesq). The approximate (nonlinear) dispersion relations were derived in [1]. The 'new wave' of interest in finite amplitude wave theory was initiated by the development of the inverse problem method in 1967 and rapid progress in the integration of a class of model evolution equations [7].

The successes of the physical theory include the development of experimental soliton dynamics and the investigation of specially organized nonlinear phenomena. Examples are investigations of surface and internal waves in hydro-channels and rotating vessels [8–10], and modeling of nonlinear wave phenomena in electric transmission lines with a given nonlinearity, dispersion, and dissipation [11]. New

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discoveries in electromagnetic nonlinear wave physics [12] have led to new technological devices such as the splendid example of single-mode solitons propagating 10,000 km in fiber optic waveguides [13, 14]. Progress in the design of powerful sources and control devices, necessary for investigations of large-amplitude media disturbances, has allowed the global control of geophysical parameters [15].

These experiments provided ample information for theoretical interpretation. As a result, a gap has formed between the development of the physics and the mathematical methods. In one-dimensional problems, notable progress has been made through the efforts of many authors [14, 16, 17], but the essentially multi-dimensional nonlinear wave theory is far from being complete [1, 18]. The propagation of waves in guides is one such problem.

We should add that the rapid perfection of numerical methods and computers allows us to solve complex problems that model the physical situation ever more exactly. Development of analytical techniques combined with numerical simulation allow mutual testing and broaden the range of validity of the theory. The methods of reduction of systems of equations introduces new evolution equations into mathematical physics. Thus, the classic waveguide propagation theory for surface and internal waves has given birth to a variety of integrable nonlinear equations, including Korteweg–de Vries (KdV), Kadomtsev–Petviashvili (KP), nonlinear Schröinger (NS), Benjamin–Ono (BO), Joseph, Johnson, and so on, whose universal applicability has influenced all areas of physics. Recently, new integrable equations such as the sine–Gordon and short pulse equations (SPE) have been discovered in the theory of electromagnetic wave propagation in dispersive media by Schafer–Wayne [19] and Ampilogov–Leble for metamaterials [20], with two-component generalizations to account for polarization and direction [21].

Two basic overall ideas may be formulated as alternatives to the long wave and wave packet descriptions. The first leads to the coupled KdV system, the second to the coupled NS equations. This demonstrates the basic tool used for the new description of two- and three-dimensional nonlinear waves in guides. As a universal tool for such a derivation of the model systems, we continue to develop the dynamic projection operator technique [22].

We encounter two kinds of projection. The first involves a combination of the basic fields in the first order linear approximation, i.e., the basic wave modes are introduced. These modes are projections in the eigen subspace of a Sturm–Liouville problem for transverse variables $\boldsymbol{\rho}$ for functions $\mathbf{Z}_n$. The vector field $\mathbf{u}$ is represented by a series

$$\mathbf{u} = \sum_n \mathbf{Z}_n(\boldsymbol{\rho}) B_n(x,t) \ , \tag{1.1}$$

where $\boldsymbol{\rho}$ is transverse and x is the longitudinal coordinate. The sum in (1.1) corresponds to the case of the discrete spectrum of the Sturm–Liouville problem. It is this property that leads to classic waveguide propagation. In the case of quasi-waveguides, there is a possibility of energy leakage and the spectrum may contain a continuous part. In the particular case of long waves, for a problem in hydrodynamics with typical nonlinearity of second order, the mode coefficient functions $B_n(x,t)$ satisfy the coupled KdV equations (CKdV)

$$B_{nt} + c_n B_{nx} + \sum_{m,k} N_n^{mk} B_m B_{kx} + D_n B_{nxxx} = 0, n, m, k = 0, 1, 2, \ldots \quad (1.2)$$

where N_n and D_n are the nonlinear and dispersion constants, respectively. The equations are obtained by plugging the expansion (1.1) into both the linear and nonlinear terms and finalizing the derivation of (1.2) by projecting onto a transverse mode subspace.

Let us turn to the role of the boundary conditions that determine the basis functions Z_n for the expansion (1.1). In the simple cases of classic waveguides, these are the uniform conditions of the first or second kind that give the standard spectral problem for a finite interval. In the case of optical fibers, the matching conditions replace the 'solid' ones. Otherwise, if a quasi-waveguide is formed and the waves are captured in any transverse coordinate interval, due to significant inhomogeneity in the stratification in the direction of some transverse variable z, the boundaries are conventional. This leads to a change in the dispersion type, which is usually nonlocal. Problems that lead to model equations with pseudodifferential operators are considered in Sect. 3.1.

The study of interaction of waves with a nonlocal dispersion law is one of the current directions in waveguide propagation theory [1, 23]. It includes the derivation and investigation of BO and Joseph equations, as well as their short wave, two-dimensional, and multi-mode generalization. Separating the propagation medium into regions of varying stratification scales improves the convergence of the expansions and allows one to get a compact representation for the quasi-waveguide propagation solutions.

The application of Fourier analysis and perturbation theory to nonlinear oscillations began with the well-known studies by Poincare, Bogolyubov, and Galerkin. Their results were generalized by Taniuti's group [24–27], Maslov, Dobrokhotov [28–30], Grimshaw [31], Ostrovsky [32], and Pelinovsky [4, 33]. The concept of interacting guide modes was further developed by Miropolsky [34] and the present author [1, 23, 35, 36]. The difficulty with this problem lies in a certain ambiguity in the correct choice of sequence in the perturbation technique for solving the nonlinear equation for which the simple perturbation theory shows a singularity (instability, effect accumulation). The presence of two or three space variables obviously complicates the description. Moreover, the question arises whether the expansion (1.1) converges at all orders of the amplitude and dispersion parameters. The answer could be obtained only from the arbitrary mode decomposition term. One must therefore explore the dependence of the solution on the same small parameters used to derive this sequence of evolution systems. This is one issue on the program in this book.

We now give a short review of results directly related to this approach. The appropriate CKdV system was introduced in [37] and discussed in [38]. Another version of this system as well as single- and two-soliton solutions have been given by Hirota and Satsuma [39] without any discussion of the physical implications. The general CKdV derivation is reproduced here for hydrodynamical systems, describing oceanic internal gravity waves (IGW) (Sect. 6.4) [1, 35, 40]. The explicit form is obtained for nonlinear and dispersion constants. The approach to CKdV integration was developed in a collaboration with Kshevetsky. This allows one to reduce the wave disturbance solution, which is nearly a single-mode wave, to the solution of a combined KdV–MKdV equation [41] (with the same amplitude parameter as in the derivation of CKdV), which is integrable [7]. The solution of the latter equation and formulas for intermode couplings gives the representation of the multi-mode problem. Amplitudes of mode contributions decrease rapidly enough with mode number n to ensure convergence of the expansion (1.1). The limiting case $D_n \to 0$ of the CKdV (1.2) was considered in [41]. Solutions have been constructed with characteristics given in the same approximation as was used for the perturbation scheme. Analysis of the solution demonstrates a tendency to self-localization due to wave mode interaction [1]. Independent investigations have shown that similar results are obtained in the interaction of two Riemann waves [42].

The integrability of the CKdV system was studied using the Lax and Wahlquist–Estabrook methods [Kshevetsky and Leble, unpublished]. It was shown that there exist two-mode systems generalizing the Hirota–Satsuma equation that have a Lax pair. Independently, Dodd and Fordy found an L-A pair for the Hirota–Satsuma equations [17]. The methods of nonlinear evolution equations, including waveguide systems, are presented in [4]. The material in that volume is similar to the first stage of our method, which also follows Ostrovsky and Pelinovsky [32, 33]. The subsequent stages, i.e., interactions in small terms and determination of dispersion branches, were not pursued there, however. The simplest evolution equation was derived in [4] for a single mode case. The development of nonlinear evolution equations and progress in the theory of integrable systems [43], as well as the discovery of new integrable systems (e.g., the Benney–Kaup and Ito systems [44]), suggest good reason to hope for further propagation of the method.

The important stage in the simplification of the description of nonlinear wave dynamics is the determination of dispersion branches (i.e., separation of waves according to their types). The formulation of this problem appeared in the pioneering work [45], in a search for a connection between the solutions of the Boussinesq and KdV equations. The KdV equation is traditionally derived by separating the 'left' and 'right' waves by introducing a small parameter in the wave function argument. This allows one to get the equation for a single-directed wave, but does not give a unique form for interaction terms. Traditionally, separation methods for any individual case follow the characteristic (often spectral) properties of a given branch. The conditions prohibiting another branch might be transversality in the case of an electromagnetic wave or incompressibility in the case of an internal gravity wave in a stratified fluid. The wave separation decreases the order of the basis equations. Thus, the possibility arises of including the various waves identified in physical experiments in the description. Currently, increasing attention is being paid to different types of wave interaction [46, 47]. In the case of electromagnetic waves, either the components of the field intensity or the states of definite polarization can be similarly described [48]. There is very rich set of various types of plasma waves [49].

The first attempts to develop a universal approach to this problem in 1D were made by Novikov [50, 51] and, for higher dimensions, by Leble [52, 53]. The correct introduction of branch subspaces in the wave dispersion relation is necessary even in a linear theory. This is not a simple problem and it is important for correct separation of the corresponding contributions in given initial data, a boundary regime, or sources. Progress here is reviewed in the book [22]. The approach there is based on a dynamic projection operator technique which is a convenient tool for deriving equations in their explicit form. This problem has an important physical aspect: identification of the contribution of each wave type involves measuring a fundamental set of dynamical variables and studying the connection between them (polarization relations). The modern nomenclature is based on linear equations. Therefore it is convenient to preserve the traditional terminology, at least in the case of weak non-linearity. The classification principle within the projection operator approach is general and thus allows generalization to the nonlinear case. In this book, however, only the linear couplings are used.

The complexity of the wave separation problem depends on the number of dynamical variables and orders of derivatives in the equations of the fundamental problem. The separation into right- and left-traveling waves may be obtained via a multi-scale decomposition [4, 31, 54], and also by the reductive perturbation method [55], as has been shown for the nonlinear string case. However, the separation of internal and sound waves in the atmosphere cannot be so simply treated. It is shown in [1] that, with the discrete Silin–Tikhonchuk equations written in a form that includes interacting unidirectional Langmuir waves, the system can be integrated exactly.

The rapid developments in the theory of integrable systems have often led to the proposal of new equations and their solutions before such equations are derived from physics. This was the case with the two-dimensional Joseph equation, a nonlocal analogue of the Kadomtsev–Petviashvili equation, which was investigated in [56] and derived in [35, 57]. Nonlocal operators in the theory of electromagnetic wave dispersion are introduced when there is spatial dispersion or a strong inhomogeneity in the stratification in the medium of propagation [49]. Such strong variations in the stratification scale of liquid media lead to nonlocal dispersion of internal waves, too. In these cases the waveguide regions appear in a propagation medium where the solutions oscillate and exponentially decrease outside the medium. Then the basis of the linear problem is introduced inside the waveguide interval, matching the solutions at the region's boundaries. The solution outside the guide interval may be found without mode expansion, due to the simplification of the structure of the basic equation operator.

This problem was first studied at the level of a dispersion relation by Phillips [58, 59] and Whitham [60]. The explicit form of the pseudo-differential operators appeared in the studies by Benjamin [61], Ono [62], and Levikov [63] for the fluid pycnoclyne guide (layer with density varying in z) surrounded by infinite homogeneous layers in the single-mode case. For a finite-depth fluid, the nonlocal analogue of the KdV equation was derived by Joseph [64]. The two-dimensional multi-mode wave has also been studied [35]. The theory of nonlocal equations led to algebraic solitons [62]. The theory of algebraic and algebro-geometric integration was developed for nonlocal dispersion in [65]. The soliton solutions of single-mode evolution equations are generally a rather crude approximation in waveguide theory. However, they can be the starting point for the averaging methods of Whitham [60]. The solution of the multi-mode problem is to look for a soliton form with parameters that weakly depend on the coordinates. Knowledge of the behavior of the solutions of simplified equations allows one to apply the iteration procedure to the system considered, going over to the model equation in a power series of a small parameter [1]. The fundamental statement of the problem has been discussed by Maslov and Dobrokhotov [30, 66] and also in [67].

An important case of waveguide propagation is when only one mode is possible, due to the waveguide dimensions. This is the case for electromagnetic waves in metal tubes, dielectric layers, and fibers (Chap. 3). The broad range of possible applications in physics and engineering has resulted in a proliferation of publications on model evolution equations of NS type with various kinds of higher order nonlinear dispersion and dissipation [14, 68]. The total dispersion operator preserves the soliton form and the existence of multi-soliton decay gives the possibility of shortening the pulse time to femtoseconds [68].

Elimination of arbitrariness in the choice of boundary conditions [34, 68] requires an additional analysis of basic physical assumptions. For example, the transition from hydrodynamic flow to the collisionless regime in gaseous flow may cause waveguide propagation [69]. The boundary conditions for such a wave propagation problem are formulated within the kinetic theory. The statement of the problem in the hydrodynamic regime follows from the general kinetic formulation.

There are many situations when the boundary regions along the longitudinal coordinates should be taken into account. In this case new salient features appear. For example, periodic conditions may arise in problems with a torus (ring) guide geometry, as is the case in experiments with rotating vessels [9, 70] and tokamak plasma installations [47]. The fundamental mathematical elements of these problems are the Riemann theta-functions or finite-gap solutions [71]. The problem of extracting the class of real nonsingular periodic solutions from a general finite-gap solution has been discussed in [71, 72]. Large-scale atmospheric wave propagation, where the periodicity along the planetary longitude is natural, allows for the application of the finite-gap integration method. For long internal waves, the simplest two-dimensional single-mode system is the Kadomtsev–Petviashvili (KP) equation [36]. Various aspects of the application of the functional KP solution are specified in [73, 74]; see also [1]. The very wide field of applications at the ‘quantum border’ of this book are outlined in the rather recent multi-author paper [75].

1.2 Overview

1.2.1 On Mathematics. Evolution and Projection

In this chapter we sketch the basic mathematical notions used in this book, starting from general relations and illustrating them with the simplest examples. We pay particular attention to the waveguide aspects of these fundamentals: eigenfunctions of the chosen transverse coordinate operator and expansion in these basic mode series; and formulation of a problem in terms of the longitudinal variable and time, defining a propagation with dispersion and nonlinear effects. The first problem involves the boundary/matching condition formulation. The second is studied from two points of view: the initial (classic Cauchy) problem and a boundary regime propagation picture. Both problems for multicomponent fields require special techniques that can be used to fix a generalized polarization and direction of propagation. We start from the dynamic projection operator technique developed for the corresponding evolution operators [22], applying it to the initial and boundary regime propagation problems. The peculiarities of the technique arising from introduction of a small parameter are illustrated in Sect. 2.4 with an account of weak nonlinearity and Sect. 2.5 for the weak inhomogeneity case.

1.2.2 Electromagnetic Waveguides

Electromagnetic (EM) waves in waveguide propagation have a specific feature that comes from the Maxwell equation formulation, which contains stationary equations (like div $\mathbf{B} = 0$). Such equations are in fact ‘projecting’ equations, keeping the EM field $\mathbf{E}, \mathbf{B}$ in a subspace that does not change under the evolution. We should take this into account when writing the time evolution operator or the evolution of the longitudinal variable. Another important issue relates to EM waves in a medium, where additional information about atomic charges should be taken into account. Quantum or model material relations are necessary. These relate the magnetisation/polarisation vectors with the EM field vectors, seriously complicating the statement of the problem in the case of nanostructures and metamaterials (see Chaps. 7 and 8).

1.2.3 Solitonics

Soliton theory has an important place in modern physics. In particular, electromagnetic waveguides such as optical fibers or photonic crystals have demonstrated its physical relevance and opened the way to technical applications [13, 17]. The multi-soliton formalism is intimately related to and inspired by the integrable systems of mathematical physics. Its practical construction is based on one of the iteration methods, called the dressing method [76]. In this chapter, we outline a version of the method, known as the Darboux transformation technique. We present here results that are not included in that book. We describe ‘inclined’ solitons for the three-wave interaction with asynchronism, solutions of the Maxwell–Bloch system, and solitons for the integrable version of the coupled nonlinear Schrödinger equation (Manakov case).

1.2.4 Hydrodynamics

Hydrodynamics is conventionally based on a system of conservation laws for problems without losses. The losses are described by the corresponding equations of balance (mass, energy, and momentum). We focus our attention on determination of the wave type using the dynamical projection technique. Next we apply the method to atmospheric wave problems and develop this by including waveguide features of the propagation [77]. We start from acoustic gravity waves in a 2D exponential atmosphere and continue with the study of Rossby and Poincaré waves.

1.2.5 Guide Propagation and Interaction of Plasma Waves. Metamaterials

The main peculiarity of wave theory for an interesting medium like a plasma is its rich spectrum. The reason for the abundance of such dispersion branches is the multicomponent nature of the medium, which contains electrons and ions. The mixture oscillates in space in a way that naturally requires us to account for the electromagnetic field.

Even in a very simplified version of multi-stream hydrodynamics, unified with the Maxwell equations, this leads to a system of equations with an evolution operator of high dimension. Direct investigation is extremely difficult even after simplification, as attested by the magneto-acoustic wave theory with 7×7 and 8×8 evolution operators and the nonlinear heating effect in [78]. The kinetic description is based on the Vlasov kinetic equation, which is of a higher level than the hydrodynamic one; special tools are required to derive the wave equation, especially in an inhomogeneous medium. In Sect. 7.4.1, such transformations are illustrated by 1D examples. The plasma confinement problem is briefly outlined using the example of a rarefied plasma. Confinement creates a quasi-waveguide. Wave propagation and the possibility of flute instability within such a region are illustrated in Sect. 7.4.3.

1.2.6 Nanowaveguides. Bloch Waves

Minimizing the waveguide dimension increases the relative contribution of surface atoms in energy, compared to bulk atoms. Quantum foundations are definitely needed to understand any phenomenon on such a length scale. When we consider a world involving such quantum notions as electron exchange, which depend on spin variables, a microscopic approach should be the basis of any consideration that introduces averaged (mean) values, including parameters of distribution functions. In its limiting case, mesoscopic physics is relevant to the study of quantum waveguides [79].

In this chapter the problem of nanowire conductivity is studied from the kinetic point of view for quasi-classical Bloch electrons in an electric field. Several problems with cylindrical symmetry are formulated for the integro-differential Kolmogorov equation: a dynamical Cauchy problem and two problems with stationary boundary regimes. The first is for an empty cylinder with scattering of the conduction electrons on the walls, while the second takes into account scattering by defects inside the wire. The integro-differential equations are transformed to integral equations and solved by iteration. There are two types of expansion with the leading terms on the right and left sides. The iteration series is constructed and its convergence is investigated. Under such conditions, the pseudopotential [76] and other solvable models that include resonances [80] are effective tools for understanding various phenomena.

1.2.7 Microwaveguides Versus Nanowaveguides. Domain Wall Propagation

Microwires (MW) [81] do not only differ from nanowires in terms of size. Their characteristics have 3D properties that influence the dynamics of the magnetic moment density (MMD). In an amorphous MW, we use either the Landau–Lifshitz–Gilbert equation [82] or direct applications of an alternative Hamiltonian approach [83]. In this chapter we review three important aspects of the theory: the origin of the basic equations, their explicit solutions, and stability. The latter is investigated for a 1D model, where we consider the so-called Walker solution/instability [84]. Applications to nanowires are also considered, including an approach by direct application of the original discrete Heisenberg chain equations. We start from the Heisenberg chain equations (see also [82]), and particular explicit solutions as in [85, 86], touching upon the results of [83].

1.2.8 Kinetics of Charges in Waveguides. Charge Transport

The propagation of 'charge waves' in waveguides is naturally described by a kinetic equation for a distribution function in phase space. In this chapter we continue to develop the technique introduced in Chap. 7 on plasma dynamics, based on the collisionless Vlasov equation. Here, we take into account collisions of charged particles with particles of the propagation medium and scattering by the waveguide walls, introducing the collision term in the Kolmogorov equation from [87]. We consider the non-stationary kinetic equation (containing a time derivative). This allows us to study the propagation of charge pulses and the dynamics of resistance [88]. Stationary cases are also studied. Expressions for the resistivity of electron transport as a function of temperature and wire radius are derived by solving the resulting Fredholm equation using the resolvent method.

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Chapter 2
Evolution Operator and Projectors to Its Eigenspaces

In this chapter we sketch the basic mathematical notions of dynamical projection used in this book, starting from general relations and illustrating them by the simplest examples, following [1–3] and paying particular attention to the impact of inhomogeneities and accompanying effects. As mentioned in the introduction [see (1.1)], in the waveguide propagation, after expanding all the fields in series over the transverse coordinate basis, the coefficients ψ_k of the expansions will depend on the unique longitudinal space coordinate, say x, and time. Let $\partial = \partial/\partial x$ denote the space derivative. We shall write the basic evolution equation in 1D for the components $\psi_k(x,t)$ as

$$\psi_t(x) - L(\partial, x)\psi(x) = N(\psi(x)) \ , \tag{2.1}$$

where the multicomponent vector of state represents a set of physical variables of the system under consideration. The transverse mode index k will henceforth be omitted. It is written as

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} . \tag{2.2}$$

We pick up the linear terms on the left-hand side and the nonlinear ones on the right-hand side, denoted $N(\psi)$. There is an option to place small linear terms on the right-hand side:

$$\psi_t(x) - L_0(\partial, x)\psi(x) = N(\psi(x)) + \epsilon L_1(\partial, x)\psi(x) \ , \tag{2.3}$$

treating $\epsilon L_1(\partial, x)\psi(x)$ as a perturbation. The origin of such terms may be related to dissipation or gain . The joint account of dissipation and nonlinearity leads to such important phenomena in acoustics as heating [4–6] and streaming [7, 8].

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