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Standard Monomial Theory

Invariant Theoretic Approach

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To the memory of
Professor C. Musili

Preface

The goal of this book is to present the results of Classical Invariant Theory (abbreviated CIT) and those of Standard Monomial Theory (abbreviated SMT) in such a way as to bring out the connection between the two theories. Even though there are many recent books on CIT, e.g., [25, 35, 53, 97, 99], none of them discusses SMT: there is but only a passing mention of the main papers of SMT towards the end of [53]. Details about the connection are also not to be found in the comprehensive treatment of SMT [59] that is in preparation. Hence the need was felt for a book that describes in some detail this natural and beautiful connection.

After presenting SMT for Schubert varieties—especially, for those in the ordinary, orthogonal, and symplectic Grassmannians—it is shown (using SMT) that the categorical quotients appearing in CIT may be identified as "suitable" open subsets of certain Schubert varieties. Similar results are presented for certain canonical actions of the special linear and special orthogonal groups. We have also included some important applications of SMT: to the determination of singular loci of Schubert varieties, to the study of some affine varieties related to Schubert varieties—ladder determinantal varieties, quiver varieties, variety of complexes, etc.—and to toric degenerations of Schubert varieties.

Prerequisite for this book is some familiarity with commutative algebra, algebraic geometry and algebraic groups. A basic reference for commutative algebra is [27], for algebraic geometry [37], and for algebraic groups [7]. We have also included a brief review of GIT (geometric invariant theory), a reference for which is [87] (and also [96]).

We have mostly used standard notation and terminology, and have tried to keep notation to a minimum. Throughout the book, we have numbered Theorems, Lemmas, Propositions etc., in order according to their section and subsection; for example, 3.2.4 refers to fourth item in the second subsection of third section of the present chapter. The chapter number is also mentioned if the item appears in another chapter.

This book may be used for a year long course on invariant theory and Schubert varieties. The material covered in this book should provide adequate preparation for graduate students and researchers in the areas of algebraic geometry and algebraic groups to work on open problems in these areas.

A Homage & an acknowledgement: In the original plan for this book, Musili was one of the co-authors. Unfortunately, Musili passed away suddenly on Oct 9, 2005. We dedicate this book to the memory of Musili. We also would like to thank Ms. Bhagyavati (Musili's wife) and Ms. Lata (Musili's daughter) for providing us with the files that Musili had prepared.

We wish to thank the referees for their comments.

Boston, Trieste
October 2007

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Introduction

1.1 The subject matter in a nutshell

This book aims to describe the beautiful connection between Schubert varieties and their STANDARD MONOMIAL THEORY (SMT for short) on the one hand and CLASSICAL INVARIANT THEORY (CIT for short) on the other. This connection was discovered by Lakshmibai and Seshadri [67]. Here is the opening paragraph of [67] in a slightly edited form:

> The main aim of this paper is to show how the work of De Concini and Procesi [22] on classical invariant theory can be interpreted to suggest a generalization of the Hodge-Young theory of standard monomials (cf. Hodge [41] and Hodge and Pedoe [42]). This generalization is given as a set of conjectures (which have now been proved in collaboration with C. Musili [61]). On the other hand, we also show that the results of De Concini and Procesi follow as consequences of the generalization.

SMT is the name given to this generalization of the Hodge-Young theory. And the term CIT in this book refers, for the most part, to the results of De Concini and Procesi mentioned in the quote above. These results, when the characteristic of the ground field is zero, are classical—cf. Weyl's book [115].

As we will see, determinantal varieties form the bridge connecting CIT and SMT, and Schubert varieties become relevant because they are natural compactifications of determinantal varieties.

1.1.1 What is CIT?

CIT concerns certain canonical actions of classical groups G on affine spaces, namely, cases **A**, **B**, **C**, **D**, and **E** to be discussed in §1.2 below. It describes, in each case, a presentation for the ring of G-invariant polynomial functions on the affine space. This description comprises of two theorems known as the *first* and *second fundamental theorems*. The first fundamental theorem specifies a finite set of algebra generators, over the ground field, for the ring of invariants. Note that for the action of a reductive group G on an affine variety, the ring of G-invariant polynomial functions is a finitely generated algebra (classically, i.e., in characteristic zero, this result goes back to Weyl [115]; in positive characteristic, this follows from well-known results of Nagata [93, 94] and Haboush [36]: see Chapter 9 for details). The second fundamental theorem specifies a finite set of generators for the ideal of relations among the algebra generators: that there always exists a finite set of generators for the ideal follows from Hilbert's basis theorem [79].

1.1.2 What is SMT?

The roots of SMT are to be found in the work of Hodge [41, 42], who described nice bases for the homogeneous co-ordinate rings of Schubert varieties of the Grassmannian in the Plücker embedding (over a field of characteristic 0). Grassmannians being precisely the homogeneous spaces that arise as quotients of special linear groups by maximal parabolic subgroups, it is natural to try to generalize Hodge's work to natural projective embeddings of other quotients G/Q where G is a semi-simple algebraic group and Q a parabolic subgroup. In the early '70s Seshadri initiated this generalization and called it SMT.

1.1.3 The SMT approach to CIT

The main idea in this approach is to relate a certain subring of the ring of invariants (which will turn out to be in fact the ring of invariants) as the ring of functions on an affine variety related to a Schubert variety. This allows the use of SMT to prove the first and second fundamental theorems.

1.2 The subject matter in detail

In this section, we shall explain the SMT-approach [67] as well as the approach of De Concini-Procesi [22] for the actions of general linear, symplectic, and the orthogonal groups. In both approaches, one makes a guess on the ring of invariants; to be more precise, there are some obvious invariants (we call these the *basic invariants*), and one shows (in both approaches) that the ring of invariants is in fact generated (as an algebra over the base field) by these invariants.

Fix a field K, algebraically closed of arbitrary characteristic, and V a finite dimensional vector space over K.

A. The general linear group $G := GL(V)$ of invertible linear transformations acts naturally on both V and its dual V^*. Consider G acting diagonally on $Z := V^{\oplus m} \oplus V^{*\oplus q}$; here $V^{\oplus m}$ denotes the direct sum of m copies of V.

Let R be the ring of polynomial functions on Z. There is a natural action of G on R. Let R^G be the subring of R consisting of those polynomial functions that are invariant under G. The "scalar products" $(v_1, \ldots, v_m; f_1, \ldots, f_q) \mapsto \varphi_{ij} := f_j(v_i)$, for $1 \leq i \leq m$ and $1 \leq j \leq q$, are evidently invariant—these are the "basic invariants."

B. Let now V be equipped with an alternating non-degenerate bilinear form (,)—note that this forces V to be even dimensional. Let $Sp(V)$ be the symplectic group, namely, the subgroup of $GL(V)$ consisting of those linear automorphisms of V that preserve the form. Consider the diagonal action of the symplectic group $Sp(V)$ on $Z := V^{\oplus m}$.

Let R be the ring of polynomial functions on Z. The "basic invariants" in this case are the bilinear products $(v_1, \ldots, v_m) \mapsto \varphi_{ij} := (v_i, v_j)$ for $1 \leq i, j \leq m$.

Now suppose that instead of an alternating form we have a symmetric form. More precisely:

C. Let the characteristic of K be different from 2 and let V be equipped with a non-degenerate symmetric bilinear form (,). Let $O(V)$ be the orthogonal group, namely, the sub group of $GL(V)$ consisting of those linear automorphisms of V that preserve the form. Consider the diagonal action of the orthogonal group $O(V)$ on $Z := V^{\oplus m}$.

Let R be the ring of polynomial functions on Z. The "basic invariants" are still the bilinear products $\varphi_{ij} := (v_i, v_j)$.

In all of the three cases **A, B, C** above, let us denote by S the subalgebra of R^G generated by the basic invariants. The goal is

(1) to show $S = R^G$ which will yield immediately the First Fundamental Theorem.

(2) to construct a "nice" basis for $S(= R^G)$ which will yield the "straightening relations" (see §1.5) and hence the Second Fundamental Theorem.

To explain the proof of De Concini-Procesi, it suffices to quote in a slightly edited form from [22]:

> The line of the proof is the following: we have an algebraic group G acting on an affine variety E with coordinate ring R and we have a subring S of R^G, namely the one generated by the basic invariants, which we want to show equals R^G. First we show that on an open set $U \subseteq W$ (where an element s of S is invertible) the group action is a product action; thus the localized invariant ring $R^G[1/s]$ turns out to be $S[1/s]$. Then we have to find a way to cancel s: i.e., if $sa \in S$ and $a \in R^G$ we must show that $a \in S$. This is accomplished by finding an explicit basis of the ring S and deducing the cancellation result from this. This part is the main contribution of the paper.

1.2.1 Proof of (1), (2) by SMT approach (cf. [67])

Let us denote the basic invariants by $f_1, \cdots, f_N$. We have a natural map

$$\psi : X \to \mathbb{A}^N, \ x \mapsto (f_i(x))$$

In all of the three cases **A, B, C** above, it is not difficult to see that $\psi(X)(= Spec\ S)$ gets identified with a determinantal variety $\mathbb{D}$: In each case, let G denote the group in question, namely, $G = GL(V),\ Sp(V),\ O(V)$ in cases **A**, **B**, **C** respectively. Let n be the dimension of V.

• In **A**, $\mathbb{D}$ is the subvariety $D_{n+1}(M_{m,q})$ of $M_{m,q}(K)$ (the space of $m \times q$ matrices with entries in K) consisting of matrices of rank at most n, i.e., the matrices all of whose $(n+1)$-minors vanish.

• In **B**, $\mathbb{D}$ is the subvariety $D_{n+1}(Sk\ M_m)$ of $Sk\ M_m(K)$ (the space of skew symmetric $m \times m$ matrices with entries in K) consisting of matrices of rank at most n.

• In **C**, $\mathbb{D}$ is the subvariety $D_{n+1}(Sym\ M_m)$ of $Sym\ M_m(K)$ (the space of symmetric $m \times m$ matrices with entries in K) consisting of matrices of rank at most n.

Now $f_1, \cdots, f_N$ being G-invariants, the morphism ψ goes down to a morphism

$$\psi_{\mathbb{D}} : Spec\ R^G \rightarrow \mathbb{D}(= Spec\ S)$$

The main idea behind the proof of the First Fundamental Theorem in [67] is to show that $\psi_{\mathbb{D}}$ satisfies the hypotheses of ZMT (Zariski's Main Theorem):

(i) $\psi_{\mathbb{D}}$ is surjective with finite fibers
(ii) $\psi_{\mathbb{D}}$ is birational
(iii) $\mathbb{D}$ is normal.

It would then follow by ZMT that $\psi_{\mathbb{D}}$ is in fact an isomorphism, and we would obtain that the inclusion $S \hookrightarrow R^G$ is in fact an equality. The verifications in [67] of (i) & (ii) turn out to be rather straight forward in view of certain geometric invariant theoretic considerations (see Chapter 10 for details). Thus to conclude that $S = R^G$, proving normality of $Spec\ S(= \mathbb{D})$ is the only non-trivial part. Here is where the Schubert variety connection is used in the approach in [67]. To make this more precise, in case **A** (respectively, **B**, **C**) above, it turns out that $M_{m,q}(K)$ (respectively $Sk\ M_m(K),\ Sym\ M_m(K)$) gets identified with the "opposite cell" $\mathcal{O}^-$ in the Grassmannian $G_{q,m+q}$ (respectively the orthogonal Grassmannian $SO(2m)/P_m$, the symplectic Grassmannian Sp_{2m}/P_m); and $\mathbb{D}$ gets identified with $\mathcal{O}^- \cap X$, for a suitable Schubert variety X in $G_{q,m+q}$ (respectively in $SO(2m)/P_m$, Sp_{2m}/P_m). Hence the normality of $Spec\ S(= \mathbb{D})$ follows, once we know the normality of Schubert varieties. Normality of Schubert varieties is a consequence of SMT.

Using the SMT-basis for the homogeneous co-ordinate ring of X, we obtain a basis for $\mathbb{D}$ (by the process of dehomogenization). This in turn yields the Second Fundamental Theorem.

Thus using Schubert varieties and their SMT, we obtain in one stroke, the proofs of the first and second fundamental theorems.

At the time of the appearance of [22], SMT was developed only for "minuscule" G/P's. To be more precise, in his thesis written under Seshadri's guidance, Musili [88] had extended Hodge's results to arbitrary characteristics. Soon after, Seshadri [111] had generalized Hodge's results to quotients by *minuscule*[1] maximal

[1] A maximal parabolic subgroup P is *minuscule* if the Weyl group translates of a highest weight vector span the space of global sections on G/P of the ample generator of the Picard group of G/P. The geometry of a minuscule G/P, i.e., when P is minuscule, is very similar to that of a Grassmannian.

parabolics. As is easily seen, the G/P's appearing in cases **A, B** are minuscule; and the bases for R^G as given by [22] on the one hand and SMT on the other are the same.

But the G/P arising in case **C** is not minuscule. Given this, the following problem cried out for an answer: shouldn't there be an approach to CIT in case **C** similar to that of cases **A**, **B** above? Analysing carefully the work of De Concini and Procesi in case **C**, Lakshmibai and Seshadri [67] arrived at a conjectural SMT for Schubert varieties in quotients of classical semi-simple groups by maximal parabolic subgroups (technically speaking a little more generally but we will ignore that here for the sake of simplicity). These conjectures were later proved by them in collaboration with C. Musili in [61, 62]. Taking for granted such an SMT, case **C** too can be handled in a fashion entirely analogous to cases **A** and **B**.

1.2.2 $SL_n(K)$, $SO_n(K)$ actions

Replacing the general linear group in case **A** discussed in §1.2 by the special linear group, consider

D. The special linear group $SL(V)$ acting diagonally on $V^{\oplus m} \oplus V^{*\oplus q}$.

The basic invariants in this case are the scalar products $\varphi_{ij} := f_j(v_i)$ (as for the general linear group—cf. case **A** above) and the determinants $u(I) := \det[v_{i_1}, \ldots, v_{i_n}]$, $\xi(J) := \det[f_{j_1}, \ldots, f_{j_n}]$, where $1 \le i_1 < \ldots < i_n \le m$, $1 \le j_1 < \ldots < j_n \le q$—these appear only in the case when $m \ge n$.

Replacing in case **C** the orthogonal group by the special orthogonal group, consider

E. The diagonal action of the special orthogonal group $SO(V)$ on $V^{\oplus m}$.

The basic invariants in this case are the bilinear products as above and the determinants $u(I) := \det[v_{i_1}, \ldots, v_{i_n}]$, where $1 \le i_1 < \ldots < i_n \le m$—these appear only if $m \ge n$.

De Concini and Procesi [22] treat CIT in cases **A**, **B**, and **C** fully. In cases **D** and **E**, they prove the first fundamental theorem, but there are no details about the second fundamental theorem.

As for cases **D** and **E**, the invariant rings are not—not in any obvious way at least—rings of functions on open parts of Schubert varieties. So the approach to CIT described in §1.2.1 does not work in exactly the same way. Nevertheless Schubert varieties remain relevant. And an SMT theoretic approach to CIT in cases D and E has recently been worked out in [63, 72] respectively.

In Case **D**, the normality of the ring S generated by the basic invariants is deduced by degenerating its spectrum to a toric variety. As a first step towards the degeneration, a basis for S is constructed. Straightening relations are then written down and the degeneration is carried out using the straightening relations. The Cohen-Macaulayness of S also follows immediately as a corollary of the degeneration. That the poset structure of Schubert varieties in the Grassmannian form a distributive lattice is used in the proof crucially.

In case **E**, the ring S generated by basic invariants arises as the ring of functions on a branched covering of degree 2 over the symmetric determinantal variety $D_{n+1}(Sym\, M_m)$. The normality of this ring is proved by showing that it is non-singular in codimension one and also Cohen-Macaulay (in view of Serre's criterion—see [79] for example). The first property follows by noting that the branch locus is contained in the singular locus of $D_{n+1}(Sym\, M_m)$ and that $D_{n+1}(Sym\, M_m)$ being an open part of a Schubert variety is normal (and in particular non-singular in codimension one). The Cohen-Macaulay property however takes some work to prove. As a first step, one obtains a basis for the ring S. Then a deformation argument due to De Concini and Lakshmibai [21] is applied. They used it originally to show that Schubert varieties in quotients of classical groups by maximal parabolics are arithmetically Cohen-Macaulay for the embedding given by the ample generator of the Picard group.

This finishes the discussion of the SMT approach to CIT. There is a lot more that has happened in SMT beyond what is described above. A brief history of SMT is given in §1.4 below. But the scope of this book is confined only to that part of SMT that has been discussed above. In this book we treat in detail all cases A–E via SMT.

1.3 Why this book?

The main subject matter of this book, namely the connection between Schubert varieties on the one hand and CIT on the other described in §1.2 above, has not come until now within the scope of any book. The books on CIT—and there are quite a few recent ones among them, e.g., [25, 35, 53, 97, 99]—come no where near discussing this connection. In fact, except in [53] where there is a quick mention of the main papers of SMT towards the end, the books are totally silent about the connection.

On the other hand, looking from the SMT side, the book [59] under preparation aims at being a comprehensive account of SMT and is authored by the main players in the creation of that theory. But here too there is very little discussion on the connection between Schubert varieties and CIT. The connection is of course mentioned in the book's introduction which recounts in detail the history of the development of SMT. But the emphasis of the book being elsewhere, this connection is not treated in the book itself.

The raison d'etre for the present book is thus clear.

As seen in §1.2.1, the fact that determinantal varieties arise as open sets of Schubert varieties plays a crucial role in the connection between CIT and SMT. Although the relationship between determinantal varieties and Schubert varieties is quite classical, except for [40, 51, 89], there is not much in the literature about it. Further, to the best of our knowledge, [67] is the first work in the literature which discusses the relationship between determinantal varieties in the space of symmetric and skew-symmetric matrices on the one hand and Schubert varieties on the other. These aspects are treated in detail in the present book.

1.4 A brief history of SMT

We have discussed above the origins of SMT and how the work of De Concini and Procesi on CIT influenced the development of SMT. What follows is a brief account of the history of SMT. The book [59] under preparation aims to give a comprehensive account of SMT. Its introduction gives a detailed account of the history of SMT. But, since the present book will in all likelihood appear before [59], the following account may still be of some value.

We may divide SMT into four stages. The first stage consists of the work done up to the appearance of the paper of De Concini and Procesi on CIT. As explained above, this stage comprises of the work of Hodge [41,42], Musili [88], and Seshadri [111].

The second stage consists of the work done in roughly over a decade after the appearance of De Concini-Procesi's paper [22] on CIT. This stage begins with the paper of Lakshmibai and Seshadri [67] which describes SMT conjecturally for Schubert varieties in quotients of classical groups by maximal parabolic subgroups. These conjectures are proved in [61,62]. Finally, in [69], SMT is established for quotients of classical groups by parabolic subgroups (maximal or otherwise).

Classical groups having been satisfactorily addressed, the third stage attempts to handle exceptional groups: [55] handles G_2, [64] handles E_6, and [70] handles the case of the affine Kac Moody group $\hat{sl}_2$. Finally, as a culmination of all the work, the conjectures for a general SMT (for Schubert varieties in G/Q for an arbitrary parabolic subgroup Q of a symmetrizable Kac-Moody Group G) are formulated by Lakshmibai-Seshadri in [71].

Solving the conjectures of [71] required new ideas. This is what Littelmann has done in his work. His papers [74,75,77] form the fourth and final stage of SMT. An important idea of Littelmann is to view "standard tableaux" (the indexing set for the SMT basis) as certain paths—the so-called *Lakshmibai-Seshadri paths*. Littelmann uses the theory of quantum groups. The technique of [103] is also crucially used. Thus SMT is now complete thanks to Littelmann's work!

Our goal being the description of the connection between SMT and CIT, it suffices for us to consider SMT for quotients of classical groups by maximal parabolic subgroups. So in this book we develop SMT in the spirit of [61,62,67]. In particular, we do not discuss Littelmann's work or quantum group theory. For details of Littelmann's work, one could refer to the original papers [74,75,77], or the book [59].

1.5 Some features of the SMT approach

Let us now discuss some features and advantages of the Schubert variety theoretic approach to CIT. This approach is more conceptual than that of [22], the reason being that Schubert varieties provide a powerful inductive tool facilitating the proofs. It also opens doors for generalization to other groups/situations.

An important feature in the Schubert-variety-theoretic approach is the qualitative description of quadratic straightening relations—a straightening relation is the expression for a non-standard monomial as a linear sum of standard monomials. In this approach, the generation of the space of all monomials by standard monomials hinges on the qualitative description of certain quadratic straightening relations; further, the proof of the linear independence of standard monomials is also carried out by using these quadratic relations (and induction on the dimension of a Schubert variety). In the context of CIT, in this approach, such relations are first established on Schubert varieties, and then are specialized to rings of invariants.

Even though, there are some combinatorial description of relations of any degree on a determinantal variety in [26, 101], one cannot deduce the required qualitative description of the quadratic relations from the relations found in loc.cit; the reason for this is that in [26, 101], a typical relation (on a determinantal variety) expresses a given monomial as a linear sum of monomials (not necessarily standard) which are greater than the given monomial for a suitable order (which is different from the order used in SMT) on the set of all monomials. Thus the Schubert-variety-theoretic approach seems to be indispensable from this perspective. Also, the Schubert-variety-theoretic approach seems to be the only approach which can yield the desired qualitative description of the quadratic relations - a crucial step in SMT. Similar remarks apply to the symplectic and the orthogonal group actions also.

Yet another important advantage in this approach is that we obtain the proof of Cohen-Macaulayness for these rings of invariants, something one does not get in the approach of [22]. The rings of invariants in cases **A**, **B**, **C** described in §1.2 above being identified with open subsets of suitable Schubert varieties, the Cohen-Macaulayness for these rings of invariants follows at once from the Cohen-Macaulayness properties for Schubert varieties. The proof, via SMT, of the Cohen-Macaulayness of Schubert varieties in the Grassmannian, namely those that are related to the categorical quotients in case **A**, is proved in [88] (see also Chapter 4). The proof, via SMT, of the general case of Schubert varieties in G/P, where G is semi-simple and P a maximal parabolic subgroup of classical type (in particular for those related to the categorical quotients in cases **A**, **B**, **C** is proved by De Concini-Lakshmibai [21]. While the proof in [88] uses commutative algebra arguments, that in [21] uses "deformation technique". Chirivi [17] has extended the latter technique to the case when G is any semi-simple algebraic group and P is any parabolic subgroup.

The deformation technique consists in constructing a flat family over $\mathbb{A}^1$, with the given variety as the generic fiber (corresponding to $t \in K$ invertible). If the special fiber (corresponding to $t = 0$) is Cohen-Macaulay, then one may conclude the Cohen-Macaulayness of the given variety. Hodge algebras (cf. [20]) are typical examples where the deformation technique affords itself very well. Deformation technique is also used in [12, 33, 46]. The philosophy behind these works is that if there is a "standard monomial basis" for the co-ordinate ring of the given variety, then the deformation technique will work well in general (using the "straightening relations").

In recent times, among the several techniques of proving the Cohen-Macaulayness of algebraic varieties, particularly those that are related to Schubert varieties, two techniques have proved to be quite effective, namely, Frobenius splitting technique and deformation technique. Frobenius splitting technique is used in [104], for example, for proving the (arithmetic) Cohen-Macaulayness of Schubert varieties; further, in [83], the normality of Schubert varieties is proved using the Frobenius splitting technique. This technique is also used in [80, 81, 84] for proving the Cohen-Macaulayness of certain varieties.

1.6 The organization of the book

Turning now to the organization of the book, we have tried to make it self-contained keeping in mind the needs of prospective graduate students and young researchers. After reviewing some basics in Algebraic Geometry and Algebraic Groups in Chapters 2 and 3, we first present SMT for the Grassmannian and its Schubert varieties (Musili's thesis (cf. [88])) in Chapter 4. We then discuss the relationship between determinantal varieties and Schubert varieties in the Grassmannian in Chapter 5. Similar relationships between determinantal varieties in the space of symmetric (respectively skew-symmetric) matrices and Schubert varieties in the symplectic (respectively orthogonal) Grassmannian are established in Chapter 6 (respectively Chapter 7). The main results of SMT are stated in Chapter 8 and are proved in the Appendix. Chapter 9 is a review of GIT. In Chapter 10, using the results of Chapter 5, we describe a basis for the ring of invariants for the $GL_n(K)$-action (cf. Example **A** above) thus giving a Standard Monomial Theoretic proof for DeConcini-Procesi's results in this case. We also discuss the $SL_n(K)$-action on the space considered in (A) above, and describe a basis for the corresponding ring of invariants in Chapter 11. Chapters 6 and 7 describe similar results for the symplectic and orthogonal group actions (cf. Examples **B, C** above) at the same time giving a Standard Monomial Theoretic proof for DeConcini-Procesi's results in these cases. In Chapter 12, we discuss the $SO_n(K)$-action on the space considered in (C) above, and describe a basis for the corresponding ring of invariants; we further deduce some related results on the moduli space M of equivalence classes of semistable rank 2 vector bundles on a smooth projective curve of genus > 2. We also describe a characteristic-free basis for the ring of invariants for the (diagonal) adjoint action of $SL_2(K)$ on $\underbrace{sl_2(K) \oplus \cdots \oplus sl_2(K)}_{m \text{ copies}}$.

In Chapter 13, we discuss some important applications of SMT; we first present a discussion of singular loci of Schubert varieties. Next, we discuss the relationship of ladder determinantal varieties, Quiver varieties and variety of complexes to Schubert varieties, and deduce results for these varieties; in this chapter, as another application of SMT, we present the results of toric degenerations of Schubert varieties in the Grassmannian.

2

Generalities on algebraic varieties

In this chapter, we recollect some basic facts on commutative rings and algebraic varieties. For details, we refer the reader to [27, 37, 79].

2.1 Some basic definitions

Definition 2.1.0.1 *The* Krull dimension*: (or simply the* dimension*) of a commutative ring R (with 1) is the supremum of the lengths of chains of prime ideals of R and it is denoted by dim R. It need not be finite even for Noetherian rings. It is however finite for Noetherian* local *rings.*

If R is a Noetherian local ring with its maximal ideal $\mathbf{m}$ and residue field $\mathbf{k} = R/\mathbf{m}$, then the least cardinality of a set of generators of $\mathbf{m}$ equals $\dim_{\mathrm{k}}(\mathbf{m}/\mathbf{m}^2)$; it is called the *embedding dimension* of R, denoted $e(R)$, and is an upper bound for the dimension of R.

In case R is an integral domain with its field of fractions (or quotients) $K = Q(R)$ and R is also a finitely generated algebra over a field F, then K is a finitely generated field extension of F and $\dim R$ is precisely the *transcendence degree* of K over F. (This is not obvious but follows as an immediate consequence of the *Noether's Normalisation Lemma* - see [86] for a statement and proof of Noether's Normalisation Lemma.)

The localisation of a ring R at a prime ideal $\mathbf{p}$ is denoted by $R_{\mathbf{p}}$. The dimension of $R_{\mathbf{p}}$ is called the *height* of $\mathbf{p}$ and is denoted by $\mathrm{ht}_R(\mathbf{p})$. For a ring R, dimension of R is the supremum of the heights of its maximal ideals and in particular, for a local ring it is the height of its maximal ideal.

Definition 2.1.0.2 *A* regular local ring *is a Noetherian local ring R for which the Krull dimension is the same as its embedding dimension, i.e., dim(R)* $= e(R)$. *We say that a Noetherian ring is* regular *if the localisations at all of its maximal ideals are regular local rings.*

Definition 2.1.0.3 Normal domains: *If R is an integral domain and $K = Q(R)$ (its field of fractions), then we say that R is* normal *if it is* integrally closed *in K, i.e., an element of K which is a root of a* monic *polynomial with coefficients in R must be already in R.*

Normality is a *local property* in the sense that R is normal if and only if $R_{\mathbf{p}}$ (resp.$R_{\mathbf{m}}$) is normal for all prime ideals $\mathbf{p}$ (resp. maximal ideals $\mathbf{m}$) of R. The ring of quotients $S^{-1}R$ of a regular (resp. normal) ring R at any multiplicatively closed subset S is regular (resp. normal).

2.2 Algebraic varieties

2.2.1 Affine varieties

Let K be an algebraically closed field of arbitrary characteristic. For a non–negative integer $n \in \mathbb{Z}^+$, the linear space K^n is called the *Affine* n–space over K and is denoted by $\mathbb{A}^n$ (by convention, $\mathbb{A}^0 = (0)$, a point). Let $A = K[X_1, \cdots, X_n]$ be the ring of polynomial functions on $\mathbb{A}^n$. The field $K(X_1, \cdots, X_n)$ which is the field of fractions of A is called the field of *rational functions* on $\mathbb{A}^n$ and is also denoted by $K(\mathbb{A}^n)$.

For a point $P = (a_1, \cdots, a_n) \in \mathbb{A}^n$, the a_i's are called the coordinates of P. An element $f \in A$ defines a K-valued function on $\mathbb{A}$ by evaluation, namely, $P \mapsto f(P) = f(a_1, \cdots, a_n)$. The subring $\mathcal{O}_P = \mathcal{O}_P(\mathbb{A}^n) := \{f/g \in K(\mathbb{A}^n) \mid g(P) \neq 0\}$ which is a *local ring* is called the local ring of $\mathbb{A}^n$ at P.

Recall (see [86] for example) that the ideal $\mathbf{m} =< X_1 - a_1, \cdots, X_n - a_n >$ is maximal in A for all $(a_1, \cdots, a_n) \in K^n$ (for every field K, algebraically closed or not). The converse is true only for algebraically closed fields K (which is known as the **Hilbert's Nullstellensatz** - see [86] for further details). Thus points of $\mathbb{A}^n$ for K algebraically closed can be identified with the set $\text{Spm}(A)$ of all maximal ideals of the ring A. Given $f \in A$, of total degree $d \geq 1$, the set of zeros of f in K^n, namely,

$$V(f) := \{(a_1, \cdots, a_n) \in \mathbb{A}^n \mid f(a_1, \cdots, a_n) = 0\}$$

is called the *(affine) hypersurface* whose equation is f and d is also called the degree of $V(f)$. If $d = 1$, $V(f)$ is called a *hyperplane*. It can be seen that a point $P = (a_1, \cdots, a_n) \in V(f) \iff f \in \mathbf{m}_P$ where $\mathbf{m}_P =< X_1 - a_1, \cdots, X_n - a_n >$, i.e., $V(f)$ can be identified with the set of all maximal ideals of the quotient ring $A/ < f >$.

Given $T \subseteq A$, the set $V(T) = \cap_{f \in T} V(f)$ of all **common zeros** of elements of T is called an *affine algebraic* (or simply an *algebraic*) subset of $\mathbb{A}^n$. The family of all algebraic subsets satisfy the axioms of closed sets and the corresponding topology is called the *Zariski topology* on $\mathbb{A}^n$.

Coordinate Rings

Let $J(T)$ be the ideal generated by T in A and $\sqrt{J(T)}$, the *radical* of $J(T)$: recall the definition of the radical of an ideal J, denoted $\sqrt{J}$,

$$\sqrt{J} := \{f \in K[X_1, \cdots, X_n] | f^r \in J \text{ for some } r \geq 1\}$$

We say that J is a *radical ideal* if $J = \sqrt{J}$.

Since A is Noetherian, any ideal in A is finitely generated. Let then $J(T) =< f_1, \cdots, f_m >$ and $\sqrt{J(T)} =< g_1, \cdots, g_\ell >$. Now it follows that

$$V(T) = \bigcap_{f \in T} V(f) = \bigcap_{j=1}^{m} V(f_j) = \bigcap_{f \in J(T)} V(f) = \bigcap_{f \in \sqrt{J(T)}} V(f) = \bigcap_{j=1}^{\ell} V(g_j),$$

i.e., every algebraic subset is a finite intersection of hypersurfaces. In fact, we have

$$V(T) = \bigcap_{f \in J(T)} V(f) = V(f_1, \cdots, f_m) = V(g_1, \cdots, g_\ell) = \bigcap_{f \in \sqrt{J(T)}} V(f).$$

Given a subset $Y \subseteq \mathbb{A}^n$, let $I(Y) = \{f \in A^n \mid f(y) = 0, \ \forall\, y \in Y\}$, which is the largest ideal in A vanishing on Y, called *the ideal of* Y. We have the following basic facts:

$$V(T) \supseteq V(S) \iff \sqrt{J(T)} \subseteq \sqrt{J(S)}$$
$$V(I(Y)) = \overline{Y}$$
$$I(V(T)) = \sqrt{J(T)}$$

Thus $\sqrt{J(T)}$ is the ideal of $V(T)$. If $V = V(T)$ and $I = \sqrt{J(T)}$, the ring A/I is called the *coordinate ring* of V and is denoted by $K[V]$. We also write $V = \operatorname{Spec} K[V]$ and the points of V can be identified with the set of all maximal ideals of $K[V]$, i.e., with the set of all maximal ideals in A containing the ideal I of V.

The Local Rings $\mathcal{O}_P(V)$

For $P = (a_1, \cdots, a_n) \in V$, the ideal $\mathbf{m}_P =< X_1 - a_1, \cdots, X_n - a_n >$ (mod I) is called the maximal ideal of V at P and the local ring $\mathcal{O}_P(V) := \mathcal{O}_P(\mathbb{A})/I =$ the localization of $K[V]$ at the maximal ideal $\mathbf{m}_P$ is called the *local ring* of V at P, or also the *stalk* at P.

An algebraic subset V whose ideal is a prime ideal $\mathbf{p}$ is called an *affine variety* (defined by $\mathbf{p}$) so that its coordinate ring $K[V] = A/\mathbf{p}$ is an integral domain. The field $Q(K[V])$ of fractions of $K[V]$ is called the field of *rational functions* on V and it is denoted by $K(V)$.

Definition 2.2.1.1 *An affine variety V is said to be* non-singular *or* smooth at a point *$P \in V$ if its local ring $\mathcal{O}_P(V)$ at P is regular. We say that V is non-singular or smooth if it is so at all of its points.*

Definition 2.2.1.2 *An affine variety V is said to be* normal at a point *$P \in V$ if its local ring $\mathcal{O}_P(V)$ at P is a normal domain. We say that V is* normal *if it is so at all of its points.*

Definition 2.2.1.3 *An affine variety V is said to be* factorial at a point $P \in V$ *if its local ring $\mathcal{O}_P(V)$ at P is a unique factorization domain (i.e., every element in $\mathcal{O}_P(V)$ has a unique factorization as a product of prime elements in $\mathcal{O}_P(V)$). We say that V is* factorial *if it is so at all of its points.*

Definition 2.2.1.4 *The* topological dimension *of an algebraic subset V is defined as the supremum of the lengths of chains of algebraic varieties contained in V and it is denoted by $\dim V$.*

The topological dimension of V is the same as the Krull dimension of its coordinate ring $K[V]$. For example, one can see that the dimension of $\mathbb{A}^n$ is n and dimension of $V(f)$ is $n-1$ (if degree of f is positive).

Projective Varieties

Definition 2.2.1.5 ***Projective Space:*** *Given a non-zero vector space L over K, the set of all lines (i.e., one dimensional subspaces) in L is called the* projective space *associated to L and is denoted by $\mathbb{P}(L)$. It is also called the projective n–space if L is of dimension $n+1$ over K.*

Choosing a basis of L and writing $L = K^{n+1}$ $(n \geq 0)$, (with respect to that basis), we write $\mathbb{P}^n = \mathbb{P}^n_K$ for $\mathbb{P}(K^{n+1})$ or simply $\mathbb{P}$ if there is no ambiguity about K and n.

Convention: $\mathbb{P}^n =$ a point if $n = 0$.

A point P in $\mathbb{P}^n$ has $n+1$ **homogeneous** coordinates $(a_0, a_1, \cdots, a_n)$, $a_j \in K$, in the sense that (1) not all a_j's are zero and (2) both $(a_0, a_1, \cdots, a_n)$ and $(\lambda a_0, \lambda a_1, \cdots, \lambda a_n)$ represent the same point for all $\lambda \in K^\star$.

Let $S = K[X_0, X_1, \cdots, X_n]$ be the polynomial ring in $n+1$ variables over K. Unlike the affine case, an element $f \in S$ does not define a function on $\mathbb{P}^n$. Nevertheless, we can talk about the vanishing or non-vanishing of a *homogeneous polynomial* $f \in S$ at a point $P = (a_0, a_1, \cdots, a_n) \in \mathbb{P}^n$ because we have $f(\lambda(a_0, a_1, \cdots, a_n)) = \lambda^d f((a_0, a_1, \cdots, a_n))$ if d is the total degree of f. This allows us to define the *(projective) hypersurface* $V_+(f)$ whose *equation* is a homogeneous polynomial f of total degree d, i.e., $V_+(f) = \{P = (a_0, a_1, \cdots, a_n) \in \mathbb{P}^n \mid f(P) = 0\}$. As before, the degree of f is called the degree of $V_+(f)$. It is called a *hyperplane* if $d = 1$, etc.

Given $T \subseteq S$ but consisting of only homogeneous polynomials of possibly different degrees, the set $V_+(T) = \cap_{f \in T} V_+(f)$ of all *common zeros* of elements of T is called a *projective algebraic* (or simply *an algebraic*) subset of $\mathbb{P}^n$. The family of all algebraic subsets satisfy the axioms of closed sets and the corresponding topology is again called the *Zariski topology* on $\mathbb{P}^n$.

Now keep the natural gradation on $S = \sum_{m \geq 0} S_m$ (where $S_0 = K$) and proceed exactly as before replacing "ideals" by "homogeneous ideals", "generators" of ideals by "homogeneous generators", etc., and define *projective varieties* as $V_+(\mathbf{p})$ for homogeneous prime ideals $\mathbf{p}$ of S. The graded ring $S/\mathbf{p}$ is called the *homogeneous coordinate ring* of $V_+(\mathbf{p})$. We write $V_+(\mathbf{p})$ (or just V_+) = Proj$(S/\mathbf{p})$. Consider the affine variety $V = V(\mathbf{p}) = \text{Spec } S/\mathbf{p}$ defined by $\mathbf{p}$ in K^{n+1}. We note that the origin $(0, \cdots, 0)$ of K^{n+1} is a point of the affine variety V because $\mathbf{p}$ is a homogeneous ideal of S. The affine variety V is called the *cone over* V_+, and origin is the *vertex* of the cone.

Nonsingularity, normality, factoriality etc. are defined for a projective variety in the same way as for the affine varieties. Some of the basic facts are the following:

1. A variety which is both affine and projective is a single point.
2. Every projective variety in $\mathbb{P}^n$ has a finite open covering by suitable affine subvarieties of K^n.
3. The affine variety V is a *cone* over the projective variety V_+ as the *base* and origin as the *vertex*.
4. $\dim(V) = 1 + \dim(V_+)$.
5. The linear space K^{n+1} is the cone over $\mathbb{P}^n$.
6. The projective variety V_+ is non–singular (resp. normal, factorial) at all of its points $\Leftrightarrow$ V is non–singular (resp. normal, factorial) at all points outside the vertex.
7. V is non–singular at its vertex $\Leftrightarrow$ $V = K^m$ is linear $\Leftrightarrow$ $V_+ = \mathbb{P}^{m-1}$ for some $m \geq 1$.
8. V is normal (resp. factorial) at its vertex $\Leftrightarrow$ V is so at all of its points $\Leftrightarrow$ $S/\mathbf{p}$ is so.

Definition 2.2.1.6 ***Projective Normality:*** *A projective variety V_+ is said to be* projectively *(or* arithmetically*)* normal *(resp.* factorial*) if its cone V is normal (resp. factorial) at its vertex, i.e., the stalk at the vertex is a normal domain (resp. a unique factorization domain).*

The following example shows that projective normality is a property of the particular projective embedding of the variety (unlike the affine varieties). The projective line $\mathbb{P}^1$ is obviously projectively normal since its cone is the affine plane K^2 (which is non-singular). However, it can be also embedded in $\mathbb{P}^3$ as the **quartic curve**, namely,

$$V_+ = \{(a^4, a^3b, ab^3, b^4) \in \mathbb{P}^3 \mid (a, b) \in \mathbb{P}^1\},$$

i.e., $V_+ = V_+(XT - YZ, TY^2 - XZ^2)$, but the coordinate ring of its cone V which is $K[X, Y, Z, T]/(XT - YZ, TY^2 - XZ^2)$ is not normal.

Using the language of Weil divisors, Cartier divisors and line bundles on V_+ (as in the affine case), we define the divisor class group $\text{Cl}(V_+)$ and the Picard group $\text{Pic}(V_+)$ of V_+.

In $\mathbb{P}^n = V_+(\mathbf{0})$, any hypersurface (of degree d) is a Cartier divisor. Two hypersurfaces define the same divisor class if and only if they have the same degree. The line bundle corresponding to the divisor class of a hyperplane is called the *hyperplane bundle* or the *tautological line bundle* on $\mathbb{P}^n$ and it is denoted by $\mathbf{O}_{\mathbb{P}}(1)$. The line bundle corresponding to the divisor class of a hypersurface of degree d is then the d^{th} tensor power of $\mathbf{O}_{\mathbb{P}}(1)$ and it is denoted by $\mathbf{O}_{\mathbb{P}}(d)$ $(= \mathbf{O}_{\mathbb{P}}(1)^{\otimes d})$. The space $H^0(\mathbb{P}^n, \mathbf{O}_{\mathbb{P}}(d))$ of *global sections* of $\mathbf{O}_{\mathbb{P}}(d)$ turns out to be S_d, the homogeneous component of S of degree d (see [37] for details).

For a projective subvariety $X = V_+(\mathbf{p})$ of $\mathbb{P}^n$, the restriction of $\mathbf{O}_{\mathbb{P}}(1)$ to X gives a line bundle, called the *(induced) hyperplane bundle* on X (for this embedding of X in $\mathbb{P}^n$) and is denoted by $\mathbf{O}_X(1)$. It follows that the space of sections $H^0(X, \mathbf{O}_X(d))$ contains the homogeneous component $(S/\mathbf{p})_d$ but need not be equal (it is well-known that equality holds for d sufficiently large). However, if X is projectively normal, then equality holds for all d.

More general varieties: Let X be a locally closed (i.e., an open subset of a closed subset) in the ambient space Z (Z being $\mathbb{A}^n$ (respectively $\mathbb{P}^n$) for some $n \in \mathbb{N}$). Then X has a natural variety structure and one refers to X as a *locally closed subvariety* of Z or also as a *quasi-affine* (respectively *quasi-projective*) variety. All of the above concepts extend naturally to these more general varieties. In the sequel, by an "algebraic variety", we shall mean an affine or quasi affine or projective or quasi projective variety.